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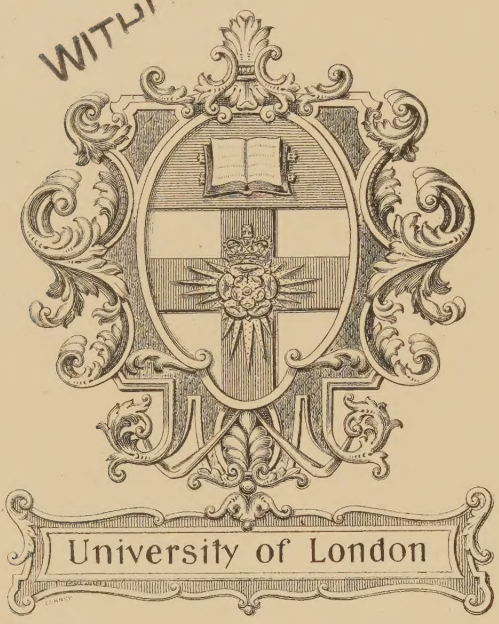

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PAPERS

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THE MEAN VALUES OF CERTAIN DIRICHLET SERIES, II

By H. S. A. POTTER.

[Received 16 September, 1939.—Read 15 December, 1939.]

1. *Introduction.*

Hecke has recently discussed a class of Dirichlet series which satisfy a functional equation. We say that the function $\phi(s)$ has the signature $\{\lambda, k, \gamma\}$, where $\lambda > 0$, $k > 0$ and $\gamma = \pm 1$, if

(I) $(s-k)\phi(s)$ is an integral function of finite genus,

(II) $R(s) = \gamma R(k-s)$, where

$$R(s) = (2\pi/\lambda)^{-s}\,\Gamma(s)\,\phi(s), \tag{1}$$

and

(III) $\phi(s)$ is expressible as an ordinary Dirichlet series $\Sigma a_n n^{-s}$ convergent in some half-plane.

Hecke has found* the number of functions having a given signature. He

* E. Hecke, **1**.

has shown* that under certain conditions $\phi(s)$ has an infinity of zeros on its critical line $\sigma = \frac{1}{2}k$. In my first paper† I discussed the asymptotic value of

$$\frac{1}{2T}\int_{-T}^{T} |\phi(\sigma+it)|^2\,dt$$

when $\sigma > \frac{1}{2}k$ and when $\sigma < \frac{1}{2}k$. In the present paper I study the mean value of $\phi(s)$ on the critical line $\sigma = \frac{1}{2}k$. By a method used by Titchmarsh‡ I prove

THEOREM 1. *Suppose that*

$$\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

has the signature $\{\lambda, k, \gamma\}$ *and that*

$$\sum_{n<x} |a_n|^2 \sim x^k L(x), \tag{2}$$

where $L(x)$ *is a slowly increasing function*§. *Let*

$$L_1(x) = \int_1^x \frac{L(t)}{t}\,dt \tag{3}$$

and suppose that $L_1(x) \to \infty$ *as* $x \to \infty$. *Then*

$$\int_0^T |\phi(\tfrac{1}{2}k+it)|^2\,dt \sim 2kTL_1(T). \tag{4}$$

We may define a wider class of Dirichlet series in the following manner. Suppose that

(I′) $(s-k)\phi_1(s)$, $(s-k)\phi_2(s)$ are integral functions of finite genus,

(II′) $WR_1(s) = \bar{W}R_2(k-s)$,

where

$$R_\nu(s) = (2\pi/\lambda)^{-s}\Gamma(s)\phi_\nu(s) \quad (\nu = 1, 2), \quad \lambda > 0, \quad k > 0, \quad |W| = 1, \tag{5}$$

---

* E. Hecke, **3**.
† H. S. A. Potter, **1**.
‡ E. C. Titchmarsh, **1**.
§ This is defined in § 2.

and

(III′) $\phi_1(s)$, $\phi_2(s)$ are expressible as ordinary Dirichlet series,

$$\phi_1(s) = \sum_{n=1}' a_n n^{-s}, \qquad \phi_2(s) = \sum_{n=1}' b_n n^{-s}, \tag{6}$$

convergent in some half-plane.

In this case we have

THEOREM 2. *Suppose that $\phi_1(s)$, $\phi_2(s)$ satisfy the conditions* (I′), (II′) (III′). *Suppose further that the coefficients $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are all real and*

$$\sum_{n<x} a_n^2 \sim a x^k L(x) \tag{7}$$

and

$$\sum_{n<x} b_n^2 \sim b x^k L(x), \tag{8}$$

where $L(x)$ satisfies the conditions of Theorem 1. *Then*

$$\int_0^T |\phi_\nu(\tfrac{1}{2}k+it)|^2 dt \sim k(a+b) T L_1(T) \quad (\nu = 1, 2). \tag{9}$$

In § 5 I prove some related theorems to deal with the case where the coefficients a_n, b_n are not real. Finally, in § 6, I give several illustrations.

2. *Slowly increasing functions.*

We shall say that $L(x)$ is a *slowly increasing* function if $L(x)$ is positive and continuous for all $x > 0$ and

$$L(cx) \sim L(x) \tag{10}$$

for every $c > 0$, as $x \to \infty$ [$L(x)$ need not tend to infinity]. We use the following properties of such functions*:

(i) For any fixed numbers a, b such that $b > a > 0$,

$$L(ux) \sim L(x)$$

uniformly in $a \leqslant u \leqslant b$, as $x \to \infty$.

(ii) $t^{-\epsilon} L(t) \to 0$, as $t \to \infty$.

---

* See J. Karamata, **1**.

LEMMA 1. *As* $x \to \infty$,

$$L(xu) = O\Big(u^{\epsilon} L(x)\Big)$$

uniformly for $u \geqslant 1$, *and as* $t \to \infty$

$$L(t) = O\Big(u^{-\epsilon} L(t/u)\Big)$$

uniformly for $0 < u < 1$.

Proof. By (i) there is a number x_0 such that

$$e^{-\epsilon} < L(tx)/L(x) < e^{\epsilon}$$

uniformly for $1/e \leqslant t \leqslant e$, if $x > x_0$. First suppose that $u \geqslant 1$ and choose the integer N so that $e^N < u \leqslant e^{N+1}$. Then

$$\frac{L(xu)}{L(x)} = \frac{L(xue^{-N})}{L(x)} \prod_{n=1}^{N} \frac{L(xue^{1-n})}{L(xue^{-n})}$$

$$< e^{\epsilon(N+1)} < e^{\epsilon} u^{\epsilon},$$

provided that $x > x_0$. Now suppose that $0 < u < 1$ and choose N so that $e^N < 1/u \leqslant e^{N+1}$. Then

$$\frac{L(xu)}{L(x)} = \frac{L(xue^{N})}{L(x)} \prod_{n=1}^{N} \frac{L(xue^{n-1})}{L(xue^{n})}$$

$$< e^{\epsilon(N+1)} < e^{\epsilon} u^{-\epsilon},$$

provided that $xu > x_0$.

LEMMA 2. *Let* $$L_1(x) = \int_1^x \frac{L(t)}{t}\, dt,$$

where $L(t)$ *is a slowly increasing function. Then* $L_1(x)$ *is also a slowly increasing function and*

$$L(x) = o\Big(L_1(x)\Big).$$

Proof. By (i), $L(t) > \frac{1}{2} L(x)$ uniformly for $\lambda x \leqslant t \leqslant x$ if x is large enough. Hence, for sufficiently large x,

$$L_1(x) > \int_{\lambda x}^{x} \frac{L(t)}{t}\, dt > \tfrac{1}{2} \log(1/\lambda) \cdot L(x) = \frac{1}{2\epsilon}\, L(x)$$

if $\lambda = e^{-1/\epsilon}$. That is to say, $L(x) = o\Big(L_1(x)\Big)$.

Also by (i), if $c>0$,

$$L_1(cx)-L_1(x)=\int_x^{cx}\frac{L(t)}{t}\,dt\sim\log c\,.\,L(x)=o\Big(L_1(x)\Big).$$

That is $L_1(cx)\sim L_1(x)$, so that $L_1(x)$ is slowly increasing.

3. *Various lemmas.*

LEMMA 3. *If*

$$F_1(z)=\sum_{n=1}^{\infty}a_n e^{-(2\pi/\lambda)nz},\qquad F_2(z)=\sum_{n=1}^{\infty}b_n e^{-(2\pi/\lambda)nz},$$

and $M=(2\pi/\lambda)^{-k}\Gamma(k)$ *and* a_ν $(\nu=1,\ 2)$ *is the residue of* $\phi_\nu(s)$ *at* $s=k$, *then*

$$F_1(z)-\frac{Ma_1}{z^k}=\frac{W^{-2}}{z^k}\Big\{F_2\Big(\frac{1}{z}\Big)-Ma_2z^k\Big\}. \tag{11}$$

Also, if $\theta=\frac{1}{2}\pi-\delta$, *then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|R_1(\tfrac{1}{2}k+it)|^2e^{2\delta t}\,dt \tag{12}$$

$$=\int_1^{\infty}\Big|F_1(xe^{i\vartheta})-\frac{Ma_1}{x^ke^{ik\theta}}\Big|^2x^{k-1}\,dx+\int_1^{\infty}\Big|F_2(xe^{-i\vartheta})-\frac{Ma_2}{x^ke^{-ik\vartheta}}\Big|^2x^{k-1}\,dx.$$

Proof. We have, if $\Re(z)>0$,

$$F(z)=\frac{1}{2\pi i}\sum_{n=1}^{\infty}a_n\int_{c-i\infty}^{c+i\infty}\{(2\pi/\lambda)nz\}^{-s}\Gamma(s)\,ds$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}R_1(s)\,z^{-s}\,ds.$$

provided that the line $\sigma=c$ lies in the half-plane of absolute convergence of $\phi_1(s)$. Put $z=xe^{i\theta}$, $\theta=\frac{1}{2}\pi-\delta$, $\delta>0$, $s=\sigma+it$. Then

$$R_1(s)\,z^{-s}=O(|t|^Ke^{-\delta|t|})$$

uniformly in $\frac{1}{2}k\leqslant\sigma\leqslant c$ as $|t|\to\infty$. Hence we may move the contour from $\sigma=c$ to $\sigma=\frac{1}{2}k$, and, since there is a pole at $s=k$, we obtain

$$F_1(z)-Ma_1z^{-k}=\frac{1}{2\pi i}\int_{\frac{1}{2}k-i\infty}^{\frac{1}{2}k+i\infty}R_1(s)\,z^{-s}\,ds. \tag{13}$$

By the functional equation II′,

$$\frac{1}{2\pi i}\int_{\frac{1}{2}k-i\infty}^{\frac{1}{2}k+i\infty} R_1(s)\,z^{-s}\,ds = \frac{W^{-2}}{2\pi i}\int_{\frac{1}{2}k-i\infty}^{\frac{1}{2}k+i\infty} R_2(k-s)\,z^{-s}\,ds$$

$$= \frac{W^{-2}}{2\pi i z^k}\int_{\frac{1}{2}k-i\infty}^{\frac{1}{2}k+i\infty} R_2(s)\,z^s\,ds.$$

Hence

$$F_1(z) - Ma_1 z^{-k} = \frac{W^{-2}}{z^k}\left\{F_2\left(\frac{1}{z}\right) - Ma_2 z^k\right\}. \tag{14}$$

Again, if we put

$$G_1(x) = F_1(xe^{i\theta}) - Ma_1(xe^{i\theta})^{-k},$$

$$g_1(s) = R_1(s)\,e^{i\theta s},$$

(13) becomes

$$G_1(x) = \frac{1}{2\pi i}\int_{\frac{1}{2}k-i\infty}^{\frac{1}{2}k+i\infty} g_1(s)\,x^{-s}\,ds,$$

whence we have the Parseval equation*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |g_1(\tfrac{1}{2}k+it)|^2\,dt = \int_0^{\infty} |G_1(x)|^2\,x^{k-1}\,dx. \tag{15}$$

Also, by the functional equation (14),

$$\int_0^{\infty} |G_1(x)|^2\,x^{k-1}\,dx = \int_1^{\infty}\left|F_1(xe^{i\theta}) - \frac{Ma_1}{x^k e^{ik\theta}}\right|^2 x^{k-1}\,dx \tag{16}$$

$$+\int_0^1 |F_2(e^{-i\theta}/x) - Ma_2 x^k e^{ik\theta}|^2\,x^{-k-1}\,dx$$

$$= \int_1^{\infty}\left|F_1(xe^{i\theta}) - \frac{Ma_1}{x^k e^{ik\theta}}\right|^2 x^{k-1}\,dx$$

$$+\int_1^{\infty}\left|F_2(xe^{-i\theta}) - \frac{Ma_2}{x^k e^{-ik\theta}}\right|^2 x^{k-1}\,dx.$$

Equation (12) follows from (15) and (16).

LEMMA 4. *If $\phi_1(s)$ satisfies the conditions of Theorem* 1 *or* 2 *then, as* $\delta \to +0$,

$$\int_1^{\infty} |\phi_1(\tfrac{1}{2}k+it)|^2\,t^{\rho-1}\,e^{-\delta t}\,dt = O(\delta^{-\kappa-\epsilon}),$$

where $\kappa = \max(\rho,\ 0)$.

* Bochner, **1**, § 36.

Proof. In a previous paper* it was proved that

$$\int_1^x |\phi_1(\sigma+it)|^2\,dt = O(x)$$

for $\sigma > \frac{1}{2}k$. Since the mean order function is continuous, we have

$$p(x) = \int_1^x |\phi(\tfrac{1}{2}k+it)|^2\,dt = O(x^{1+\varepsilon}).$$

Thus, if $\delta > 0$, by partial integration,

$$\begin{aligned}\int_1^\infty |\phi_1(\tfrac{1}{2}k+it)|^2\,t^{\rho-1}e^{-\delta t}\,dt &= \int_1^\infty p(t)\,t^{\rho-2}(\delta t+1-\rho)\,e^{-\delta t}\,dt\\ &= O\Big(\int_1^\infty t^{\rho-1+\varepsilon}(\delta t+|1-\rho|)\,e^{-\delta t}\,dt\Big)\\ &= O\Big(\int_0^\infty t^{\kappa-1+\epsilon}(\delta t+|1-\rho|)\,e^{-\delta t}\,dt\Big)\\ &= O(\delta^{-\kappa-\varepsilon}),\end{aligned}$$

where $\kappa = \max(\rho, 0)$.

LEMMA 5. *Suppose that $p(t)$ is a positive function such that*

$$\int_0^\infty p(t)\,t^{k-1}e^{-\delta t}\,dt \quad (k>0)$$

is convergent when $\delta > 0$.

(*a*) *If, as $\delta \to 0$,*

$$\int_0^\infty p(t)\,t^{k-1}e^{-\delta t}\,dt \sim \Gamma(k)\,\delta^{-k}\,L(1/\delta), \tag{17}$$

where $L(x)$ is a slowly increasing function, then

$$\int_0^T p(t)\,dt \sim TL(T). \tag{18}$$

(*b*) *If*

$$\int_0^\infty p(t)\,t^{k-1}e^{-\delta t}\,dt = O\big(\delta^{-k}\,L(1/\delta)\big), \tag{19}$$

then

$$\int_0^T p(t)\,dt = O\big(TL(T)\big). \tag{20}$$

---

* H. S. A. Potter, **1**.

Proof. (*a*) This follows immediately from a general Tauberian theorem due to Karamata*.

(*b*) Let $$P(x) = \int_0^x p(t)\, t^{k-1}\, dt.$$

Then

$$P(x) < e \int_0^x p(t)\, t^{k-1}\, e^{-t/x}\, dt < e \int_0^\infty p(t)\, t^{k-1}\, e^{-t/x}\, dt < K x^k\, L(x),$$

by (19). Hence

$$\begin{aligned} \int_0^T p(t)\, dt &= \int_0^T t^{1-k}\, P'(t)\, dt \\ &= (k-1) \int_0^T P(t)\, t^{-k}\, dt + T^{1-k}\, P(T) \\ &= O\Big(\int_0^T L(t)\, dt \Big) + O\big(T L(T) \big) \\ &= O\big(T L(T) \big). \end{aligned}$$

LEMMA 6. (*a*) *Suppose that* $\Re(z) > 0$, $k > 0$. *Then*

$$\int_1^\infty e^{-zt}\, t^{k-1}\, dt = \Gamma(k)\, z^{-k} + O(1).$$

(*b*) *If* $|z| > h$ *and* $|\arg z| \leqslant \pi - \epsilon < \pi$, *then*

$$\int_1^\infty e^{-zt}\, t^{k-1}\, dt = \frac{e^{-z}}{z} \Big\{ 1 + O\Big(\frac{1}{|z|} \Big) \Big\},$$

where the constant implied in the O depends on h, ϵ, *and k, but not on z.*

Proof. (*a*) We have

$$\begin{aligned} \Big| \int_1^\infty e^{-zt}\, t^{k-1}\, dt - \Gamma(k)\, z^{-k} \Big| &= \Big| \int_0^1 e^{-zt}\, t^{k-1}\, dt \Big| \\ &< \int_0^1 t^{k-1}\, dt = \frac{1}{k}. \end{aligned}$$

* Karamata, **2**, main Theorem 1, 28.

(*b*) This result concerning the asymptotic behaviour of the incomplete gamma function is proved, for instance, by Whittaker and Watson, *Modern Analysis* (1927), 341–343.

LEMMA 7. *Let $L(x)$ be a steady increasing function and let*

$$L_1(x) = \int_1^x \frac{L(t)}{t}\,dt,$$

and suppose that $L_1(x) \to \infty$, as $x \to \infty$.

(*a*) *If*
$$A(x) \equiv \sum_{n<x} |a_n|^2 = O\big(x^k L(x)\big),$$

where $k > 0$, then

$$\sum_{n=1}^{\infty} n^\rho |a_n|^2 e^{-\delta n} = O\Big(\delta^{-k-\rho} L(1/\delta)\Big)$$

provided that $\rho > -k$ and

$$\sum_{n<x} n^{-k} |a_n|^2 = O\big(L_1(x)\big).$$

(*b*) *If*
$$A(x) \sim x^k L(x),$$

then
$$\sum_{n<x} n^{-k} |a_n|^2 \sim k L_1(x).$$

Proof. (*a*) We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^\rho |a_n|^2 e^{-\delta n} \\
&= \int_1^\infty t^\rho e^{-\delta t}\,dA(t) \\
&= \int_1^\infty A(t)\, t^{\rho-1} e^{-\delta t}(\delta t - \rho)\,dt \\
&= O\Big(\int_1^{1/\delta} t^{k+\rho-1} L(t)\,dt\Big) + O\Big(\delta \int_{1/\delta}^\infty t^{k+\rho} L(t) e^{-\delta t}\,dt\Big) \\
&= O\Big(\delta^{-k-\rho} L(1/\delta) \int_\delta^1 u^{k+\rho-1}\,du\Big) + O\Big(\delta^{-k-\rho} L(1/\delta) \int_1^\infty u^{k+\rho+\epsilon} e^{-u}\,du\Big) \\
&= O\Big(\delta^{-k-\rho} L(1/\delta)\Big),
\end{aligned}$$

by Lemma 1, if we choose $\epsilon < k+\rho$.

Also
$$\sum_{n<x} n^{-k}|a_n|^2 = \int_1^x t^{-k}\,dA(t)$$
$$= k\int_1^x A(t)\,t^{-k-1}\,dt + x^{-k}A(x)$$
$$= O\big(L_1(x)\big) + O\big(L(x)\big)$$
$$= O\big(L_1(x)\big),$$
by Lemma 2.

(*b*) If $A(x)\sim x^k L(x)$, we have, since $L_1(x)\to\infty$,
$$\sum_{n<x} n^{-k}|a_n|^2 = k\int_1^x A(t)\,t^{-k-1}\,dt + x^{-k}A(x)$$
$$\sim kL_1(x) + L(x) \sim kL_1(x),$$
by Lemma 2.

4. *Proof of Theorems* 1 *and* 2.

Since
$$|\Gamma(\tfrac{1}{2}k+it)|^2 = 2\pi e^{-\pi|t|}|t|^{k-1}\{1+O(|t|^{-1})\},$$
we have
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |R_1(\tfrac{1}{2}k+it)|^2 e^{(\pi-2\delta)t}\,dt$$
$$= \left(\frac{\lambda}{2\pi}\right)^k \int_0^\infty |\phi_1(\tfrac{1}{2}k+it)|^2 t^{k-1}e^{-2\delta t}\,dt + O\left(\int_1^\infty |\phi_1(\tfrac{1}{2}k+it)|^2 t^{k-2}e^{-2\delta t}\,dt\right) + O(1)$$
$$= \left(\frac{\lambda}{2\pi}\right)^k \int_0^\infty |\phi_1(\tfrac{1}{2}k+it)|^2 t^{k-1}e^{-2\delta t}\,dt + O(\delta^{-\kappa-\epsilon}),$$
by Lemma 4, where $\kappa = \max(k-1, 0)$. Hence, by Lemma 3,

(21)
$$\left(\frac{\lambda}{2\pi}\right)^k \int_0^\infty |\phi_1(\tfrac{1}{2}k+it)|^2 t^{k-1}e^{-2\delta t}\,dt + O(\delta^{-k})$$
$$= \int_1^\infty \left| F_1(xe^{i\theta}) - \frac{Ma_1}{x^k e^{ik\theta}}\right|^2 x^{k-1}\,dx + \int_1^\infty \left| F_2(xe^{-i\theta}) - \frac{Ma_2}{x^k e^{-ik\theta}}\right|^2 x^{k-1}\,dx$$
$$= I_1 + I_2 + O(1),$$

where
$$I_1 = \int_0^\infty \big(|F_1(xe^{i\theta})|^2 + |F_2(xe^{-i\theta})|^2\big)\,x^{k-1}\,dx$$

and

$$I_2 = -2M\Re\left[\bar{a}_1 e^{ik\theta}\int_1^\infty F_1(xe^{i\theta})\frac{dx}{x} + \bar{a}_2 e^{-ik\theta}\int_1^\infty F_2(xe^{-i\theta})\frac{dx}{x}\right]$$

$$= -2M\Re\left[\bar{a}_1 e^{ik\theta} I_{21} + \bar{a}_2 e^{-ik\theta} I_{22}\right],$$

say. First of all, by the second mean-value theorem, we have, for some $\xi > 1$,

$$\Re(I_{21}) = \int_1^\xi \Re\{F_1(xe^{i\theta})\}\,dx$$

$$= \frac{\lambda}{2\pi}\,\Re\left[\sum_{n=1}^\infty a_n \frac{\exp\{-(2\pi/\lambda)nxe^{i\theta}\}}{ne^{i\theta}}\right]_1^\xi$$

$$= O\left(\sum_{n=1}^\infty \frac{|a_n|}{n}\exp\{-(2\pi/\lambda)n\sin\delta\}\right)$$

$$= O\left\{\left(\sum_{n=1}^\infty |a_n|^2 e^{-(2\pi/\lambda)n\sin\delta}\right)^{\frac{1}{2}}\right\}$$

$$= O\left\{\left(\delta^{-k}L(1/\delta)\right)^{\frac{1}{2}}\right\} = O(\delta^{-k}),$$

by Lemma 7 and § 2(ii). Similarly $\Im(I_{21}) = O(\delta^{-k})$, so that $I_{21} = O(\delta^{-k})$. By the same method we find that $I_{22} = O(\delta^{-k})$. Hence

$$I_2 = O(\delta^{-k}). \tag{22}$$

Also

$$I_1 = \sum_{n<T}(|a_n|^2+|b_n|^2)\int_1^\infty \exp\{-(4\pi/\lambda)xn\sin\delta\}x^{k-1}\,dx$$

$$+\sum_{n\geqslant T}(|a_n|^2+|b_n|^2)\int_1^\infty \exp\{-(4\pi/\lambda)xn\sin\delta\}x^{k-1}\,dx$$

$$+\sum\sum_{m\neq n}(a_m\bar{a}_n-\bar{b}_m b_n)\int_1^\infty \exp[-(2\pi/\lambda)x\{(m+n)\sin\delta+i(m-n)\cos\delta\}]x^{k-1}\,dx$$

$$= S_1+S_2+S_3,$$

say. Putting $T = 1/2\delta$, we have, by Lemma 6 (*b*),

$$S_2 = O\left(\sum_{n\geqslant T}(|a_n|^2+|b_n|^2)\frac{e^{-(4\pi/\lambda)n\sin\delta}}{n\sin\delta}\right)$$

$$= O\left(\sum_{n=1}^\infty (|a_n|^2+|b_n|^2)e^{-(4\pi/\lambda)n\sin\delta}\right)$$

$$= O\left(\delta^{-k}L(1/\delta)\right).$$

Again, by Lemma 6 (*b*),

$$S_3 = \frac{\lambda}{2\pi} \sum_{m \neq n} \sum (a_m \bar{a}_n + \bar{b}_m b_n) \frac{\exp\left[-(2\pi/\lambda)\{(m+n)\sin\delta + i(m-n)\cos\delta\}\right]}{(m+n)\sin\delta + i(m-n)\cos\delta}$$

$$+ O\left(\sum_{m \neq n} \sum (|a_m a_n| + |b_m b_n|) \frac{e^{-(2\pi/\lambda)(m+n)\sin\delta}}{(m+n)^2 \sin^2\delta + (m-n)^2 \cos^2\delta} \right)$$

$$= S_{31} + O(S_{32}),$$

say. Now

$$\sum_{m \neq n} \sum |a_m a_n| \frac{e^{-(2\pi/\lambda)(m+n)\sin\delta}}{(m+n)^2 \sin^2\delta + (m-n)^2 \cos^2\delta}$$

$$< 2 \sum_{m>n} \sum \frac{|a_m a_n|}{(m-n)^2} e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$= 2 \sum_{g=1}^{\infty} g^{-2} \sum_{m=g+1}^{\infty} |a_m| e^{-(2\pi/\lambda) m \sin\delta} |a_{m-g}| e^{-(2\pi/\lambda)(m-g)\sin\delta}$$

$$\leqslant 2 \sum_{g=1}^{\infty} g^{-2} \left(\sum_{m=g+1}^{\infty} |a_m|^2 e^{-(4\pi/\lambda) m \sin\delta} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |a_n|^2 e^{-(4\pi/\lambda) n \sin\delta} \right)^{\frac{1}{2}}$$

$$< \frac{\pi^2}{3} \sum_{n=1}^{\infty} |a_n|^2 e^{-(4\pi/\lambda) n \sin\delta} = O\left(\delta^{-k} L(1/\delta)\right).$$

Hence $$S_{32} = O\left(\delta^{-k} L(1/\delta)\right).$$

When $a_n = b_n$ for all values of n or a_n and b_n are real for all values of n, $a_m \bar{a}_n + \bar{b}_m b_n$ is real for all m and n. Hence

$$S_{31} = \frac{\lambda}{\pi} \sum_{m>n} \sum (a_m \bar{a}_n + \bar{b}_m b_n) \frac{(m+n)\sin\delta \cos\{(2\pi/\lambda)(m-n)\cos\delta\}}{(m-n)^2 + 4mn \sin^2\delta}$$

$$\times e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$- \frac{\lambda}{\pi} \sum_{m>n} \sum (a_m \bar{a}_n + \bar{b}_m b_n) \frac{(m-n)\cos\delta \sin\{(2\pi/\lambda)(m-n)\cos\delta\}}{(m-n)^2 + 4mn \sin^2\delta}$$

$$\times e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$= \Sigma_1 - \Sigma_2.$$

Now

$$\Sigma_1 | < \frac{2\lambda}{\pi} \delta \sum_{m>n} \sum |a_m a_n| \frac{m}{(m-n)^2} e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$+ \frac{2\lambda}{\pi} \delta \sum_{m>n} \sum |b_m b_n| \frac{m}{(m-n)^2} e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$= \Sigma_{11} + \Sigma_{12}$$

and

$$\Sigma_{11} = \frac{2\lambda}{\pi}\delta \sum_{g=1}^{\infty} g^{-2} \sum_{m=g+1}^{\infty} m|a_m| e^{-(2\pi/\lambda) m \sin\delta} |a_{m-g}| e^{-(2\pi/\lambda)(m-g)\sin\delta}$$

$$< \frac{2\lambda}{\pi}\delta \sum_{g=1}^{\infty} g^{-2}\Big(\sum_{m=g+1}^{\infty} m^2 |a_m|^2 e^{-(4\pi/\lambda) m \sin\delta}\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} |a_n|^2 e^{-(4\pi/\lambda) n \sin\delta}\Big)^{\frac{1}{2}}$$

$$< K\delta\{\delta^{-k-2} L(1/\delta)\}^{\frac{1}{2}} \{\delta^{-k} L(1/\delta)\}^{\frac{1}{2}} = K\delta^{-k} L(1/\delta).$$

Similarly $$\Sigma_{12} = O\big(\delta^{-k} L(1/\delta)\big),$$

so that $$\Sigma_1 = O\big(\delta^{-k} L(1/\delta)\big).$$

Since

$$\frac{m-n}{(m-n)^2+4mn\sin^2\delta} = \frac{1}{m-n} - \frac{4mn\sin^2\delta}{(m-n)\{(m-n)^2+4mn\sin^2\delta\}}$$

$$= \frac{1}{m-n} + O\Big(\frac{m^2\delta^2}{(m-n)^3}\Big),$$

if $m > n \geqslant 1$, we have

$$\Sigma_2 = \frac{\lambda\cos\delta}{\pi} \sum\sum_{m>n} (a_m \bar{a}_n + \bar{b}_m b_n) \frac{\sin\{(2\pi/\lambda)(m-n)\cos\delta\}}{m-n} e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$+ O\Big\{\delta^2 \sum\sum_{m>n} (|a_m a_n| + |b_m b_n|) \frac{m^2}{(m-n)^3} e^{-(2\pi/\lambda)(m+n)\sin\delta}\Big\}$$

$$= \Sigma_{21} + O(\Sigma_{22}).$$

Now

$$\sum\sum_{m>n} |a_m a_n| \frac{m^2}{(m-n)^3} e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$= \sum_{g=1}^{\infty} g^{-3} \sum_{m=g+1}^{\infty} m^2 |a_m| e^{-(2\pi/\lambda) m \sin\delta} |a_{m-g}| e^{-(2\pi/\lambda)(m-g)\sin\delta}$$

$$< \sum_{g=1}^{\infty} g^{-3}\Big(\sum_{m=g+1}^{\infty} m^4 |a_m|^2 e^{-(4\pi/\lambda) m \sin\delta}\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} |a_n|^2 e^{-(4\pi/\lambda) n \sin\delta}\Big)^{\frac{1}{2}}$$

$$= O\Big\{\Big(\delta^{-k-4} L(1/\delta)\Big)^{\frac{1}{2}} \Big(\delta^{-k} L(1/\delta)\Big)^{\frac{1}{2}}\Big\} = O\Big(\delta^{-k-2} L(1/\delta)\Big),$$

so that $$\Sigma_{22} = O\big(\delta^{-k} L(1/\delta)\big).$$

It remains to discuss Σ_{21}. We have

$$\Sigma_{21} = \frac{\lambda \cos\delta}{2\pi} \sum\sum_{m \neq n} (a_m \bar{a}_n + \bar{b}_m b_n) \frac{\sin\{(2\pi/\lambda)(m-n)\cos\delta\}}{m-n} e^{-(2\pi/\lambda)(m+n)\sin\delta}$$

$$= \tfrac{1}{2}\cos\delta \int_{-\cos\delta}^{\cos\delta} \Big(|F_1(\sin\delta + ix)|^2 + |F_2(\sin\delta + ix)|^2 \Big)\, dx$$

$$- \cos^2\delta \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)\, e^{-(4\pi/\lambda) n \sin\delta}$$

$$< \tfrac{1}{2}\cos\delta \int_{-l\lambda}^{l\lambda} \Big(|F_1(\sin\delta + ix)|^2 + |F_2(\sin\delta + ix)|^2 \Big)\, dx + O\Big(\delta^{-k} L(1/\delta)\Big),$$

where $l = [1/\lambda] + 1$. Now

$$\int_{-l\lambda}^{l\lambda} |F_1(\sin\delta + ix)|^2\, dx = 2l\lambda \sum_{n=1}^{\infty} |a_n|^2 e^{-(4\pi/\lambda) n \sin\delta}$$

$$= O\Big(\delta^{-k} L(1/\delta)\Big),$$

so that $$\Sigma_{21} = O\Big(\delta^{-k} L(1/\delta)\Big),$$

and therefore $$S_3 = O\Big(\delta^{-k} L(1/\delta)\Big).$$

Finally, by Lemmas 6 (a) and 7,

$$S_1 = \left(\frac{\lambda}{4\pi}\right)^k \frac{\Gamma(k)}{\sin^k\delta} \sum_{n<T} (|a_n|^2 + |b_n|^2)\, n^{-k} + O\Big(\sum_{n<T} (|a_n|^2 + |b_n|^2) \Big)$$

$$= \left(\frac{\lambda}{4\pi}\right)^k \frac{\Gamma(k)}{\delta^k} \sum_{n<T} (|a_n|^2 + |b_n|^2)\, n^{-k} + O\Big(T^k L(T) \Big).$$

Collecting our results, we have

$$\int_0^{\infty} |\phi_1(\tfrac{1}{2}k + it)|^2 t^{k-1} e^{-2\delta t} = \frac{\Gamma(k)}{(2\delta)^k} \sum_{n<1/2\delta} (|a_n|^2 + |b_n|^2)\, n^{-k} + O\Big(\delta^{-k} L(1,\delta)\Big).$$

Replacing 2δ by δ, we obtain

$$(23) \quad \int_0^{\infty} |\phi_1(\tfrac{1}{2}k + it)|^2 t^{k-1} e^{-\delta t}\, dt$$

$$= \Gamma(k)\, \delta^{-k} \sum_{n<1/\delta} (|a_n|^2 + |b_n|^2) n^{-k} + O\Big(\delta^{-k} L(1/\delta)\Big).$$

By Lemmas 7 (b) and 2,

$$\int_0^{\infty} |\phi_1(\tfrac{1}{2}k + it)|^2 t^{k-1} e^{-\delta t}\, dt \sim (a+b)\, \Gamma(k+1)\, \delta^{-k} L_1(1/\delta).$$

Finally, by Lemma 5 (*a*),

$$\int_0^T |\phi_1(\tfrac{1}{2}k+it)|^2\,dt \sim k(a+b)\,TL_1(T).$$

5. *Related theorems.*

So far we have found the mean value of $\phi_1(s)$, $\phi_2(s)$ on the critical line $\sigma = \frac{1}{2}k$ only when $a_n = b_n$ or a_n, b_n are real for all n. In the more general case where the coefficients are not so simply related we have

THEOREM 3. *Suppose that $\phi_1(s)$, $\phi_2(s)$ are ordinary Dirichlet series satisfying the conditions* I′, II′, III′, *and that*

$$\sum_{n<x} |a_n|^2 \sim ax^k L(x), \quad \sum_{n<x} |b_n|^2 \sim bx^k L(x), \tag{24}$$

where $L(x)$ is a slowly increasing function such that

$$L_1(x) = \int_1^x \frac{L(t)}{t}\,dt \to \infty$$

as $x \to \infty$. Then

$$\int_0^T \left(|\phi_1(\tfrac{1}{2}k+it)|^2 + |\phi_2(\tfrac{1}{2}k+it)|^2\right) dt \sim 2k(a+b)\,TL_1(T). \tag{25}$$

Proof. Put $c_n = \frac{1}{2}(Wa_n + \bar{W}b_n)$, $d_n = \frac{1}{2}(Wa_n - \bar{W}b_n)$,

$$\begin{cases} \psi_1(s) = \sum_{n=1}^{\infty} c_n n^{-s} = \frac{1}{2}\{W\phi_1(s) + \bar{W}\phi_2(s)\}, \\ \psi_2(s) = \sum_{n=1}^{\infty} d_n n^{-s} = \frac{1}{2}\{W\phi_1(s) - \bar{W}\phi_2(s)\}. \end{cases} \tag{26}$$

Then $\psi_1(s)$, $\psi_2(s)$ have the signatures $\{\lambda, k, 1\}$, $\{\lambda, k, -1\}$ respectively and

$$|a_n|^2 + |b_n|^2 = 2(|c_n|^2 + |d_n|^2) \tag{27}$$

and

$$|\phi_1(s)|^2 + |\phi_2(s)|^2 = 2\left(|\psi_1(s)|^2 + |\psi_2(s)|^2\right). \tag{28}$$

Also, by (24) and (27),

$$\sum_{n<x} |c_n|^2 = O\left(x^k L(x)\right), \quad \sum_{n<x} |d_n|^2 = O\left(x^k L(x)\right).$$

Thus, as in the proof of Theorem 1, we obtain [*cf.* (23)]

$$\int_0^\infty |\psi_1(\tfrac{1}{2}k+it)|^2 t^{k-1} e^{-\delta t}\,dt = 2\Gamma(k)\,\delta^{-k} \sum_{n<1/\delta} |c_n|^2 n^{-k} + O\Big(\delta^{-k} L(1/\delta)\Big)$$

and

$$\int_0^\infty |\psi_2(\tfrac{1}{2}k+it)|^2 t^{k-1} e^{-\delta t}\,dt = 2\Gamma(k)\,\delta^{-k} \sum_{n<1/\delta} |d_n|^2 n^{-k} + O\Big(\delta^{-k} L(1/\delta)\Big).$$

Addition of these equations gives, by (27) and (28),

$$\begin{aligned}(29)\quad \int_0^\infty \Big(|\phi_1(\tfrac{1}{2}k+it)|^2 + |\phi_2(\tfrac{1}{2}k+it)|^2\Big)\, t^{k-1} e^{-\delta t}\,dt \\ = 2\Gamma(k) \sum_{n<1/\delta} (|a_n|^2+|b_n|^2)\, n^{-k} + O\Big(\delta^{-k} L(1/\delta)\Big) \\ \sim 2(a+b)\,\Gamma(k+1)\,\delta^{-k} L_1(1/\delta).\end{aligned}$$

The result (25) now follows by Lemma 5 (*a*).

THEOREM 4. *If $\phi_1(s)$, $\phi_2(s)$ satisfy the conditions of Theorem* 3 *and $b_n = \bar{a}_n$ for all n, then*

$$\int_{-T}^{T} |\phi_1(\tfrac{1}{2}k+it)|^2\,dt \sim 4kaT L_1(T).$$

Proof. Since $\phi_2(s) = \overline{\phi_1(\bar{s})}$, we have

$$\int_0^T |\phi_2(\tfrac{1}{2}k+it)|^2\,dt = \int_{-T}^{0} |\phi_1(\tfrac{1}{2}k+it)|^2\,dt,$$

so that the result is an immediate consequence of Theorem 3.

THEOREM 5. *Suppose that $\phi_1(s)$, $\phi_2(s)$ are ordinary Dirichlet series satisfying the conditions* I′, II′, III′, *and that*

$$\sum_{n<x} |a_n|^2 = O\Big(x^k L(x)\Big), \quad \sum_{n<x} |b_n|^2 = O\Big(x^k L(x)\Big),$$

where $L(x)$ is a slowly increasing function such that

$$L_1(x) \equiv \int_1^x \frac{L(t)}{t}\,dt \to \infty$$

as $x \to \infty$. Then

$$\int_0^T |\phi_\nu(\tfrac{1}{2}k+it)|^2\,dt = O\Big(T L_1(T)\Big) \quad (\nu = 1, 2).$$

Proof. As in (29) we have

$$\int_0^\infty \left(\phi_1(\tfrac{1}{2}k+it)|^2+|\phi_2(\tfrac{1}{2}k+it)|^2\right) t^{k-1}e^{-\delta t}\,dt$$

$$= 2\Gamma(k)\sum_{n<1/\delta}(|a_n|^2+|b_n|^2)\,n^{-k}+O\left(\delta^{-k}L(1/\delta)\right)$$

$$= O\left(\delta^{-k}L_1(1/\delta)\right)$$

by Lemma 7 (*a*). The result follows by Lemma 5 (*b*).

6. *Applications.*

For several types of function with signature $\{\lambda, k, \gamma\}$ the asymptotic behaviour of $\sum_{n<x}|a_n|^2$ can be determined. We give a few examples.

(*a*) Since $\zeta(2s)$ has the signature $\{2, \frac{1}{2}, 1\}$, we may apply Theorem 1 and obtain the well-known result

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^2\,dt \sim T\log T.$$

(*b*) Let

$$\zeta(s, Q) = \sum_{m,n}{}' Q(m, n)^{-s} = \sum_{n=1}^{\infty} r(n, Q)\,n^{-s},$$

where $Q(m, n) = am^2+bmn+cn^2$ is a positive definite form with integer coefficients and $r(n, Q)$ denotes the number of integer solutions of

$$ax^2+bxy+cy^2 = n.$$

Let $d = b^2-4ac$; then $\zeta(s, Q)$ has the signature $\{\sqrt{(-d)}, 1, 1\}$ and, as Kober has shown*,

$$\sum_{n<x}\{r(n, Q)\}^2 = O(x\log x).$$

From Theorem 5 we deduce Kober's result†,

$$\int_0^T|\zeta(\tfrac{1}{2}+it, Q)|^2\,dt = O(T\log^2 T).$$

(*c*) If $\chi(n)$ is a proper character modulo q $(q>2)$ and $\chi(-1) = -1$, then the series

$$L(s, \chi) = \sum_{n=1}^{\infty}\chi(n)\,n^{-s}$$

* H. Kober, **1**. This is a trivial deduction from his result, 132–135.

† H. Kober, **1**.

has the property that $L(2s-1, \chi)$ satisfies the conditions I′, II′, III′ with $\lambda = 2q$, $k = \frac{3}{2}$. Applying Theorem 4, we obtain*

$$\int_{-T}^{T} |L(\tfrac{1}{2}+it, \chi)|^2 dt \sim 2\frac{\phi(q)}{q} T \log T.$$

(*d*) If $\chi(n)$ is a proper character modulo q ($q > 2$) and $\chi(-1) = 1$, then $L(2s, \chi)$ satisfies the conditions I′, II′, III′ with $\lambda = 2q$, $k = \frac{1}{2}$. On applying Theorem 4 we obtain the same result as in (*c*).

(*e*) We have the more general

THEOREM 6. *If $\phi(s)$ has the signature $\{\lambda, k, \gamma\}$, where $0 < \lambda < 2$, and $\phi(s)$ is regular at $s = k$, then*

$$\int_0^T |\phi(\tfrac{1}{2}k+it)|^2 dt = O(T \log T).$$

Proof. Hecke has proved† that in this case

$$\sum_{n<x} |a_n|^2 = O(x^k),$$

so that the result follows from Theorem 5.

THEOREM 7. *Suppose that*

$$F(r) = \sum_{n=0}^{\infty} a_n e^{2\pi i n r}$$

is a cusp form‡ of the kind $(-k, S)$ (i.e. of dimension $-k$ and related to the subgroup S of the modular group) and that the corresponding Dirichlet series $\phi(s) = \sum_1^\infty a_n n^{-s}$ has the signature $\{\lambda, k, \gamma\}$. Then

$$\int_0^T |\phi(\tfrac{1}{2}k+it)|^2 dt = O(T \log T).$$

Proof. In this case also Hecke§ has proved that

$$\sum_{n<x} |a_n|^2 = O(x^k),$$

so the result follows from Theorem 5.

* Here $\phi(q)$ denotes Euler's function.

† E. Hecke, **2**, 8.

‡ This is defined in the previous paper and by Hecke, **3**.

§ E. Hecke, **3**, 82.

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FLEXURE AND TORSION OF A CIRCULAR SHAFT WITH TWO CRACKS

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1. *Introduction.*

In this paper the Saint-Venant flexure and torsion problem is solved for the cross-section bounded by a circle and two slits extending from the ends of a diameter to any depths along the diameter.

The solution of the flexure problem in which the load is perpendicular to the slits involves the determination of three functions of the complex variable z referring to the plane of the cross-section. These functions are regular within the cross-section and their imaginary parts are continuous throughout the cross-section and have given values on its boundary.

The solution is obtained by means of a conformal transformation of the area on to a half plane followed by an application of Schwarz's integral formula.

It is of some interest that all the results, including the flexure and torsion moments, are expressed in terms of elementary functions, as those given by Shepherd (**2**) for one slit and for two equal slits are in terms of infinite series, the coefficients of which are themselves infinite series.

2. *Conformal transformation of the cross-section.*

The area in the z-plane is bounded by the circle $|z| = a$ and the slits ACA', BDB' extending from $z = ac$ to $z = a$ and from $z = -ad$ to $z = -a$ respectively.

The first slit is taken to be the longer, and it does or does not extend beyond the centre of the circle according as c is negative or positive.

The area is represented conformally on the upper half of the t-plane by means of the transformation

$$t = -\left[\frac{(z+ad)(a+zd)}{(z-ac)(a-zc)}\right]^{\frac{1}{2}}, \tag{2.1}$$

where those branches of $(z+ad)^{\frac{1}{2}}$, etc. are chosen which are positive when z lies on CA.

It is also convenient to make use of the variable s defined by the equations

$$s = \tfrac{1}{2}(z/a+a/z), \tag{2.2}$$

$$z/a = s-(s^2-1)^{\frac{1}{2}}, \tag{2.3}$$

where that branch of $(s^2-1)^{\frac{1}{2}}$ is chosen which is positive when s is real and greater than unity.

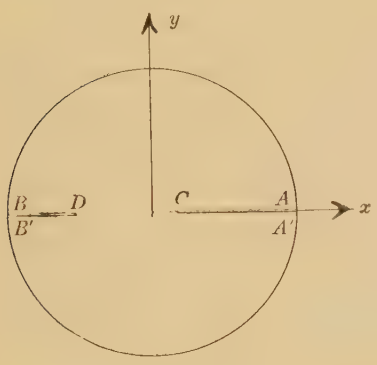

Fig. 1. z-plane.

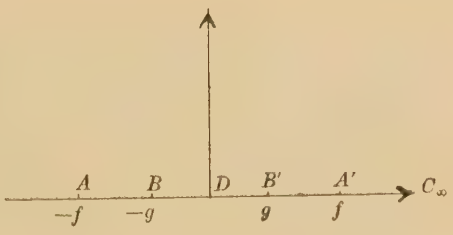

Fig. 2. t-plane.

From (2.1) and (2.2) we have

$$t = -\left[\frac{d(n+s)}{c(m-s)}\right]^{\frac{1}{2}}, \tag{2.4}$$

where m and n are given by

$$m = \tfrac{1}{2}(c+1/c), \tag{2.5}$$

$$n = \tfrac{1}{2}(d+1/d). \tag{2.6}$$

The points A, A' correspond to $t = \pm f$ and the points B, B' to $t = \pm g$, where f and g are given by

$$f = (1+d)/(1-c), \tag{2.7}$$

$$g = (1-d)/(1+c). \tag{2.8}$$

The correspondence between the z- and t-planes is shown in Figs. 1 and 2.

3. *The Schwarz integral formula.*

If $F(z)$ is regular for $|z|<1$ and $IF(z)$ is continuous for $|z|\leqslant 1$, then the value of $F(z)$ at a point z within the circle is given by

$$F(z)=b+\frac{1}{2\pi}\int_C IF(z')\,\frac{(z'+z)\,dz'}{(z'-z)\,z'}, \tag{3.1}$$

where b is a real constant and C is the circle $|z|=1$.

This formula appears to be due to Schwarz*. It is given with a slightly different notation in Forsyth's *Theory of functions* (3rd ed., 1918), 467.

In view of later applications of (3.1), we put it into a form suitable for reference to the upper half of the t-plane. This is accomplished by means of the transformation

$$z=\frac{t-t_0}{t-\bar{t}_0}, \tag{3.2}$$

where t_0, $\bar{t}_0$ are conjugate complex numbers and $It_0>0$.

Here it is as well to point out that the variable z of this section makes only a temporary appearance and is not to be confused with the z of the previous and subsequent sections.

The transformation (3.2) represents the interior of the circle $|z|=1$ conformally on the upper half of the t-plane.

If we write $F(z)=f(t)$ and use t' to correspond to z' we have, from (3.1) and (3.2),

$$f(t)=b'+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{If(t')\,dt'}{t'-t}, \tag{3.3}$$

where b' is a real constant.

In the applications of (3.3) to the problems of this paper, we are concerned only with cases in which $If(t')$ is an even function of t' and we then have, from (3.3),

$$f(t)=b'+\frac{2t}{\pi}\int_0^{\infty}\frac{If(t')\,dt'}{t'^2-t^2}. \tag{3.4}$$

This gives the value of $f(t)$ at a point t such that $It>0$ when $f(t)$ is regular in this region, $If(t)$ is continuous for $It\geqslant 0$ and $If(t')$ is an even function of t'.

4. *Statement of the flexure and torsion problems.*

In the flexure problem we adopt the method of subdivision introduced in a recent paper (Stevenson, 4). When the axes are principal axes at the

* *Ges. Math. Abhandlungen*, 2 (1890), 190 *et seq.*

centroid of the cross-section, which is the case here, the solution when the load is W' parallel to the axis of y involves the three canonical flexure functions ϕ_1, χ_2, and ϕ_3.

ϕ_3 is the classical torsion function and ϕ_1, χ_2 are related to the classical flexure function χ' (Love, **1**, 316, 348) by the equation

$$\chi' = -(1+\eta)\chi_2 - \eta\phi_1 + (1+\tfrac{1}{2}\eta)(\tfrac{1}{3}y^3 - x^2 y), \tag{4.1}$$

where η is Poisson's ratio.

These functions are harmonic, and the physical conditions of finite, one-valued, and continuous displacements and stresses require them, together with their first derivatives, to be finite, one-valued and continuous over the cross-section and on its boundary.

The functions ψ_1 and ψ_3 conjugate to ϕ_1 and ϕ_3 satisfy the boundary equations

$$\psi_1 = \tfrac{1}{3}x^3 + \text{const.}, \tag{4.2}$$

$$\psi_3 = \tfrac{1}{2}(x^2+y^2) + \text{const.}, \tag{4.3}$$

and χ_2 satisfies the boundary equation

$$\frac{\partial \chi_2}{\partial n} = \tfrac{1}{3}\frac{\partial y^3}{\partial n}, \tag{4.4}$$

where $\partial/\partial n$ denotes differentiation along the normal to the boundary

If $\chi_2{}^*$ is the function conjugate to χ_2, we have

$$\frac{\partial \chi_2}{\partial n} = \frac{\partial \chi_2{}^*}{\partial s} \quad \text{and} \quad \frac{\partial y}{\partial n} = -\frac{\partial x}{\partial s}, \tag{4.5}$$

where $\partial/\partial s$ denotes differentiation along the boundary.

From (4.4) and (4.5) we have

$$\chi_2{}^* = -\int y^2\, dx. \tag{4.6}$$

When the boundary consists of two or more portions with different analytical descriptions, as is the case here, the integral in (4.6) has different forms in these portions and its continuity at the points of subdivision must be ensured. The possibility of this implies the equation $\int_C y^2\, dx = 0$, where C denotes the complete boundary.

This is true when the centroid of the cross-section lies on the axis of x, as is the case here.

We now introduce the complex flexure functions defined by

$$\omega_1 = \phi_1 + i\psi_1 = f_1(t), \tag{4.7}$$

$$\Omega_2 = \chi_2 + i\chi_2^* = f_2(t), \tag{4.8}$$

$$\omega_3 = \phi_3 + i\psi_3 = f_3(t). \tag{4.9}$$

In finding them we make use of (3.4) together with the boundary equations (4.2), (4.3), (4.5) and, where necessary, we verify that they satisfy the physical conditions.

We write

$$\omega_{11} = \phi_{11} + i\psi_{11} = (\omega_1 - \tfrac{1}{3}iz^3)/a^3, \tag{4.10}$$

$$\omega_{31} = \phi_{31} + i\psi_{31} = (\omega_3 - \tfrac{1}{2}iz^2)/a^2, \tag{4.11}$$

$$\omega = \phi + i\psi = -\tfrac{3}{2}\Omega_2/a^3 - \tfrac{1}{2}\omega_{11}. \tag{4.12}$$

From (4.2), (4.10), and (4.3), (4.11) we have the boundary equations

$$\psi_{11} = xy^2/a^3, \tag{4.13}$$

$$\psi_{31} = y^2/a^2, \tag{4.14}$$

when the additive constants in (4.2) and (4.3) are taken to be zero.

From (4.6) we have on the circle

$$\begin{aligned}\chi_2^* &= \int (x^2 - a^2)\, dx \\ &= \tfrac{1}{3}x^3 - a^2 x.\end{aligned}$$

Since y is zero on the slits it follows that χ_2^* has the constant values $-\frac{2}{3}a^3$ on ACA' and $+\frac{2}{3}a^3$ on BDB'.

Hence from (4.12) the boundary values of ψ are given by

$$\psi = x/a \quad \text{on the arcs } AB,\ A'B', \tag{4.15}$$

$$\psi = 1 \quad \text{on } ACA', \tag{4.16}$$

$$\psi = -1 \quad \text{on } BDB'. \tag{4.17}$$

Putting $z = ae^{i\theta}$ on the circle, we have, from (2.2),

$$s = \cos\theta = x/a. \tag{4.18}$$

Hence in terms of s the boundary conditions (4.13), etc., are

$$\psi = s, \quad \psi_{31} = 1 - s^2, \quad \psi_{11} = s - s^3 \quad \text{on the arcs } AB,\ A'B'; \tag{4.19}$$

$$\psi = 1, \quad \psi_{31} = 0, \quad \psi_{11} = 0 \quad \text{on } ACA'; \tag{4.20}$$

$$\psi = -1, \quad \psi_{31} = 0, \quad \psi_{11} = 0 \quad \text{on } BDB'. \tag{4.21}$$

We can now apply (3.4) to find ω, ω_{31}, and ω_{11}.

Using s' to correspond to t', we have

$$\omega = -\frac{2t}{\pi}\int_0^g \frac{dt'}{t'^2-t^2} + \frac{2t}{\pi}\int_g^f \frac{s'\,dt'}{t'^2-t^2} + \frac{2t}{\pi}\int_f^\infty \frac{dt'}{t'^2-t^2}, \tag{4.22}$$

$$\omega_{31} = \frac{2t}{\pi}\int_g^f \frac{1-s'^2}{t'^2-t^2}\,dt', \tag{4.23}$$

$$\omega_{11} = \frac{2t}{\pi}\int_g^f \frac{s'-s'^3}{t'^2-t^2}\,dt'. \tag{4.24}$$

The additive constant in (3.3) has been omitted, since it is immaterial to the torsion and flexure problems.

From (4.22), etc. we have

$$\omega = \frac{1}{\pi}\log\frac{(f+t)(t+g)}{(f-t)(t-g)} + \frac{s}{\pi}\log\frac{(f-t)(g+t)}{(f+t)(g-t)} + \frac{2t}{\pi}\int_g^f \frac{s'-s}{t'^2-t^2}\,dt', \tag{4.25}$$

$$\omega_{31} = \frac{1-s^2}{\pi}\log\frac{(f-t)(g+t)}{(f+t)(g-t)} + \frac{2t}{\pi}\int_g^f \frac{s^2-s'^2}{t'^2-t^2}\,dt', \tag{4.26}$$

$$\omega_{11} = \frac{s-s^3}{\pi}\log\frac{(f-t)(g+t)}{(f+t)(g-t)} + \frac{2t}{\pi}\int_g^f \frac{s'-s+s^3-s'^3}{t'^2-t^2}\,dt'. \tag{4.27}$$

Now, from (2.4) and the corresponding result when s' and t' replace s and t, we find that

$$\frac{2t\,dt'}{t'^2-t^2} = \left[\frac{P(s)}{P(s')}\right]^{\frac{1}{2}}\frac{ds'}{s'-s}, \tag{4.28}$$

where

$$P(s) = (m-s)(n+s). \tag{4.29}$$

For brevity write

$$J_n = \int_{-1}^{1} \frac{s'^n}{\sqrt{P(s')}}\,ds'. \tag{4.30}$$

Then, from (4.25), (4.26), (4.27) and (4.28), we have

$$\omega = \frac{s}{\pi}\log\frac{(f-t)(g+t)}{(f+t)(g-t)} + \frac{1}{\pi}\log\frac{(f+t)(t+g)}{(f-t)(t-g)} + \frac{J_0}{\pi}\sqrt{P(s)}, \tag{4.31}$$

$$\omega_{31} = \frac{1-s^2}{\pi}\log\frac{(f-t)(g+t)}{(f+t)(g-t)} - \frac{\sqrt{P(s)}}{\pi}(J_0 s + J_1), \tag{4.32}$$

$$\omega_{11} = \frac{s-s^3}{\pi}\log\frac{(f-t)(g+t)}{(f+t)(g-t)} + \frac{\sqrt{P(s)}}{\pi}(J_0 - J_2 - J_1 s - J_0 s^2). \tag{4.33}$$

The integrals J_0, J_1, J_2 are found to be

$$J_0 = 4\tan^{-1}\sqrt{(cd)}, \tag{4.34}$$

$$J_1 = -\frac{p}{4cd}\left[4\sqrt{(cd)} - qJ_0\right], \tag{4.35}$$

$$J_2 = \tfrac{1}{2}J_0 + \frac{1}{8cd}\left[(p^2+q^2)J_0 + 3pqJ_1 - 4q\sqrt{(cd)}\right], \tag{4.36}$$

where

$$p = d-c, \quad q = 1-cd. \tag{4.37}$$

Formulae (4.31), (4.32), (4.33) together with (4.10), (4.11) and (4.12) give the solution of the flexure problem when the load is parallel to the axis of y; the solution of the torsion problem is included since it involves only ω_3.

It is easy to verify that all the physical conditions are satisfied except that the stresses at the ends of the slits have infinite values. This occurs because dt/dz is infinite at these points and the derivatives of the flexure functions are correspondingly infinite.

The case when the load is along the axis of x is of less interest since the slits have no effect. The reason for this is that the axis is one of symmetry and it is then known that the solution for the complete cross-section also satisfies the boundary conditions along the axis of symmetry. (Stevenson, 4, 213). The solution for this case is therefore the well-known result for the circular cross-section. (Love, 1, 340; Stevenson, 4, 183, 184.)

5. *The flexure and torsion moment integrals.*

In order to complete the solution we require the values of the moment integrals given by

$$M_3 = \int (x^2+y^2)\,dS + \int\left(x\frac{\partial\phi_3}{\partial y} - y\frac{\partial\phi_3}{\partial x}\right)dS, \tag{5.1}$$

$$M_1 = \int x^3\,dS + \int\left(x\frac{\partial\phi_1}{\partial y} - y\frac{\partial\phi_1}{\partial x}\right)dS, \tag{5.2}$$

$$L_2 = \int xy^2\,dS + \int\left(x\frac{\partial\chi_2}{\partial y} - y\frac{\partial\chi_2}{\partial x}\right)dS, \tag{5.3}$$

the integrals being taken over the cross-section.

By an application of Green's theorem these area integrals can be transformed into line integrals round the boundary, whence

$$M_3 = -\int_C \phi_3(x\,dx+y\,dy)+\tfrac{1}{2}\pi a^4, \tag{5.4}$$

$$M_1 = -\int_C \phi_1(x\,dx+y\,dy), \tag{5.5}$$

$$L_2 = -\int_C \chi_2(x\,dx+y\,dy), \tag{5.6}$$

since the first terms in (5.1), (5.2), and (5.3) have the values $\frac{1}{2}\pi a^4$, 0, and 0 respectively.

In (5.4), etc., C denotes the whole boundary and the sense of description is such that the area is kept on the left.

For brevity let C_1 denote the circular part and C_2 the slits, each slit being taken twice.

Then, since $x\,dx+y\,dy$ is zero on C_1 and y is zero on C_2, we have, from (4.10), (4.11), (4.12), and (5.4), etc.,

$$M_3 = -a^2\int_{C_2} \phi_{31}\,x\,dx+\tfrac{1}{3}\pi a^4 \tag{5.7}$$

$$M_1 = -a^2\int_{C_2} \phi_{11}\,x\,dx, \tag{5.8}$$

$$L_2 = \tfrac{2}{3}a^3\int_{C_2} \phi\,x\,dx-\tfrac{1}{3}M_1. \tag{5.9}$$

Now $\omega_{31}z$ is a function of z regular within C and regular on C except at the ends of the slits, where it is continuous.

Hence, by Cauchy's theorem,

$$0 = \int_C \omega_{31}\,z\,dz \tag{5.10}$$

$$= \int_{C_1} \omega_{31}\,z\,dz+\int_{C_2} \omega_{31}\,z\,dz.$$

Writing $z = ae^{i\theta}$ on C_1 and taking real parts, we have

$$\int_{C_2} \phi_{31}\,x\,dx = a^2\int_{C_1} (\phi_{31}\sin 2\theta+\psi_{31}\cos 2\theta)\,d\theta. \tag{5.11}$$

Now, from (4.18) and (4.19), ψ_{31} is an even function of θ on C_1, having the value $\sin^2\theta$, so that ϕ_{31} must be an odd function of θ.

Hence, from (5.11),

$$\int_{C_2} \phi_{31}\, x\, dx = 2a^2 \int_{-\pi}^{0} (\phi_{31} \sin 2\theta + \sin^2\theta \cos 2\theta)\, d\theta \tag{5.12}$$

$$= -4a^2 \int_{-1}^{1} \phi_{31}\, s\, ds - \tfrac{1}{2}\pi a^2.$$

Similarly

$$\int_{C_2} \phi_{11}\, x\, dx = -4a^2 \int_{-1}^{1} \phi_{11}\, s\, ds. \tag{5.13}$$

In the case of ϕ we can use the equation

$$\int_C \phi\, d\psi_3 = \int_C \phi_3\, d\psi.$$

In virtue of (4.3), (4.19), etc. and (4.11) this becomes

$$\int_{C_2} \phi\, x\, dx = \int_{C_1} \phi_3\, ds \tag{5.14}$$

$$= 2a^2 \int_{-1}^{1} \phi_{31}\, ds.$$

From (5.7), etc. and (5.12), etc. we have

$$\frac{M_3}{a^4} = 4 \int_{-1}^{1} \phi_{31}\, s\, ds + \pi, \tag{5.15}$$

$$\frac{M_1}{a^5} = 4 \int_{-1}^{1} \phi_{11}\, s\, ds, \tag{5.16}$$

$$\frac{L_2}{a^5} = \tfrac{4}{3} \int_{-1}^{1} \phi_{31}\, ds - \frac{M_1}{3a^5}. \tag{5.17}$$

Now from (4.23), (4.24), and (4.28) we have

$$\omega_{31} = \frac{1}{\pi} \sqrt{P(s)} \int_{-1}^{1} \frac{1-s'^2}{s'-s} \frac{ds'}{\sqrt{P(s')}}, \tag{5.18}$$

$$\omega_{11} = \frac{1}{\pi} \sqrt{P(s)} \int_{-1}^{1} \frac{s'-s'^3}{s'-s} \frac{ds'}{\sqrt{P(s')}}. \tag{5.19}$$

In evaluating the integrals in (5.15), etc. we use the values of ϕ_{31} and ϕ_{11} in their integral forms obtained from (5.18) and (5.19), since this reduces the algebra involved.

The substitution leads in each case to a repeated improper integral and we therefore use the result that, if

$$F(s) = g(s) \int_{-1}^{1} \frac{f(s')}{s'-s}\, ds' \tag{5.20}$$

and $f(s)$ and $g(s)$ are real in the range of integration, then

$$R \int_{-1}^{1} F(s)\, ds = \tfrac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \frac{f(s')\, g(s) - f(s)\, g(s')}{s'-s}\, ds\, ds'. \tag{5.21}$$

This is not difficult to prove when

$$[f(s')-f(s)]/(s'-s) \quad \text{and} \quad [g(s')-g(s)]/(s'-s)$$

are continuous for $-1 \leqslant s \leqslant 1$ and $-1 \leqslant s' \leqslant 1$; and in the present applications these conditions are satisfied.

In the first place write

$$g(s) = s\sqrt{P(s)} \quad \text{and} \quad f(s) = (1-s^2)/\sqrt{P(s)}. \tag{5.22}$$

Then from (4.29) we find that

$$\frac{f(s')\, g(s) - f(s)\, g(s')}{s'-s} = -\frac{(mn-1)(1+ss') + (m-n)(s+s') + (1-s^2)(1-s'^2)}{\sqrt{\{P(s)\, P(s')\}}}. \tag{5.23}$$

Hence, from (5.21) and (5.23) together with (4.30),

$$R \int_{-1}^{1} F(s)\, ds = -\tfrac{1}{2}[(mn-1)(J_0^2 + J_1^2) + 2(m-n)\, J_0 J_1 + (J_0 - J_2)^2]. \tag{5.24}$$

Now from (5.18), (5.20), and (5.22) we have

$$F(s) = \pi s\, \omega_{31},$$

from which it follows that

$$R \int_{-1}^{1} F(s)\, ds = \pi \int_{-1}^{1} \phi_{31}\, s\, ds. \tag{5.25}$$

Finally, from (5.15), (5.24), and (5.25) we have

$$\frac{M_3}{a^4} = -\frac{2}{\pi}\, [(mn-1)(J_0^2 + J_1^2) + 2(m-n)\, J_0 J_1 + (J_0 - J_2)^2] + \pi. \tag{5.26}$$

The calculation of M_1 and L_2 is very similar and we find that

$$\frac{M_1}{a^5} = -\frac{2}{\pi}\,[2(mn-1)\,J_2\,J_1+(m-n)(J_2^2+J_1^2)], \tag{5.27}$$

$$\frac{L_2}{a^5} = -\frac{2}{3\pi}\,[2(mn-1)\,J_0\,J_1+(m-n)(J_0^2+J_1^2)]-\frac{M_1}{3a^5}. \tag{5.28}$$

Expressing m and n in terms of p and q by means of (2.5), (2.6), and (4.37), we find from (5.26), etc.,

$$\frac{M_3}{a^4} = \pi-\frac{1}{2\pi}\,[(pI_0-qI_1)^2+(qI_0-pI_1)^2+4cd(I_0-I_2)^2], \tag{5.29}$$

$$\frac{M_1}{a^5} = -\frac{1}{\pi}\,(pI_2-qI_1)(qI_2-pI_1), \tag{5.30}$$

$$\frac{L_2}{a^5} = -\frac{1}{3\pi}\,(pI_0-qI_1)(qI_0-pI_1)-\frac{M_1}{3a^5}, \tag{5.31}$$

where, from (4.34), etc.,

$$I_0 = \frac{J_0}{\sqrt{(cd)}} = \frac{4}{\sqrt{(cd)}}\tan^{-1}\sqrt{(cd)}, \tag{5.32}$$

$$I_1 = -\frac{J_1}{\sqrt{(cd)}} = \frac{p}{4cd}\,(4-qI_0), \tag{5.33}$$

$$I_2 = \frac{J_2}{\sqrt{(cd)}} = \frac{I_0}{2}+\frac{1}{8cd}\,[(p^2+q^2)\,I_0-3pqI_1-4q]. \tag{5.34}$$

When c is negative and the longer slit extends beyond the centre, (5.32) is replaced by

$$I_0 = \frac{4}{\sqrt{(-cd)}}\tanh^{-1}\sqrt{(-cd)}. \tag{5.35}$$

6. *The associated twist.*

It is known that, when the load in the flexure problem is W' through the centroid of the end section and perpendicular to the axis of symmetry (*i.e.* the axis of x), the associated twist τ_0' is given by

$$-\tau_0'(EI'\,M_3)/W' = (1+\eta)\,L_2+\eta M_1, \tag{6.1}$$

where E is Young's modulus, η is Poisson's ratio and I' is the second moment of area of the section about the axis of x. (Stevenson, **4**, 178.)

This result can be written in the form

$$-\tau_0'(EI')/W' = a(p_0+q_0\eta), \tag{6.2}$$

where p_0 and q_0 are pure numbers given by

$$p_0 = L_2/(aM_3), \quad q_0 = p_0+M_1/(aM_3). \tag{6.3}$$

7. *The centre of flexure.*

This point lies on the axis of symmetry at $(f_0, 0)$, where

$$f_0 = (L_2+\sigma M_1)/2I', \tag{7.1}$$

and σ is related to η by the equation $(1-\sigma)(1+\eta) = 1$. (Stevenson, 4, 178.)

Since $I' = \frac{1}{4}\pi a^4$, we have from (7.1)

$$f_0 = a(p_1+q_1\sigma), \tag{7.2}$$

where p_1 and q_1 are pure numbers given by

$$p_1 = \frac{2L_2}{\pi a^5}, \quad q_1 = \frac{2M_1}{\pi a^5}. \tag{7.3}$$

8. *The torsional rigidity.*

In the case of torsion the couple per unit twist or torsional rigidity is μM_3, where μ is the rigidity of the material. (Stevenson, 4, 174.)

In the absence of the slits this would have the well-known value $\frac{1}{2}\pi a^4$, so that the fractional reduction in the torsional rigidity is $1-r$, where

$$r = 2M_3/(\pi a^4). \tag{8.1}$$

9. *Some particular cases.*

The case of one slit and the case of two equal slits are of interest both in themselves and in that they have been dealt with before (Shepherd, 2). The method used by Shepherd led to results in terms of infinite series, but from the results given in the present paper it appears that these series can be summed in terms of elementary functions.

The third particular case is that in which the longer slit is a complete radius; its interest lies in the simplicity of the expressions for M_1, M_3, and L_2.

We take these three cases in order.

One slit. In this case $d = 1$, $p = q = 1-c$, and, if we write

$$C = \frac{1}{c}\left[(1+c)^2 \frac{\tan^{-1}\sqrt{c}}{\sqrt{c}} - (1-c)\right], \tag{9.1}$$

$$D = \frac{(1-c)}{c}\left[2 - \tfrac{3}{4}(1-c)\,C\right], \tag{9.2}$$

we find from (5.32), etc., that

$$I_0 - I_2 = \tfrac{1}{2}(C+D), \tag{9.3}$$

$$I_2 - I_1 = \tfrac{1}{2}(C-D). \tag{9.4}$$

Hence, from (5.29), etc., and (9.3), (9.4),

$$\frac{M_3}{a^4} = \pi - \frac{1}{2\pi}\left[2(1-c)^2 C^2 + c(C+D)^2\right], \tag{9.5}$$

$$\frac{M_1}{a^5} = -\frac{(1-c)^2}{4\pi}(C-D)^2, \tag{9.6}$$

$$\frac{L_2}{a^5} = -\frac{(1-c)^2}{3\pi}\left[C^2 - \tfrac{1}{4}(C-D)^2\right]. \tag{9.7}$$

It is easy to deduce from (9.1) and (9.2) that when c tends to zero the limiting values of C and D are $\frac{8}{3}$ and $\frac{8}{5}$ respectively.

Hence, from (9.5), etc.,

$$\frac{M_3}{a^4} = \pi - \frac{64}{9\pi}, \quad \frac{M_1}{a^5} = -\frac{64}{225\pi}, \quad \frac{L_2}{a^5} = -\frac{512}{225\pi}.$$

These are the known values for the complete radial slit (Stevenson, **4**, 194). The difference in the signs of M_1 and L_2 occurs because the slit is to the *left* of the centre in the paper quoted.

Two equal slits. In this case $d = c$, $p = 0$, and $q = 1-c$. M_1 and L_2 are zero, as we should expect from symmetry, and

$$\frac{M_3}{a^4} = \pi - \frac{8}{\pi}\left[(1-c^2)^2 L^2 + 4c(L-M)^2\right], \tag{9.8}$$

where

$$L = \frac{1}{c}\tan^{-1} c, \tag{9.9}$$

$$M = \frac{1}{8c^2}\left[(1+c^2)^2 L - (1-c^2)\right]. \tag{9.10}$$

One slit a complete radius. Here $c = 0$, $p = d$, and $q = 1$.

From (5.32), etc., we find that

$$I_0 = 4, \tag{9.11}$$

$$I_1 = \tfrac{4}{3}d, \tag{9.12}$$

$$I_2 = \tfrac{4}{3} + \tfrac{8}{15}d^2. \tag{9.13}$$

Substituting these values into (5.27), etc., we have

$$\frac{M_3}{a^4} = \pi - \frac{8}{\pi}\,(1 - \tfrac{2}{9}d^2 + \tfrac{1}{9}d^4), \tag{9.14}$$

$$\frac{M_1}{a^5} = -\frac{32}{45\pi}\,(d^3 - \tfrac{3}{5}d^5), \tag{9.15}$$

$$\frac{L_2}{a^5} = -\frac{32}{9\pi}\,(d - \tfrac{2}{5}d^3 + \tfrac{1}{25}d^5). \tag{9.16}$$

Tables I, II, and III give the numerical values of r, p_1, and q_1 for values of c ranging from d to $-d$ at intervals of 0·1 and values of d ranging from 1·0 to 0·5 also at intervals of 0·1.

TABLE I (r).

$1-d$

$1-c$	0·0	0·1	0·2	0·3	0·4	0·5
0·0	1·0					
0·1	0·99068	0·98139				
0·2	0·96536	0·95615	0·93114			
0·3	0·92771	0·91863	0·89403	0·85762		
0·4	0·88109	0·87221	0·84819	0·81275	0·76926	
0·5	0·82852	0·81988	0·79662	0·76246	0·72077	0·67458
0·6	0·77270	0·76436	0·74203	0·70943	0·66994	0·62658
0·7	0·71596	0·70800	0·68678	0·65604	0·61914	0·57909
0·8	0·66036	0·65282	0·63289	0·60429	0·57037	0·53409
0·9	0·60757	0·60053	0·58206	0·55588	0·52529	0·49319
1·0	0·55899	0·55249	0·53636	0·51214	0·48521	0·45767
1·1	0·51565	0·50975	0·49457	0·47409	0·45109	0·42841
1·2	0·47827	0·47302	0·45989	0·44239	0·42356	0·40593
1·3	0·44726	0·44272	0·43164	0·41739	0·40185	0·39041
1·4	0·42272	0·41892	0·40997	0·39696	0·38889	0·38159
1·5	0·40441	0·40139	0·39461	0·38707	0·38118	0·37886
1·6	0·39182	0·38958	0·38499	0·38069	0·37886	
1·7	0·38414	0·38268	0·38014	0·37886		
1·8	0·38030	0·37956	0·37886			
1·9	0·37901	0·37886				
2·0	0·37886					

TABLE II (p_1).

$1-c$ \ $1-d$	0·0	0·1	0·2	0·3	0·4	0·5
0·0	0·0					
0·1	0·00506	0·0				
0·2	0·02039	0·01533	0·0			
0·3	0·04599	0·04092	0·02557	0·0		
0·4	0·08157	0·07648	0·06110	0·03549	0·0	
0·5	0·12661	0·12148	0·10602	0·08032	0·04477	0·0
0·6	0·18036	0·17518	0·15958	0·13370	0·09799	0·05311
0·7	0·24187	0·23661	0·22080	0·19464	0·15863	0·11350
0·8	0·31004	0·30465	0·28854	0·26197	0·22549	0·17992
0·9	0·38358	0·37804	0·36151	0·33435	0·29721	0·25121
1·0	0·46112	0·45537	0·43829	0·41034	0·37229	0·32513
1·1	0·54117	0·53514	0·51756	0·48838	0·44914	0·40076
1·2	0·62214	0·61576	0·59704	0·56680	0·52608	0·47615
1·3	0·70238	0·69556	0·67570	0·64387	0·60134	0·54955
1·4	0·78018	0·77280	0·75153	0·72012	0·67312	0·61913
1·5	0·85376	0·84567	0·82268	0·78669	0·73954	0·68307
1·6	0·92125	0·91225	0·88713	0·84860	0·79869	
1·7	0·98062	0·97049	0·94297	0·90146		
1·8	1·02954	1·01792	0·98756			
1·9	1·06492	1·05131				
2·0	1·08076					

TABLE III (q_1).

$1-c$ \ $1-d$	0·0	0·1	0·2	0·3	0·4	0·5
0·0	0·0					
0·1	0·00427	0·0				
0·2	0·01443	0·01017	0·0			
0·3	0·02720	0·02294	0·01279	0·0		
0·4	0·04006	0·03582	0·02570	0·01295	0·0	
0·5	0·05121	0·04700	0·03697	0·02430	0·01142	0·0
0·6	0·05948	0·05533	0·04542	0·03293	0·02021	0·00890
0·7	0·06425	0·06017	0·05047	0·03825	0·02582	0·01476
0·8	0·06534	0·06137	0·05196	0·04014	0·02816	0·01753
0·9	0·06297	0·05915	0·05011	0·03884	0·02750	0·01752
1·0	0·05764	0·05401	0·04545	0·03490	0·02440	0·01531
1·1	0·05007	0·04664	0·03844	0·02905	0·01963	0·01169
1·2	0·04108	0·03793	0·03074	0·02217	0·01406	0·00752
1·3	0·03158	0·02876	0·02244	0·01516	0·00859	0·00369
1·4	0·02243	0·01998	0·01467	0·00833	0·00402	0·00099
1·5	0·01438	0·01235	0·00817	0·00396	0·00103	0·0
1·6	0·00798	0·00642	0·00347	0·00096	0·0	
1·7	0·00355	0·00248	0·00079	0·0		
1·8	0·00105	0·00049	0·0			
1·9	0·00012	0·0				
2·0	0·0					

The graphs of r and p_1 are shown in Figs. 3 and 4. In each case the successive values of $1-d$ for the six curves, starting from the highest, are 0, 0·1, ... 0·5.

From Table III it is clear that the value of q_1 is always small, and a curious feature is that its maximum occurs when $1-c$ is about 0·8, for all the given values of d. Its greatest value is about 0·066 when $d = 1$, so that there is only one slit.

From (7.2) the term in the x-coordinate of the centre of flexure which depends on the elastic constant σ is $q_1 a$, where $\sigma = \eta/(1-\eta)$ and η is Poisson's ratio.

Since η is less than $\frac{1}{2}$, σ is less than $\frac{1}{3}$ and it follows that $q_1 \sigma$ never exceeds about 0·022, or in other words the position of the centre of flexure is practically independent of σ.

10. *Comparison with previous results.*

Shepherd (3) has given numerical results for the case of two equal slits and the case of one slit.

Two equal slits: The values of r for $c = \sqrt{\sin 45°}$ and $c = \sqrt{\sin 30°}$ obtained by Shepherd and by the present writer are

$c = \sqrt{\sin 45°}$: 0·95498 (Shepherd) and 0·95498,

$c = \sqrt{\sin 30°}$: 0·86341 and 0·86343,

so that the agreement is satisfactory.

One slit: For $c = \sin 45°$ and $c = \sin 30°$ the values of r are

$c = \sin 45°$: 0·93072 (Shepherd) and 0·93072,

$c = \sin 30°$: 0·82852 and 0·82852,

and are in agreement.

In the case of the associated twist, however, there appears to be numerical disagreement. For the above values of c the values of $E\tau'/W'$ when Poisson's ratio is 0·3 are

$c = \sin 45°$: 0·1176 (Shepherd) and 0·0887,

$c = \sin 30°$: 0·4534 and 0·2765.

[In a later paper (3, 508) Shepherd has corrected the sign of the associated twist and Stevenson (4, 199) has pointed out a further correction to Shepherd's statement of the problem.]

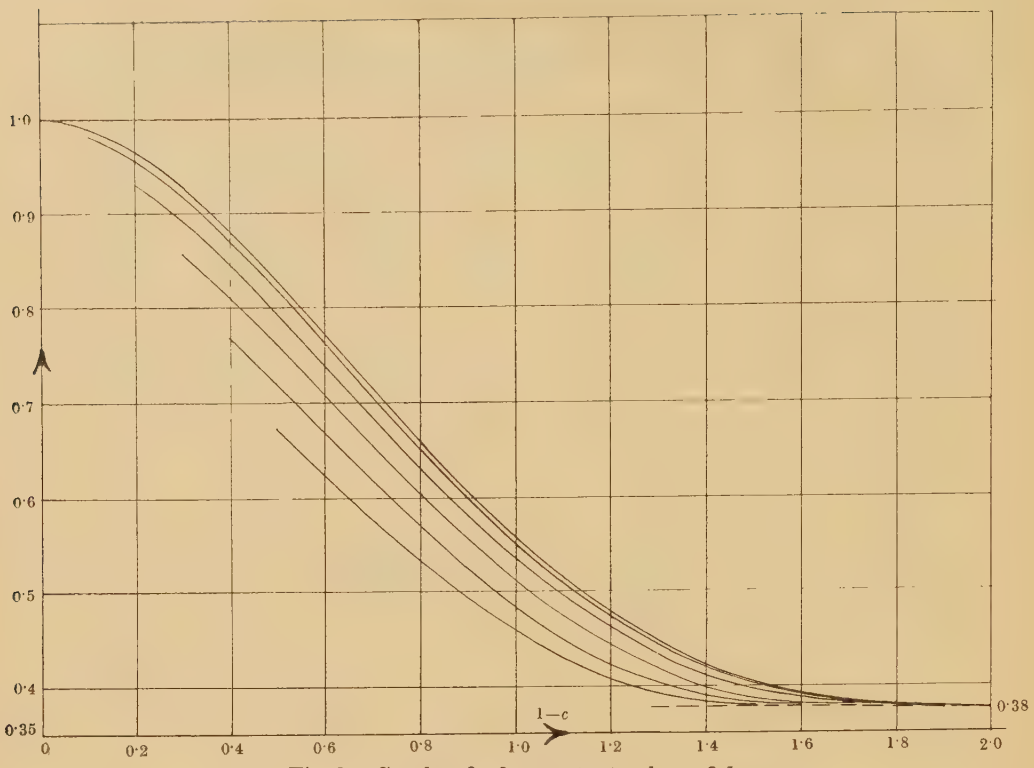

Fig. 3. Graphs of r for constant values of d.

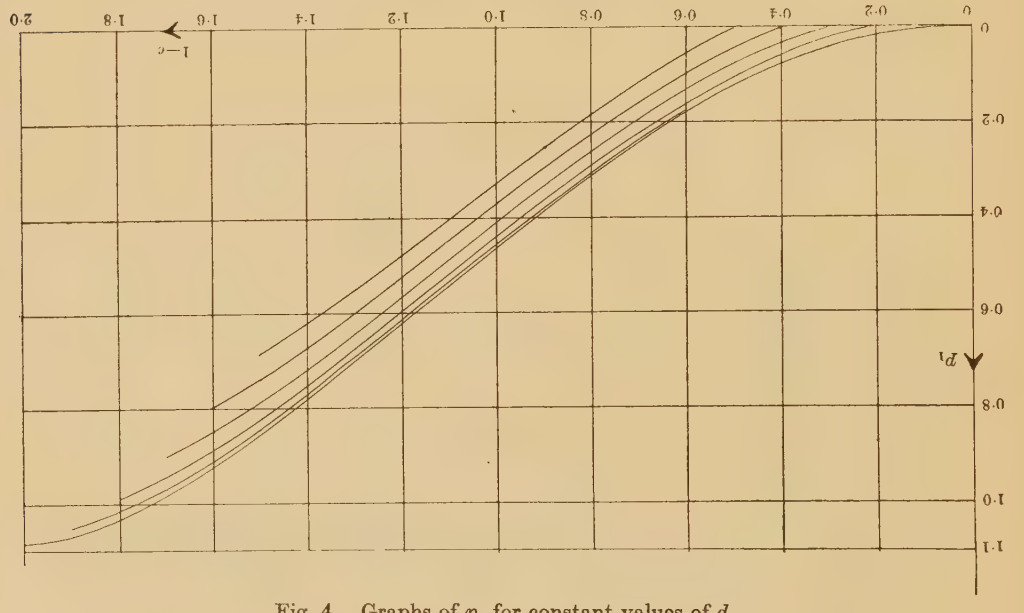

Fig. 4. Graphs of p_1 for constant values of d.

The present writer has not attempted to verify Shepherd's calculations, but it appears that the infinite series given by him (**2**, 628, 629) for his flexure function χ_1 can be summed and that his solution is then in agreement with that derived from the results of Section 4 of this paper by putting d and the radius a of the circle equal to unity.

References.

(**1**) Love, *The mathematical theory of elasticity* (Cambridge University Press, 1920).
(**2**) Shepherd, *Proc. Royal Soc.* (A), 138 (1932), 607–634.
(**3**) ———, *Proc. Royal Soc.* (A), 154 (1936), 500–509.
(**4**) Stevenson, *Phil. Trans. Royal Soc.* (A), 237 (1938), 161–229.

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ALGEBRA OF s-DIMENSIONS

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Definition 1. An algebra, G_s, is of *s dimensions* if the product of every set of s, but not less than s, elements of G_s belongs to G_s.

Ordinary mathematics is therefore two-dimensional.

In this sketch $s = 3$ and the algebra, assumed of *finite* order n, is called a *groupoid*. This term is used in a sense different from that of H. Brandt, *Math. Annalen*, 96 (1926), 360.

Example 1. The set of all odd permutations in any group is a G_3.

Example 2. The points of a harmonic range constitute a G_3 if, d being the conjugate of b with respect to a and c, we postulate that $abc = d$, $aaa = a$, $aba = a$, $abb = b$, $aab = a$ for all elements a, b, c.

Example 3. An involutorial correspondence J over a groupoid G_2 is such that $J^2 = 1$, $(ab)^J = b^J a^J$. The set of all such correspondences which satisfy the commutative law $JKL = LKJ$ is a completely associative G_3. The product of two involutions is an automorphism and not an involution.

Three-dimensional mathematics is related to that of two dimensions. Thus, to an associative G_2 in which $(ab)c = a(bc) = d$ corresponds a G_3 in which $abc = d$, and theorems in this G_3 may be regarded as theorems in the G_2. Since G_2 is associative, the following relations, in which the order of the elements is maintained, must hold for all numbers in G_3.

Partial and full associative laws.

$$\text{(I)} \quad \begin{cases} \text{(i)}\ (abc)de = a(bcd)e\,; & \text{(ii)}\ (abc)de = ab(cde)\,; \\ \text{(iii)}\ a(bcd)e = ab(cde)\,; & \text{(iv)}\ (abc)de = a(bcd)e = ab(cde). \end{cases}$$

G_3's may be constructed which satisfy none of the laws I, *e.g.* the G_3 of Example 2. Nevertheless any G_3 may be embedded in a G_2 of higher

order but, since the newly defined products of pairs of numbers in G_3 may not belong to G_3, the latter will not necessarily be a sub-groupoid of G_2. The theorems of this note may therefore be interpreted as theorems about certain sub-sets of a G_2, such sub-sets not being sub-groupoids but merely sets closed to the three-dimensional operation.

Not every G_2 contains a sub-set of this type, *e.g.*

	a	b	c
a	b	c	a.
b	b	c	a
c	b	c	a

The multiplication table of a G_3 may be represented in cubical form and may be read off in six ways each yielding a new groupoid $\bar{G}_3$, associated with G_3 in such a way that a theorem in G_3 may be interpreted as a theorem in $\bar{G}_3$ and *vice versa*. For example, if in G_3 $(abc)de = a(bcd)e$ and if $\bar{G}_3$ is such that to $abc = g$ in G_3 corresponds $a \times c \times b = g$ in $\bar{G}_3$ then the corresponding special associative law in $\bar{G}_3$ is

$$(a \times c \times b) \times e \times d = a \times e \times (b \times d \times c).$$

Thus to laws I correspond

$$(\mathrm{I}') \quad \begin{cases} \text{(i)}' \ (abc)de = ad(ceb); & \text{(ii)}' \ (abc)de = a(bde)c; \\ \text{(iii)}' \ a(bde)c = ad(ceb); & \text{(iv)}' \ (abc)de = ad(ceb) = a(bde)c. \end{cases}$$

The correspondence between the associated laws is

$$\text{(i)} \to \text{(iii)}'; \quad \text{(iii)} \to \text{(i)}'; \quad \text{(ii)} \rightleftarrows \text{(ii)}'; \quad \text{(iv)} \rightleftarrows \text{(iv)}'.$$

Hence, similarly, we have

THEOREM 1. *To any theorem T depending on the first (second) associative law there corresponds a theorem T' depending on the third (second) associative law and* vice versa, *where T' is obtained from T by writing all symbols in reverse order.*

The associative laws (I) and (I′) are independent.

Definition 2. Groupoids G, $\bar{G}$ having the same form of multiplication table but read in an order determined by a permutation of the symmetric group of order 3! will be said to be *related* by that permutation.

Thus $\bar{G}$ is related to G above by the permutation (23). Evidently groupoids exist in which all the theorems are invariant under a change of order in reading the multiplication table.

The *commutative laws* are six in number, viz. five obtained by interchange of the sequence a, b, c together with the full commutative law which is implied by any two of the partial laws and which allows all possible permutations of the numbers in the product abc.

Artifices similar to the following enable theorems about G_3's to be deduced from theorems about G_2's:

Let $a, b, c, \ldots$ denote elements of G_3 and let the ordered pairs $(a, b), \ldots$ denote elements of a G_2 in which the laws of operation are postulated:

Equality: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Multiplication: in G_2', $(a, b)(c, d) = (abc, d)$;

in G_2'', $(a, b)(c, d) = (abc, bcd)$.

THEOREM 2. *If* $(abc)de = ab(cde)$ *in* G_3 *then* e_1, e_2 *exist such that*

$$e_1 e_2 e_1 = e_1.$$

For G_2' is of finite order and is associative if $(abc)de = ab(cde)$. It therefore contains an idempotent element (e_1, e_2), *i.e.* $(e_1, e_2)^2 = (e_1, e_2)$, *i.e.* $e_1 e_2 e_1 = e_1$.

Definition 3. e_1 is called a *regular* element of G_3 and a G_3 in which all elements are regular is called a *regular groupoid.*

Similarly

(*a*) if $(abc)(bcd)e = ab(cde)$ and $a(bcd)(cde) = (abc)de$, then e_1, e_2 exist in G_3 such that $e_1 e_2 e_1 = e_1$ and $e_2 e_1 e_2 = e_2$;

(*b*) if $(abc)de = a(bde)c$, then e_1, e_2 exist in G_3 such that $e_1 e_1 e_2 = e_1$;

(*c*) if $(abc)(dbc)e = ab(cde)$ and $(abc)d(ebc) = a(bde)c$, then e_1, e_2 exist in G_3 such that $e_1 e_2 e_1 = e_1$, $e_2 e_2 e_1 = e_2$;

(*d*) if $(abc)de = a(bde)(cef)$, then e_1, e_2, e_3 exist in G_3 such that $e_1 e_1 e_2 = e_1$, $e_2 e_2 e_3 = e_2$, $e_3 e_3 e_1 = e_3$;

(*e*) if $(abc)de = a(bef)(cfd)$, then e_1, e_2, e_3 exist in G_3 such that $e_1 = e_1 e_2 e_3$, $e_2 = e_2 e_3 e_1$, $e_3 = e_3 e_1 e_2$.

Some consequences of the full associative law, (I)(iv), will now be mentioned.

All the $(2p+1)$-th powers of any number a, p being fixed, are equal and the different odd powers of a, p being varied, are fully commutative. The even powers have no significance in a G_3.

If G_3 is of finite order, the powers of any number a, viz. a, a^3, a^5, ..., are not all different. If $a^{2r+1} = a^{2s+1}$, $s > r$, then an idempotent $x^3 = x$ exists if $s-r$ is odd, while if $s-r$ is even no idempotent exists but

$$x_t = a^{2r+1+2t}$$

is such that $x_t = x_t^{2s-2r+1}$, *i.e.* all powers of a higher than the $(2r+1)$-th have the same period $2s-2r+1$.

These powers, viz. a^r, a^{r+2}, ..., a^{s-2}, form a sub-groupoid of the cyclic groupoid $\{a\}$ for the product of any three is of the form a^{3r+2t}. If

$$3r+2t = ls+m$$

then $a^{3r+2t} = a^{ls+m} = a^{lr+m}$. The process may be repeated until finally $a^{3r+2t} = a^{nr+p}$, where $r \leqslant nr+p < s$, and a^{nr+p} is therefore a member of the set which must be a sub-groupoid of $\{a\}$.

Definition 4. The powers of a, *i.e.* a^{2r+1}, a^{2r+3}, ..., a^{2s-1}, are called the *core* c of the cyclic groupoid $\{a\}$ and the numbers a, a^3, a^5, ..., a^{2r-1} are called the *stem*.

If the stem exists, then a is not the product of powers of a and the cyclic sub-groupoid $\{a\}$ is non-homogeneous.

THEOREM 3. *If the full associative law* (I) (iv) *holds, a power of* a, *viz.* $\bar{a}$, *exists such that* $\bar{a}ac = c$, *where* c *is any element of the core* C.

For $a^{2s+1} = a^{2r+1} = a^{2s-2r-1} a\, a^{2r+1} = \bar{a}aa^{2r+1}$; but $c = a^{2r+1} a^p a^p$, and multiplying on the right by $a^p a^p$ and using the associative law, we have $\bar{a}ac = c$.

THEOREM 4. *The core* C *is regular.*

Let $a^{2r+2t+1} = c$ be any number of C; then

$$a^{2r+2t+1} = a^{2r+2t+1+2\lambda(s-r)},$$

and, if $p = 2\mu+2r+1$, where μ, λ are integral solutions of

$$\mu+2r+t+1 = \lambda(s-r),$$

then $ca^p c = c$, *i.e.* c is a regular element of C, which is therefore a regular sub-groupoid of $\{a\}$.

Latin-groupoids.

There are two categories of groupoids which correspond to the Latin-groupoids of ordinary mathematics:

First category, G_L', B_L', ..., in which every number occurs in every *line* of cells of the cubical multiplication table, *i.e. each* of the equations $abx = c$, $axb = c$, $xab = c$ has a unique solution for all a, b, c in G_L'.

Second category, G_L'', B_L'', ..., in which every number occurs in every *plane* of cells of the multiplication table, *i.e. each* of the equations $xay = c$, $xya = c$, $axy = c$ has ordered pairs of solutions (x, y) for all a and c in G_L''. Since the number of cells in a plane is n^2 and there are only n symbols, the ordered pairs (x, y) are not, in general, unique and the *inverse* of the ordered pair a, c is the set of ordered pairs $[(x_1, y_1), (x_2, y_2), \ldots, (x_p, y_p)]$ satisfying $xay = c$, etc.

Every G_L' is also a G_L'' but not *vice versa*. Both G_L' and G_L'' are *homogeneous*, *i.e.* every number is the product of others.

Some of the following theorems, proved for a G_L' or a G_L'', are true for more extended classes of groupoids, since not all the equations defining a G_L' or G_L'' are used in the proofs.

THEOREM 5. *If* $(abc)de = ab(cde)$ *and if* e_1, e_2 *are the elements referred to in Theorem* 2, *then any number* g *which is of either of the forms* $g = e_1xy$ *or* $g = zwe_1$ *is such that* $g = e_1e_2g$ *or* $g = ge_2e_1$ *respectively.*

For $$g = e_1xy = (e_1e_2e_1)xy = e_1e_2(e_1xy) = e_1e_2g$$

and $$g = zwe_1 = zw(e_1e_2e_1) = (zwe_1)e_2e_1 = ge_2e_1.$$

COROLLARY. *In a* G_L'' *in which* (I)(ii) *holds* e_1, e_2 *exist such that*

$$g = e_1e_2g = ge_2e_1,$$

for all g. For, in a G_L'', x, y; z, w exist such that $g = e_1xy = zwe_1$.

THEOREM 6. *In a fully associative* G_L'' *every number* b *is such that an associated power* $\bar{b}$ *of* b *exists such that, for all* g *in* G_L'', $b\bar{b}g = \bar{b}bg = g$.

For the powers of b are not all different. Let $b^{2s+1} = b^{2r+1}$; then

$$bb^{2s-2r-1}b^{kr+1} = b^{2r+1}$$

and, since x, y exist in G_L'' such that, for a given g, $b^{2r+1}xy = g$ and since G_L'' is fully associative, $g = (bb^{2s-2r-1}b^{2r-1})xy = bb^{2s-2r-1}g = b\bar{b}g$, where $\bar{b} = b^{2s-2r-1}$.

Also the powers of b are commutative; hence $g = b\bar{b}g = \bar{b}bg$.

COROLLARY 1. $g = b\bar{b}g = gb\bar{b} = g\bar{b}b = \bar{b}bg$.

COROLLARY 2. $b = \bar{b}bb = b\bar{b}b$.

COROLLARY 3. *A fully associative G_L'' is regular.*

COROLLARY 4. *The associate of $\bar{b}$ is b.*

THEOREM 7. *If $\bar{b}$ is the associate of b then, in a fully associative G_L'', $\bar{b}_3\bar{b}_2\bar{b}_1$ is the associate of $b_1b_2b_3$.*

For $(\bar{b}_3\bar{b}_2\bar{b}_1)(b_1b_2b_3)g = \bar{b}_3\bar{b}_2[(\bar{b}_1\bar{b}_1b_2)b_3g] = \bar{b}_3\bar{b}_2(b_2b_3g)$

$$= \bar{b}_3(\bar{b}_2b_2b_3)g = \bar{b}_3b_3g = g.$$

This leads to some relations suggestive of the harmonic properties of pure geometry; for instance, from $abc = d$ we deduce that $\bar{a}d\bar{c} = b$, $c\bar{d}a = \bar{b}$, $d\bar{c}\bar{b} = a$, $\bar{d}ab = \bar{c}$, $bc\bar{d} = \bar{a}$, $\bar{b}\bar{a}d = c$.

Also, since the associate of abc is $\overline{abc}$, it follows that t can be found such that $\bar{c}\bar{b}\bar{a} = (abc)^{2t+1}$.

THEOREM 8. *The associate $\bar{a}$ of a is unique.*

For, if $g = \bar{a}ag = ga\bar{a} = \bar{\bar{a}}ag = ga\bar{\bar{a}}$, then $\bar{\bar{a}} = \bar{a}a\bar{\bar{a}} = \bar{a}$.

THEOREM 9. *In a fully associative G_L'' the e_2, e_1 of Theorem 2 are such that for all g in G_L'', $g = e_1e_2g = ge_2e_1 = e_2e_1g = g_1e_1e_2$.*

By Theorem 5 e_1, e_2 are such that $e_2 = e_2e_2e_1 = e_1e_2e_2$; hence

$$ge_1e_2 = ge_1(e_2e_2e_1) = g(e_1e_2e_2)e_1 = ge_2e_1 = g$$

and
$$e_2e_1g = (e_1e_2e_2)e_1g = e_1(e_2e_2e_1)g = e_1e_2g = g.$$

COROLLARY. *The associate e_2 of the e_1 of Theorem 5 is unique and is a power, $\bar{e}_1$ of e_1.*

For, let e_3 be another associate of e_1, then $g = e_2e_1g = ge_1e_2$, hence $e_3 = e_2e_1e_3 = e_2$, *i.e.* e_2 is unique and, by Theorem 6, it is a power of e_1.

Similar results are also true when only the second associative law, viz. (I)(ii) is postulated. Restrict the power notation as follows:

$$a,\ a^3,\ a^5 = aaa^3,\ a^7 = aaa^5,\ \ldots,\ a^{2r+1} = aaa^{2r-1},\ \ldots,$$

and suppose that (I)(ii) holds. Then in a finite groupoid the powers a, a^3, a^5, ..., are not all different. Let $a^{2s+1} = a^{2r+1}$, $s > r$, *i.e.*

$$a^{2r+1} = aaa^{2s-1} = aa(aaa^{2s-3}) = aa\{aa(aaa^{2s-s})\} = \ldots$$

$$= aa[aa\{aa\ \ldots\ (aaa^{2r+1})\}],$$

but, since the second law holds,

$$aa(aaa^{2r+1}) = a^3 aa^{2r+1}, \quad aa\{aa(aaa^{2r+1})\} = aa(a^3 aa^{2r+1})$$
$$= a^5 aa^{2r+1}, \quad aa(a^5 aa^{2r+1}) = a^7 aa^{2r+1}$$

and so on. Finally,

$$a^{2s-2r-1} aa^{2r+1} = a^{2s+1} = a^{2r+1}, \text{ i.e. } \bar{a}aa^{2r+1} = a^{2r+1}.$$

THEOREM 10. *In a G_L'', in which the second associative law holds, given any number a, a power $\bar{a}$ can be found such that $\bar{a}ag = g$ for all g.*

For x, y exist such that $a^{2r+1}xy = g$; hence, since the second associative law holds,

$$\bar{a}ag = g.$$

Note that $\bar{a}$ is the special type of power of a defined above. Also

$$a^5 = aaa^3 = aa(aaa) = a^3 aa;$$

$$a^7 = aaa^5 = aa(a^3 aa) = (aaa^3)aa = a^5 aa$$

and so on.

Hence $a^{2s-2r-1} a\, a^{2r+1} = (aa\, a^{2s-2r-3})aa^{2r+1} = (a^{2s-2r-3} aa)aa^{2r+1}$

$$= a^{2s-2r-3} a\, a^{2r+3},$$

and so on, until we finally arrive at $a^{2r+1} a\, a^{2s-2r-1}$. Hence

$$g = \bar{a}ag = ga\bar{a}.$$

THEOREM 11. *The $\bar{a}$ of Theorem* 10 *is unique.*

For, if $g = \bar{a}ag = ga\bar{a} = \bar{\bar{a}}ag = ga\bar{\bar{a}}$, then $\bar{\bar{a}} = \bar{a}a\bar{\bar{a}} = \bar{a}$.

THEOREM 12. *All the cyclic sub-groupoids of a G_n'' in which the full associative law holds are stemless.*

By Theorems 10, 11 the associate of $\bar{a}$ is a, which must therefore be a power of a; and this is possible only if the cyclic-sub-groupoid $\{a\}$ is stemless.

Evidently the associate $\bar{a}$ of a is a generalized *inverse* of a.

THEOREM 13. *Every sub-groupoid of a G_L'' in which law* (I) (ii) *holds is also a Latin-groupoid of the second category B_L''.*

Given b_1 and b_2, $\bar{b}_1$ exists such that $b_1\bar{b}_1 b_2 = b_2$, *i.e.* the equation $b_1 xy = b_2$ has a unique solution; $b_2 b_1 \bar{b}_1 = b_2$, *i.e.* $xb_1 y = b_2$ has a solution and $b_2\bar{b}_1 b_1 = b_2$, *i.e.* $xy\, b_1 = b_2$ has a solution. Hence the sub-groupoid B is a B_L''.

THEOREM 14. *In a G_L' in which* (I)(iii) *holds and in which e_1, e_2 exist such that $e_1e_2e_1 = e_1$, then also $e_2e_1e_2 = e_2$.*

$e_2e_1e_2 = e_2(e_1e_2e_1)e_2 = e_2e_1(e_2e_1e_2)$ and, since in a G_L' the equation $e_2e_1x = e_2e_1e_2$ has the unique solution $x = e_2$ it, follows that $e_2e_1e_2 = e_2$.

COROLLARY. *Corresponding results are true if law* (i) *holds.*

Regular elements.

We now give a brief discussion of some regular elements, viz. idempotents, zeros, units, and semi-units.

Zeros z are of two categories each of three kinds and are numbers z such that for all numbers a and b of G one or more of the following relations hold;

$$\text{(A)}\quad \text{(i)}\ abz = z,\quad \text{(ii)}\ azb = z,\quad \text{(iii)}\ zab = z;$$

$$\text{(B)}\quad \text{(i)}\ azz = z;\quad \text{(ii)}\ zaz = z,\quad \text{(iii)}\ zza = z.$$

Every zero of category (A) is also of category (B) but not *vice versa*.

If associative laws hold, the existence of a zero of one kind may imply that it is also of another kind. Thus, if the associative law (I)(ii) holds and if $zza = z$, then also $zbc = (zza)bc = zz(abc) = zzd = z$ and $zbz = z$. If (I) (ii) holds and if $azz = z$, then $bcz = zcz = z$.

If (I)(iii) holds and if $zza = z$, then $azc = azz$, since

$$azc = a(zzz)c = az(zzc) = azz;$$

and if $azz = z$, then $azc = zzc$.

The uniqueness properties usually associated with zeros in a G_2 may or may not depend on the associative laws in a G_3. If $abz_1 = z_1$ and $cz_2d = z_2$, then $z_1 = z_2$ and is therefore unique irrespective of any associative law. Similarly if $abz_1 = z_1$ and $z_2cd = z_2$, then $z_1 = z_2$ and is therefore unique.

If $z_1z_1a = z_1$ and $az_2z_2 = z_2$ and if (I)(ii) holds, then $z_1 = z_2$ and is therefore unique. Conversely Law (I) (ii), in so far as it applies to z_1, z_2, a, is necessary if these zeros are to be unique for $z_1 = z_1z_1z_2 = z_1z_1(az_2z_2)$ and $z_2 = z_1z_2z_2 = (z_1z_1a)z_2z_2$; hence it is necessary that

$$z_1z_1(az_2z_2) = (z_1z_1a)z_2z_2.$$

The theory of zeros is of importance in the theory of multiplicative ideals in G, a subject not discussed here.

Zeros are idempotents, $z^3 = z$.

Units e are also of two categories, namely numbers e such that, for all numbers a and b of G,

(A): (i) $abe = a$, (ii) $abe = b$, (iii) $aeb = a$,

(iv) $aeb = b$, (v) $eab = a$, (vi) $eab = b$.

(B): (i) $aee = a$, (ii) $eae = a$, (iii) $eea = a$.

Units are idempotents.

Semi-units are members e_1, e_2 such that for all a in G one of the following relations holds:

(i) $e_1 ae_2 = a$, (ii) $e_1 e_2 a = a$, (iii) $ae_1 e_2 = a$.

Semi-units are not necessarily idempotent. The numbers e_1, e_2 are said to be *associates*.

The following table shows the consequences of the various associative Laws (I) on the unit s, semi-unit, and elements:

Unit	Law (i)	Law (ii)	Law (iii)	Law (iv)
1. $eae = a$	$a = eea$; $abe = aeb$	$a = aee = eea$ $abe = eab$	$a = aee$; $aeb = eab$	$abe = aeb = eab$
2. $bea = a$	$abc = ebc$	$abc = c$	$abc = c$	$abc = ebc = c$
3. $bea = b$	$abc = a$	$abc = a$	$abc = abe$	$abc = abe = a$
4. $bac = b$	$abc = a$	No further relation	$abc = a$	$abc = a$
5. $eab = b$	$abc = c$	No further relation	$abc = c$	$abc = c$
6. $eab = a$ or $bae = a$	The groupoid contains only the single element e or is completely non-associative.			
7. $a = eea$ or $a = aee$	No further relations			
8. $e_1 ae_2 = a$	$ae_1e_2 = ae_2e_1 = e_2ae_1$ $a = e_1ae_2 = e_2e_1a$ $= e_1e_2a$ $abe_2 = ae_2b$ $e_2ae_2 = e_2e_2a = ae_2e_2$	No further relations	$e_1e_2a = e_2e_1a = e_2ae_1$ $a = e_1ae_2 = ae_2e_1 = ae_1e_2$ $e_1bc = be_1c$ $e_1ae_1 = ae_1e_1 = e_1e_1a$	a, e_1, e_2 are fully commutative with one another.

Notes on the table.

(*a*) $e_1 a e_2 = a$ is not a consequence of either $e_1 e_2 a = a$ or $a e_1 e_2 = a$ and the associative laws; neither is $eae = a$ a consequence of $a = eea$ or aee and these laws.

(*b*) If the full associative law holds, it is impossible to have two different units e_1, e_2 of kinds

$$be_1 a = a, \quad be_2 a = b; \quad be_1 a = b, \quad e_2 ab = b;$$

$$be_1 a = a, \quad bae_2 = b; \quad bae_1 = b, \quad e_2 ab = b.$$

(*c*) On the other hand, with the full associative law, the following pairs are consistent, $be_1 a = a$, $e_2 ab = b$; $be_1 a = b$, $bae_2 = b$; and the groupoid may then contain two unequal units of different kinds in which case every number is a zero of category (A).

(*d*) If only (ii) holds and units e_1, e_2 exist such that $bae_1 = b$, $e_2 ab = b$, then $e_2 ae_1 = e_2 = e_1$ and the units are unique.

(*e*) If $bea = a$, Laws (ii) or (iii) make every element a unit of the same kind; if $bea = b$, Laws (i) or (ii) make every element a unit of the same kind; if $bae = b$ or $eab = b$, Laws (i) or (iii) make every element a unit of the same kind.

(*f*) In a fully associative G, any semi-unit e_1 has at most one associate e_2; for, if e_3 is another, $e_1 e_2 e_3 = e_2 = e_3 = e_1$.

Correspondences and representations.

Correspondences can be established algebraically between elements of a groupoid which give analogues of Cayley's theorem that every group is simply isomorphic with a group of permutations.

(*a*) *Correspondences between elements.*

(i) Let $C_{x,y}$: $axy = a'$ define a correspondence $a \to a'$. Then, if the associative Law (I)(i) holds,

$$C_{x,y} C_{u,v} = C_{xyu,v}.$$

These correspondences are therefore simply isomorphic with the groupoid G_2' of § 1.

(ii) Let $C_{x,y;u,v}$: $axy = uva'$ define the correspondence $a \to a'$. Then, if the full associative Law (I) (iv) holds,

$$C_{x,y;u,v} C_{m,n;p,q} = C_{xym,n;uvp,q}.$$

(iii) Let $C_{x,y}$: $axy = xya'$ determine the correspondence $a \to a'$. Then, if Law (I) (ii) holds,

$$C_{x,y} C_{u,v} C_{r,s} = C_{xyu,\,vrs}.$$

(*b*) *Correspondence between ordered pairs.*

Let C_x: $abx = xa'b'$ define a correspondence $(a, b) \to (a', b')$ between the ordered pairs (a, b), (a', b'). Then, if the full associative law holds,

$$C_x C_y C_z = C_{xyz}.$$

Such correspondences are possible only in special G; for, given a, b, x, it is not always possible to determine a', b' to satisfy the relation.

(*c*) *Correspondence between elements and ordered pairs.*

Let $C'_{x,y;z}$: $axy = zbc$ define a correspondence $a \to (b, c)$; let

$$C''_{w;u,v}: \; bcw = uva'$$

define a correspondence $(b, c) \to a'$. Then, if the full associative law holds and if $C_{x,y;u,v}$ are as in (*a*),

$$C'_{x,y;z} C''_{w;u,v} C'_{m,n;l} C''_{r;s,t} = C_{xyw,\,mnr;\,zuv,\,lst}.$$

In certain cases these correspondences are also endomorphisms, *e.g.*, if $axy = xya'$, $bxy = xyb'$, $cxy = xyc'$, then

$$(abc)xy = ab(cxy) = ab(xyc') = a(bxy)c' = a(xyb')c' = (axy)b'c'$$
$$= (xya')b'c' = xy(a'b'c'),$$

i.e. if the full associative law holds $abc \to a'b'c'$.

As in the theory of groups, these correspondences lead to theorems of homomorphism, *e.g.*, let A and A' be two groupoids between which a homomorphism $A \sim A'$ is established, and let the second associative law hold. Then A' contains semi-units e_1', e_2' such that $e_1' e_2' a' = a'$. Hence, if H_1, H_2 denote the classes of elements of A which correspond to e', e_2' respectively, then $H_1 H_2 a \to e_1' e_2' a' = a'$, *i.e.* to the classes $H_1 H_2 a$ of A there correspond the elements a' of A'. Also to the product of any three elements from three classes corresponds the product of the corresponding elements of A' and if the product of the classes $H_1 H_2 a_1$, $H_1 H_2 a_2$, $H_1 H_2 a_3$ is defined to be $H_1 H_2$ $(a_1 a_2 a_3)$ then a simple isomorphism exists between A' and the set of classes of A; this is denoted by $A/H_1 \times H_2$. Hence

$$A' \cong A/H_1 \times H_2 \quad \text{(First theorem of isomorphism).}$$

Coset expansions in a G_L'.

Many of the theorems of this paragraph are true under much weaker conditions, but the statements and proofs would be less concise if such conditions were postulated.

THEOREM 15. *Every sub-groupoid B of a G_L' is a B_L'.*

For, s being the order of B, the numbers $b_1 b_2 b_i$, $i = 1, \ldots, s$, are all different in a G_L' and must therefore be the elements $b_1, b_2, \ldots, b_s$ of B in some order. Hence each of the equations $b_1 b_2 x = b$, $b_1 x b_2 = b$, $x b_1 b_2 = b$ has a unique solution in B, which is therefore a B_L'.

COROLLARY. *Every G_L' is regular.*

For x can be found to satisfy $gxg = g$.

Definition 5. Sets of numbers ABg, AgB, gAB; $(ABC)g_1 g_2, \ldots$, $Ag_1 g_2$, $g_1 A g_2$ $g_1 g_2 A, \ldots$, are called *cosets* of G with respect to the sub-groupoids $A, B, C, \ldots$.

Definition 6. The number g is called a *leader* of the coset ABg.

The subsequent discussion is restricted to cosets ABg; but similar theories hold for other types of cosets.

THEOREM 16. *Every number in a G_L' is in some coset ABg.*

When g and a, b are given, g' can be found such that $g = abg'$, *i.e.* g is in ABg'.

For convenience of reference certain weak forms of the associative laws are recorded, $\bar{a}$ denoting a power of a. Many theorems remain true if $\bar{a}$ is replaced by a' an element in the same sub-groupoid as a but, since the discussion is simplified if the theorems are stated so as to apply to all sub-groupoids and therefore to the cyclic sub-groupoid $\{a\}$, the more general results are not recorded.

$$\text{(II)} \quad \begin{cases} \text{(i)} \ (abc)de = \bar{a}\bar{b}(cde), & \text{(iv)} \ ab(cde) = (\bar{a}\bar{b}c)de, \\ \text{(ii)} \ a(bcd)e = (a\bar{b}\bar{c})de, & \text{(v)} \ (abc)de = a(\bar{b}\bar{c}d)e, \\ \text{(iii)} \ ab(cde) = (abc)\bar{d}\bar{e}, & \text{(vi)} \ (abc)de = ab(c\bar{d}\bar{e}). \end{cases}$$

As compared with (I), six relations are postulated instead of three, the reason being that (i) implies only that $(ABc)de \leqslant AB(cde)$ and (iv) only that $AB(cde) \leqslant (ABc)de$.

THEOREM 17. *Necessary and sufficient conditions for every number in a coset to be a leader of the coset are that given a, a_1 in A and b, b_1 in B, a_2, b_2*

can be found in A, B respectively such that

$$\text{(i) } a_1 b_1 (abg) = a_2 b_2 g; \quad \text{(ii) } a_2 b_2 (abg) = a_1 b_1 g. \tag{III}$$

For in (i), allowing a_1, b_1 to range over A, B respectively,

$$AB(abg) \leqslant ABg,$$

and from (ii) similarly $ABg \leqslant AB(abg)$. Hence $AB(abg) = ABg$, *i.e.* abg is a leader of the coset.

Conversely, if abg is a leader of the coset, then $AB(abg) = ABg$ from which (III) (i) and (ii) follow.

These conditions are included in the associative laws

$$ab(cde) = a'b'e, \quad a'b'(cde) = abe, \tag{III'}$$

where a' is in $\{a, c\}$, b' in $\{b, d\}$.

These associative Laws (III) and (III') are not consequences of (I) or (II). Thus (II) requires that $a_1 b_1 (abg) = (\bar{a}_1 \bar{b}_1 a)bg$ and, in general, $\bar{a}_1 \bar{b}_1 a$ is not in A. Conversely (III') does not imply (II) for a' is in $\{a, c\}$ in (III') whereas in (II) it is in $\{a, b, c\}$. In this respect the corresponding laws in a two dimensional groupoid are different; for $(ab)c = a(bc)$ implies that $(ab)c = ad$ where d is in $\{b, c\}$.

An example of a fully associative groupoid in which coset expansions are possible but in which the leader is not a member of the coset is furnished by the permutations $a = (12)$, $b = (23)$, $c = (31)$ for which the cosets ABa, ABb, ABc are c, a, b respectively.

COROLLARY. *If* (II) (i) *and* (III) (i) *hold and if G_L' contains an idempotent b, then* $a_1 b(abg) = (\bar{a}_1 ba)bg = a_2 bg$. Hence $\bar{a}_1 ba = a'$, *i.e.* given a_1 and a, $\bar{a}_1$ and a' can be found such that $\bar{a}_1 ba = a'$.

THEOREM 18. *If conditions* (III) *hold, the cosets are distinct and any relation such as* $a_1 b_1 g_1 = a_2 b_2 g_2$ *implies* $ABg_1 = ABg_2$.

Let ABg_1, ABg_2 have a common element $a_1 b_1 g_1 = a_2 b_2 g_2$, then from (III) for any a_α in A and b_β in B, a_i, a_α' in A and b_j, b_β' in B exist such that

$$a_\alpha b_\beta g_1 = a_i b_j (a_1 b_1 g_1) = a_i b_j (a_2 b_2 g_2) = a_\alpha' b_\beta' g_2.$$

Hence, allowing a_α, b_β to range over A, B respectively,

$$ABg_1 \leqslant ABg_2;$$

similarly $ABg_2 \leqslant ABg_1$, therefore $ABg_1 = ABg_2$.

THEOREM 19. *In a G_L' in which* (III) (i) *holds,* $ab(ABg) = ABg$.

Let $n_1, n_2, \ldots, n_s$ be the different numbers in ABg. Then in a G_L' the numbers $abn_1, abn_2, \ldots, abn_s$ are also different and, from (III)(i), given

a, $\bar{b}$, n_i, a_2, b_2 exist such that $abn_i = a_2 b_2 g$. Hence the numbers

$$abn_1,\ abn_2,\ \ldots,\ abn_s$$

are all in ABg and, being s in number, and all different, must be the numbers $n_1, n_2, \ldots, n_s$ in some order. Therefore

$$ab(ABg) = ABg. \tag{1}$$

COROLLARY 1. *In a G_L',* (III) (ii) *is a consequence of* (III) (i).

COROLLARY 2. *From equation* (1) *it follows that given a and b, then e_a, e_b can be found in A, B respectively such that $ab(e_a e_b g) = abg$ and, since the cancellation law holds in a G_L',*

$$g = e_a e_b g. \tag{2}$$

This result should be compared with that of Theorem 10.

THEOREM 20. *In a G_L' in which* (III) (i) *holds the leader of a coset belongs to the coset.*

This follows from (2).

In a G_L' there are several solutions of $g = xyg$, but Corollary 2 states that, given *any* two sub-groupoids A and B, x can be found in A and y in B to satisfy the relation $g = xyg$.

THEOREM 21. *In a G_L' in which* (III) *holds no cross-cut is void.*

Let A, B be any two sub-groupoids and let b be any number in B. Then from (2), a, $\bar{b}$ exist in A and B respectively such that $b = a\bar{b}b$ and, by Theorem 15, a must be in B and in A. Therefore a is in (A, B), which must therefore be non-void.

COROLLARY 1. *Some power $\bar{a}$ of any number a can be found in any sub-groupoid B.*

COROLLARY 2. *There exists a cyclic sub-groupoid E contained in every sub-groupoid.*

It is important to note that the existence of the sub-groupoid E which is contained in all sub-groupoids depends only on (III) and the property that the equation $b = xb_1 b_2$ has a unique solution in B.

An example shows that a sub-groupoid E common to all sub-groupoids may exist, even though no coset expansion is possible, *e.g.*, in the fully associative groupoid A, a, $a^3 = a^5$, a^3 is a minimal groupoid E, but a is not the product of other numbers of the groupoid since it is in the stem of the cyclic groupoid $\{a\}$.

COROLLARY 3. *If e is any idempotent, $e^3 = e$, then $g = eeg$, i.e. e is also a unit of Type B. Also $E = e$, which is therefore unique. Hence a G_L' in which* (III) *holds contains exactly one idempotent or none.*

In a G_L'' other types of coset expansion are necessary; for, given g and b, B being any sub-groupoid of G, then a and g' exist such that $g = abg'$. Allowing g and b to range over G and B respectively it is evident that the set of the left-hand elements serve as generators of a sub-groupoid A_B, depending on B, and that every number of G_n'' is in some coset $A_B Bg$. There is therefore a parallel but more complicated theory for coset expansions of a G_L''.

THEOREM 22. *If the associative Laws* (II) (ii) *and* (v) *hold, if $G = \Sigma ADg_2$ and $D = \Sigma BCg_1$, then $G = \Sigma(ABC)g_1g_2$.* (*Transitive property.*)

Condition (ii), viz. $a(bcd)e = (a\bar{b}\bar{c})de$ gives, on allowing b, c to range over B, C respectively, $a(BCd)e \leqslant (aBC)de$ and condition (v) gives

$$(aBC)de \leqslant a(BCd)e.$$

Hence $a(BCd) = (aBC)de$ and, allowing a to range over A,

$$A(BCg_1)g_2 = (ABC)g_1g_2.$$

Very much weaker conditions suffice to ensure that this property of transivity holds.

THEOREM 23. *In a G_L' in which the number of different elements in the cosets ABg_1, ABg_2 is the same for all g_1 and g_2 and for all sub-groupoids A, B the associative Laws* (II) (iv), (v) *and* (vi) *are consequences of* (II) (i), (ii), *and* (iii) *respectively.*

From $(abc)de = \bar{a}\bar{b}(cde)$, allowing a, b to range over A, B respectively, we have $(ABc)de \leqslant AB(cde)$; but, by hypothesis, the cosets ABc, $AB(cde)$ have the same number of different elements $g_1, g_2, \ldots, g_s$. If $g_1, g_2, \ldots, g_s$ are different elements of ABC then the numbers g_1de, g_2de, $\ldots$, g_sde each of which is in $AB(cde)$ are different and, since $AB(cde)$ has exactly s different elements, $(ABc)de = AB(cde)$; consequently $ab(cde) = (\bar{a}\bar{b}c)de$, *i.e.* (iv) is a consequence of (i) and the hypothesis. Similarly the Laws (ii) and (iii) imply (v) and (vi) respectively.

THEOREM 24. *In a G_L' in which* (II) (i) *and* (iv) *hold, the cosets ABg_1 and ABg_2 have the same number of different elements.*

From (II) (i) and (iv), $(ABc)de = AB(cde)$. Let $n_1, n_2, \ldots, n_s$ be the different elements of ABc then in a G_L' the numbers n_1de, n_2de, $\ldots$, n_sde

are also different. Hence the number of different elements in $AB(cde)$ is the same as in ABc and, since, given c, then g_2, d, e exist such that $cde = g_2$, it follows that ABc and ABg_2 have the same number of different elements.

COROLLARY 1. *If laws* (II) (i) *and* (iv) *hold,* $s \mid n$ *where* n *is the order of* G_L' (*Lagrange*).

Similarly, in a G_L' in which laws (II) (ii) and (v) hold, the cosets aBC and BCd have the same number of distinct elements for

$$a(BCd)e = (aBC)de$$

and, if $n_1, n_2, \ldots, n_s$ are the distinct elements of BCd, then $an_1e, \ldots, an_se$ are also the distinct elements of $a(BCd)e$ and are therefore the distinct numbers of $(aBC)de$. If $m_1, m_2, \ldots, m_r$ are the distinct elements of aBC then the distinct elements of $(aBC)de$ are $m_1de, m_2de, \ldots, m_rde$ and these must be $an_1e, \ldots, an_se$ in some order. Hence $r = s$, *i.e.* the cosets BCd and aBC have the same number of different elements.

COROLLARY 2. *In a* G_L' *in which* (II) (iii) *and* (vi) *hold, the cosets* g_1BC *and* g_2BC *have the same number of different elements.*

The existence of quotient groupoids demands suitable definitions of normality. Forty-six types of normal sub-groupoids may be distinguished.

Definition 8. B is said to be *normal in* G if one of the following 46 conditions is satisfied:

(α) if for all g in G and for any sub-groupoid A in G,

$$ABg = gAB. \tag{1}$$

There are fifteen kinds of normal sub-groupoids of this category obtained from (1) by permutation of the symbols A, B, g;

(β) as in (α) but with special sub-groupoid A_B, depending on B, in place of any sub-groupoid A;

(γ) if for all g_1, g_2 in G one of the following eight conditions is satisfied $g_1Bg_2 = g_2g_1B$ and the corresponding relations obtained by permuting the symbols g_1, g_2, B.

(δ) as for (γ) except that g_1 is replaced by g_B an element depending on B.

In this note only the type $ABg = gAB$ is discussed. One result of the postulate of normality is that, if B and C are of this type, then

$$BBC = BCB = CBB; \quad CCB = CBC = BCC$$

and

$$ABC = CAB.$$

The following weaker forms of the associative laws (II) (ii) and (II) (i) suffice to ensure that certain of the well-known properties of normal groups hold here.

(IV) (i) $a(bcd)e = (a\bar{b}\bar{c})\bar{d}e$; (ii) $(abc)de = \bar{a}\bar{b}(\bar{c}\bar{d}e)$.

THEOREM 25. *If B, C are normal sub-groupoids and if* (IV) *holds then every element* of $M = [B, C]$ has one of the forms b, c, bc_1c_2, cb_1b_2.*

There are twenty-four products of five elements and the conditions of normality and (IV) suffice to reduce them to one of the given forms.

$$c_1c_2(bc_3c_4) = (bc_3c_4)c_5c_6 = \bar{b}\bar{c}_3(\bar{c}_4\bar{c}_5c_6) = b_1\bar{c}_3c_7,$$

$$c_1c_2(b_1b_2c_3) = (b_1b_2c_3)c_4c_5 = \bar{b}_1\bar{b}_2(\bar{c}_3\bar{c}_4c_5) = b_3b_4c_6,$$

$$c_1(bc_2c_3)c_4 = (c_1\bar{b}\bar{c}_2)\bar{c}_3c_4 = \bar{c}_1b_2(c_5c_6c_4) = c_7b_2c_8 = b_3c_9c_{10},$$

$$c_1(b_1b_2c_2)c_3 = (c_1\bar{b}_1\bar{b}_2)\bar{c}_2c_3 = (b_3b_4c_1)\bar{c}_2c_3 = \bar{b}_3\bar{b}_4(\bar{c}_1\bar{\bar{c}}_2c_3) = b_5b_6c_4,$$

$$b_1c_1(b_2b_3c) = (b_2b_3c)b_4c_2 = \bar{b}_2\bar{b}_3(\bar{c}\bar{b}_4c_2) = (\bar{c}\bar{b}_4c_2)b_5b_6$$

$$= (c_3c_4b_7)b_5b_6 = \bar{c}_3\bar{c}_4(\bar{b}_7\bar{b}_5b_6) = c_5c_6b = b_8c_7c_8,$$

$$b_1c_1(c_2c_3b_2) = (c_2c_3b_2)b_3c_4 = \bar{c}_2\bar{c}_3(\bar{b}_2\bar{b}_3c_4) = (\bar{b}_2\bar{b}_3c_4)c_5c_6 = b_7b_8c_7,$$

$$b_1(b_2c_2c_3)c_4 = (b_1\bar{b}_2\bar{c}_2)\bar{c}_3c_4 = b_3b_4(c_5c_6c_4) = b_3b_4c_7,$$

$$b_1(b_2b_3c_1)c_2 = (b_1\bar{b}_2\bar{b}_3)\bar{c}_1c_2 = bc_3c_4,$$

$$(b_1c_1c_2)b_2c_3 = (c_2c_3b_3)b_2c_3 = \bar{c}_2\bar{c}_3(\bar{b}_3\bar{b}_2c_3) \text{ already discussed},$$

$$(b_1b_2c_1)b_3c_2 = \bar{b}_1\bar{b}_2(\bar{c}\bar{b}_3c_2) = (c_2c_3b_4)b_5b_6 = \bar{c}_2\bar{c}_3(\bar{b}_4\bar{b}_5b_6) = c_4c_5b,$$

$$(b_1c_1c_2)c_3c_4 = c_5c_6(bc_1c_2) \text{ already discussed},$$

$$(cb_1b_2)c_2c_3 = c_4c_5(cb_1b_2) \text{ already discussed}.$$

The remaining twelve cases are obtained by interchange of the letters b and c.

Next, there are three types of products of seven elements:

$$(\alpha_1\alpha_2\alpha_3)(\beta_1\beta_2\beta_3)\gamma, \quad (\alpha_1\alpha_2\alpha_3)\gamma(\beta_1\beta_2\beta_3), \quad \gamma(\alpha_1\alpha_2\alpha_3)(\beta_1\beta_2\beta_3)$$

in which the elements of each of the sets α, β are not all of the same denomination b or c.

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* $M = [B, C]$ is the *union* of B, C.

$(\alpha_1\alpha_2\alpha_3)(\beta_1\beta_2\beta_3)\gamma = \{(\alpha_1\alpha_2\alpha_3)\bar{\beta}_1\bar{\beta}_2\}\bar{\beta}_3\gamma$ and the terms in the bracket are products of five elements and are therefore of the prescribed form. Hence the whole product is now one of five elements and is therefore of the prescribed form.

$(\alpha_1\alpha_2\alpha_3)\gamma(\beta_1\beta_2\beta_3) = \bar{\alpha}_1\bar{\alpha}_2[\bar{\alpha}_3\bar{\gamma}(\beta_1\beta_2\beta_3)]$ and, using the preceding results, is also of the prescribed form.

$\gamma(\alpha_1\alpha_2\alpha_3)(\beta_1\beta_2\beta_3) = (\gamma\bar{\alpha}_1\bar{\alpha}_2)\bar{\alpha}_3(\beta_1\beta_2\beta_3) = \bar{\gamma}\bar{\bar{\alpha}}_1[\bar{\bar{\alpha}}_2\bar{\alpha}_3(\beta_1\beta_2\beta_3)]$ is also of the prescribed form.

The theorem is therefore true for products of seven elements.

There is one type of product of nine elements not immediately dependent on the previous cases, viz.

$$(\alpha_1\alpha_2\alpha_3)(\beta_1\beta_2\beta_3)(\gamma_1\gamma_2\gamma_3) = [(\alpha_1\alpha_2\alpha_3)\bar{\beta}_1\bar{\beta}_2]\bar{\beta}_3(\gamma_1\gamma_2\gamma_3),$$

which now depends on the previous cases and is therefore of the prescribed form.

The complete theorem now follows by induction.

To obtain the normal properties of the union an additional associative law is required and (II) (ii) strengthened.

(V)
$$\begin{cases} \text{(i)} & a(bcd)e = (a\bar{b}\bar{c})\bar{d}e, \\ \text{(ii)} & (abc)de = a\bar{b}(\bar{c}\bar{d}e), \\ \text{(iii)} & ab(cde) = a(\bar{b}\bar{c}\bar{d})e. \end{cases}$$

THEOREM 26. *If the associative laws* (V) *hold and if B and C are normal, then so is their union $M = [B, C]$.*

By Theorem 25 the elements of M are of one of the forms

$$b,\ c,\ bc_1c_2,\ cb_1b_2.$$

If $m = b$ then $amg = abg = g\bar{a}\bar{b}$ since B is normal. If $m = bc_1c_2$ then

$$\begin{aligned} amg &= a(bc_1c_2)g = a(c_3c_4b) = (a\bar{c}_3\bar{c}_4)\bar{b}g \\ &= (c_5c_6a)\bar{b}g = c_5\bar{c}_6(\bar{a}\bar{\bar{b}}g) = (a_3b_3g)c_5c_6 = (ga_4b_4)c_5c_6 \\ &= g\bar{a}_4(\bar{b}_4\bar{c}_5c_6) = g\bar{a}_4m_2. \end{aligned}$$

Similarly if $m = cb_1b_2$. Hence $AMg \leqslant gAM$.

Next,

$$\begin{aligned} gam &= ga(bc_1c_2) = g(\bar{a}\bar{c}_3\bar{c}_4)b = g(c_5c_6a_2)b \\ &= (gc_5\bar{c}_6)\bar{a}_2b = a_3b_2(gc_5\bar{c}_6) = a_3b_2(c_7c_8g) = a_3(\bar{b}_2\bar{c}_7\bar{c}_8)g \\ &= a_3m_2g. \end{aligned}$$

Similarly if $m = b_1 b_2 c$. Hence $gAM \leqslant AMg$. Therefore

$$AMg = gAM,$$

i.e. M is also normal.

As in group theory and in the theory of equivalence relations, the postulate of the normality of B and C may be replaced by that of permutability.

Stronger forms of the associative laws are necessary in order to preserve the properties which depend also on the existence of cosets.

The following laws are stronger forms of (V).

$$\text{(VI)} \qquad \begin{cases} \text{(i)} & a(bcd)e = (a\bar{b}\bar{c})de, \\ \text{(ii)} & (abc)de = a\bar{b}(cde), \\ \text{(iii)} & ab(cde) = a(b\bar{c}\bar{d})e. \end{cases}$$

THEOREM 27. *In a G_L' in which* (III) *and* (VI) (ii) *hold the elements of M are of the form bc_1c_2.*

For $b = be_1e_2$, where e_1, e_2 are in C and $c = bc_1c$, where b is in B and c_1 in C. Let $m = b_1b_2C$, then E exists common to all sub-groupoids, hence, given e in E; and therefore also in C, c_1c_2 exist such that $cc_1c_2 = e$. Hence $mc_1c_2 = (b_1b_2c)c_1c_2 = b_1\bar{b}_2(cc_1c_2) = b_1\bar{b}_2e = b_3$, since e is also in B. Now $mc_1c_2 = c_3c_4m$, since C is normal, and c_5c_6 exist such that $c_5c_6\ (c_3c_4m) = m$ (III). Hence $m = c_5c_6b_3$, *i.e.* every number of the form b_1b_2c is also of the form c_1c_2b.

THEOREM 28. *If B is normal and if the coset expansion exists and* (VI) *hold, then a quotient-groupoid exists.*

$$\begin{aligned}
(a_1b_1g_1)(a_2b_2g_2)(a_3b_3g_3) &= [(a_1b_1g_1)\bar{a}_2\bar{b}_2]g_2(a_3b_3g_3) && \text{(i)} \\
&= [[a_3b_3(a_1b_1g_1)]ge(a_3b_3g_3) && \text{normality of } B \\
&= (a_4b_4g_1)g_2(a_3b_3g_3) && \text{(III)} \\
&= (a_4b_4g_1)(g_2\bar{a}_3\bar{b}_3)g_3 && \text{(iii)} \\
&= (a_4b_4g_1)(a_5b_5g_2)g_3 && \text{normality of } B \\
&= [(a_4b_4g_1)\bar{a}_5\bar{b}_5]g_2g_3 && \text{(i)} \\
&= [a_6b_6(a_4b_4g_1)]g_2g_3 && \text{normality of } B \\
&= (a_7b_7g_1)g_2g_3 && \text{(III)} \\
&= a_7\bar{b}_7(g_1g_2g_3). && \text{(ii)}
\end{aligned}$$

Hence

$$(ABg_1)(ABg_2)(ABg_3) \leqslant AB(g_1 g_2 g_3).$$

As in group theory, if the above conditions hold, the cosets may be defined as elements of a *quotient-groupoid* the law of multiplication being $(ABg_1)(ABg_2)(ABg_3) = AB(g_1 g_2 g_3)$.

This quotient-groupoid is denoted by $G/A \times B$.

It is not postulated that every number in $AB(g_1 g_2 g_3)$ can be expressed as a product of elements one in each of the cosets ABg_1, ABg_2, ABg_3. One further associative relation suffices to ensure this property.

THEOREM 29. *If* $ab(cde) = (\bar{a}\bar{b}c)de$, *then every number in the coset* $AB(g_1 g_2 g_3)$ *is the product of one number from each of the cosets* ABg_1, ABg_2, ABg_3.

For $ab(g_1 g_2 g_3) = (\bar{a}\bar{b}g_1)g_2 g_3$, and e_a, e_a' and e_b, e_b' exist in A and B respectively such that $g_2 = e_a e_b g_2$, $g_3 = e_a' e_b' g_3$. Hence

$$ab(g_1 g_2 g_3) = (\bar{a}\bar{b}g_1)(e_a e_b g_2)(e_a' e_b' g_3).$$

THEOREM 30. *If* b, c *are elements of normal sub-groupoids* B, C *respectively and if* (III) *and* (V) *hold, then* $(\overline{abc}) = \bar{a}\bar{b}\bar{c}$.

Denote the various powers of a, b, c by suffixes.

$$\begin{aligned}(a_1 b_1 c_1)(a_2 b_2 c_2)(a_3 b_3 c_3) &= [(a_1 b_1 c_1)a_4 b_4]\, c_4(a_3 b_3 c_3)\\ &= [a_5 b_5(a_1 b_1 c_1)]c_4(a_3 b_3 c_3) = (a_6 b_6 c_1)\, c_4(a_3 b_3 c_3)\\ &= a_7 b_7[c_5 c_6(a_3 b_3 c_3)] = a_7 b_7[(a_3 b_3 c_3)\, c_7 c_8]\\ &= a_7 b_7[a_8 b_8(c_9 c_{10} c_8)] = a_7 b_7(a_8 b_8 c_{11}) = a_9 b_9 c_{12},\end{aligned}$$

i.e. the power of a product is the product of powers.

Assume now that the groupoid is a G_L' in which the coset expansion holds and in which (VI) is true. Then since (III) holds no cross-cut is void.

THEOREM 31. *If* B *and* C *are normal and if* $C < A$, *then*

$$(A,\ [B,\ C]) = [C,\ (A,\ B)] \quad \text{(Dedekind's axiom)}.$$

Every number in the right-hand expression belongs to the left for $(A,\ B) \leqslant (A,\ [B,\ C])$ and $C < A$.

Next, the numbers of the left-hand side are of one of the forms $a = b$, $a = c$, $a = bc_1 c_2$. Of these $a = b$ is in $(A,\ B)$, $a = c$ is in C. In $a = bc_1 c_2$

c_1, c_2 are in A hence, in a G_L', b is also in A and therefore in (A, B). Hence every number in the left-hand expression belongs to the right-hand side and *vice versa*.

It follows that the set of normal sub-groupoids, under the stated conditions, is a *Dedekind structure* to which apply the various laws of isomorphism, the extensions of the theorems of Jordan-Hölder and of the general theories of decomposition*.

In many instances the laws of isomorphism between the normal sub-groupoids which are the elements in a Dedekind structure, the operations being those of union and cross-cut, may be strengthened so as to apply to quotient-groupoids in which the elements are the cosets and the law of operation that of the multiplication of cosets. Only one example will be given.

THEOREM 32.

$$([A, B], [A, C])/A \times (B, C) \cong (C, [A, B])/(B, C) \times (A, C).$$

Let $H = (C, [A, B])$; then (B, C), (A, C) are normal sub-groupoids of H. Hence there is a coset expansion

$$H = \Sigma (A, C)(B, C) h_i. \tag{1}$$

Consider
$$P = \Sigma A(B, C) h_i. \tag{2}$$

The cosets in P are distinct; for, if $A(B, C) h_1 = A(B, C) h_2$, then, since h_1 belongs to the left-coset, a, b exist in A, (B, C) respectively such that $h_1 = abh_2$. Hence a is in $[B, H]$ and also in $[C, H]$; and, since it is also in A, it must be in $\Big(A, \Big[C, (C, [A, B])\Big]\Big) = (A, C)$; therefore, in the equality $h_1 = abh_2$, a is in (A, C), b in (B, C) and by Theorem 18.

$$(A, C)(B, C) h_1 = (A, C)(B, C) h_2.$$

Hence an equality between cosets in (2) implies an equality between the corresponding cosets in (1).

Next, to sum the cosets (2).

$[A, H] = \Big[A, (C, [A, B])\Big] = ([A, B], [A, C])$. Now every H is, from (1), of the form abh, where b is in (B, C) and a is in (A, C) and therefore

* See, for example, the various papers by O. Ore, *Annals of Math.* (2), 36 (1935), 406–437; *Duke Math. Journal*, 2 (1936), 581–596; 3 (1937), 149–174; 4 (1938), 247–269.

in A. Hence every number in H is in some coset of (2). Also, by Theorem 19, Corollary 2, b and h exist in (B, C) and H respectively such that $a = bha$ and, since A is normal, $a = ab'h'$ which is in a coset of (2). Therefore every number in $[A, H]$ is in some coset (2).

Next, every number abh of a coset $A(B, C)h$, where $h = c = [a, b]$ and $b = c$, is of the form ac_1c_2 which is in $[A, C]$ and is also of the form abh, where h is in $[A, B]$. Hence abh is in both $[A, C]$ and $[A, B]$. Therefore $P = ([A, B], (A, C])$, and P and H are simply-isomorphic as regards coset multiplication; and the theorem is proved.

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COEFFICIENT DENSITY AND THE DISTRIBUTION OF SINGULAR POINTS ON THE CIRCLE OF CONVERGENCE

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1. *Introduction.* The most general known result connecting the coefficient density of a power series with the distribution of singular points on the circle of convergence is due to Pólya† and may be stated as follows:

THEOREM. *If the maximal density of the non-zero coefficients is* Δ *then every arc of the circumference which subtends an angle* $2\pi\Delta$ *at the centre contains at least one singular point.*

It follows that *if there is an arc of central angle* α *free from singularities then the maximal density of the non-zero coefficients is not less than* $\alpha/2\pi$.

Evidently the maximal density may be as small as $1/k$ with k singular points on the circle of convergence, provided that the distribution is cyclic, and we cannot expect a sharper result than this unless some restriction is imposed on the nature of the singularities concerned. We suppose, therefore, that the power series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots \tag{1}$$

represents a function having on its circle of convergence, $|z| = 1$, exactly k singularities which are at points isolated with regard to the circumference. Of such singularities Pólya has discussed, in particular, three types‡: *isolated*, *almost isolated*, and *easily approachable* singular points.

† See **4**, 626, Satz IVa. The converse is stated on p. 552.

‡ **5**, § 69. Isolated singularities may be either critical on non-critical points. The function is uniform in the neighbourhood of a non-critical point and such points are therefore either poles or essential singularities. In the case of an isolated critical point the interpolating function for the coefficients is of a similar type to that used for almost isolated singularities and our results concerning the latter apply equally to winding points.

In the case when all k singularities are isolated and non-critical, J. M. Whittaker has proved† that not merely the maximal but also *the lower density of non-vanishing coefficients is at least* $1/k$, and R. Wilson has obtained‡ a similar result when the singularities are of algebraic-logarithmic type. What can we say of the distribution of singular points if the smallest possible density $1/k$ is attained? Must the singularities be specially distributed if the density of non-vanishing coefficients is in any sense less than unity? Even in the case of polar singularities the problem is not trivial. Mandelbrojt§ obtained the appropriate result on the coefficient density but without entering into these matters

We are able to give a complete answer in the cases in which only two singularities are involved provided that they are each of one of the three types mentioned above. If each of the two singularities is either *isolated* or *almost isolated and of finite exponential order*‖ then the density of the non-vanishing coefficients is 1 unless the two singularities are identical in form. In this case the density can take a value $(q-1)/q$, where q is an integer greater than 1, but only when the singular points subtend an angle $2m\pi/p$ at the centre, where s/q is m/p in its lowest terms.

If either of the two singularities is *almost isolated* or *easily approachable* then the upper density of the non-vanishing coefficients is 1 unless the singularities are identical in form and distributed as above, in which case similar results arise but with *upper* density in place of density.

In the case of three or more singular points the method used is different and yields the desired result only for singularities which are *isolated and of finite exponential order.* We prove that the lower density of the non-vanishing coefficients is at least $1/(k-1)$ unless the singularities are virtually identical in form and cyclically distributed. In this case the density exists and is equal to $1/k$ provided that $f(z)$ has no other singularities.

This last restriction is, in reality, an artificial one. The nature of the dissection used in (9), (20), and (29) shows that what we are really concerned with is the density of *small* coefficients, namely such coefficients as satisfy an inequality of the form $|c_n| < e^{-\gamma n}$, where γ is a suitably prescribed positive number. Thus, in the case last-mentioned, the density of small coefficients is exactly $(k-1)/k$ when $f(z)$ has k virtually identical singularities cyclically distributed on its circle of convergence but is otherwise unrestricted.

† 7.

‡ **9**, Theorem 9. The algebraic-logarithmic singularity is an isolated critical point.

§ **2**, 381–3, **3**.

‖ See **5**, 777, for a definition of this singularity.

In the same way, the statement given above for two singularities is true if $f(z)$ is regular outside the circle of convergence and it remains true without this restriction on $f(z)$ provided that we replace the term "vanishing coefficient" by the term "small coefficient".

2. *The problem of dissection.* If the function $f(z)$ of (1) has only isolated essential singularities (or poles) at the points $e^{i\alpha_\nu}$ ($\nu = 1, 2, \ldots, k$), then, as has been shown by J. M. Whittaker†,

$$c_n = \sum_{\nu=1}^{k} e^{-i\alpha_\nu n} G_\nu(n) + c_n^*, \tag{2}$$

where the $G_\nu(z)$ are integral functions of at most order 1, minimum type, and

$$\overline{\lim_{n\to\infty}} |c_n^*|^{1/n} < 1.$$

A similar result in which the singularities of $f(z)$ on $|z| = 1$ may be any of the following types, isolated (critical or non-critical), almost isolated or easily approachable, is necessary to establish the theorems which follow in §§ 3–5.

If a function has an almost isolated singularity at the point $z = 1$, then it is regular in the neighbourhood of that point save possibly at points on a segment of the x-axis terminated on the left by the point $z = 1$. If a function has an easily approachable singularity at the point $z = 1$ then‡ it is regular in the neighbourhood of that point save possibly at points within an angle 2β, where $0 < \beta < \frac{1}{2}\pi$, bisected by that portion of the real axis running outwards from the point $z = 1$.

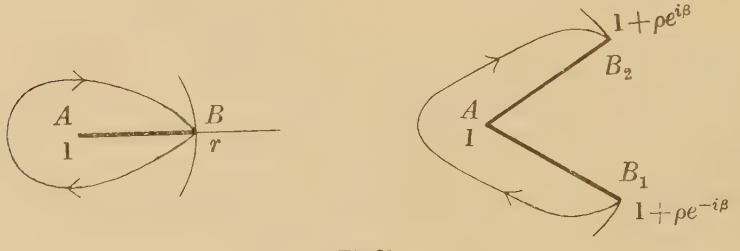


Fig. 1.

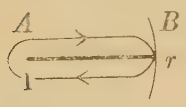
If the circle $|z| = r$, with $r > 1$, cuts the real axis in the one case, and the arms of the angle in the other case, in points in which the function is

† 7, Lemma 3. In particular, see foot-note, 298.
‡ 5, § 69.

regular, then there is no difficulty in separating the corresponding singularities by means of appropriate contours. For in this case if Γ is the loop through B enclosing AB, on the one hand, and is the contour from B_1 to B_2 inside $|z| = r$ enclosing the angle B_1AB_2 on the other, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\sum_{n=0}^{\infty} z^n \int_\Gamma \frac{f(w)\,dw}{w^{n+1}} = \sum_{n=0}^{\infty} G(n)\,z^n \tag{3}$$

is regular inside the region contained between $|z| = r$ and Γ.



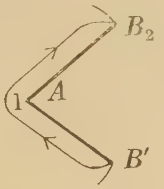


Fig. 2.

The contour Γ can be deformed into a contour arbitrarily near to AB in the one case and to the angle B_1AB_2 in the other so that the function (3) is regular in $|z| = r$ outside AB in the one case and outside the angle B_1AB_2 in the other.

The properties of the functions interpolating the coefficients in (3), namely

$$G(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)\,dw}{w^{z+1}}, \tag{4}$$

are well known†. They are integral functions of order 1, mean type, and in the first case $G(z)$ is of minimum type order 1 at most in the angle $|\theta| \leqslant \frac{1}{2}\pi$ and in the second case $G(z)$ is of minimum type in some angle‡ $|\theta| \leqslant \alpha < \frac{1}{2}\pi$, where $\alpha > 0$. This may be shown by direct evaluation of the integral (4) along contours at a small distance δ from the respective boundaries AB or B_1AB_2 as the case may be.

In dealing with a finite number of singularities at the points $e^{i\theta_\nu}$, each of which is one of the three types, we effect the separation

$$f(z) = \sum_{\nu=1}^{k} \frac{1}{2\pi i}\int_{\Gamma_\nu} \frac{f(w)\,dw}{w-z} + \frac{1}{2\pi i}\int_{C^*} \frac{f(w)\,dw}{w-z}, \tag{5}$$

† See 5, Satz VI and Satz III* together with Satz A*.

‡ The angle α is determined by β and γ.

where the Γ_ν are the contours appropriate to the respective singularities extending round the points $e^{i\alpha_\nu}$ and linked by the intervening parts of the circumference of $|z| = r$, with $r > 1$, on which $f(z)$ is supposed to be regular. The sum of these portions is the contour $C^* = C - \Sigma\Gamma_\nu$, where C is the complete contour. We write for each ν, as in (4),

$$\frac{1}{2\pi i}\int_{\Gamma_\nu}\frac{f(w)\,dw}{w^{z+1}} = \frac{e^{-i\alpha_\nu z}}{2\pi i}\int_{\Gamma_\nu'}\frac{f(ue^{i\alpha_\nu})\,du}{u^{z+1}}, \tag{6}$$

where Γ_ν' is a contour similar to Γ_ν but taken round the point $z = 1$.

This dissection is possible only if $f(z)$ remains bounded† along the paths of approach to the points in which the contours Γ_ν meet the circle $|z| = r$, and in this case also the interpolating functions

$$G_\nu(z) = \frac{1}{2\pi i}\int_{\Gamma_\nu'}\frac{f(ue^{i\alpha_\nu})\,du}{u^{z+1}} \tag{7}$$

are integral functions of order 1, mean type, with qualifying properties‡ corresponding to the kind of singularity involved. Since

$$c_n^* = \frac{1}{2\pi i}\int_{C^*}\frac{f(w)\,dw}{w^{n+1}},$$

it follows that

$$\overline{\lim_{n\to\infty}}\,|c_n^*|^{1/n} < 1;$$

and consequently, from (5), (6), and (7), that

$$c_n = \sum_{\nu=1}^{k} e^{-i\alpha_\nu n}\,G_\nu(n) + c_n^*, \tag{8}$$

as in (2).

It is not, of course, necessary that the points of dissection should lie on a circle. If, associated with each singularity, there are points of dissection outside $|z| = r'(r' > 1)$ and if these can be joined by arcs lying outside $|z| = r'$, inside and on which $f(z)$ is regular and if $f(z)$ is bounded along the different contours as z approaches the points of dissection, then the decomposition (5) and the formulae (7) and (8) remain valid.

If, however, it is impossible to find, in the neighbourhood of one or more of the singularities, points of dissection satisfying the requisite conditions

† **5**, 740.

‡ If the singularity is isolated and non-critical then $G_\nu(z)$ is of minimum type in the whole plane.

then a modification of the treatment is necessary. The method is essentially due to Poincaré†.

In the case of an almost isolated singularity at the point $z = 1$ we can find§ a function $h(z)$ which has an isolated essential singularity at the point $z = r$ but which is regular in the rest of the plane and such that, as z approaches the point $z = r$ along the contour from either side, $f(z)/h(z)$ tends to zero.

In the case of an easily approachable singularity at the point $z = 1$ we use corresponding functions $h_1(z)$, $h_2(z)$ with sole singularities at the points $z = 1+\rho e^{+i\beta}$ respectively and such that $f(z)\{h_1(z)h_2(z)\}^{-1}$ tends to zero as z approaches either of these points along the contour.

We now consider a contour C made up of the portion Γ_1 appropriate to the singularity at the point $e^{i\alpha_1}$ and a portion C^* which consists of arcs lying outside $|z| = r'$ $(r' > 1)$, with deviations inside $|z| = 1$ at the remaining points $e^{i\alpha_\nu}$, so that $f(z)$ is regular inside and on C^* except at the points of dissection. Suppose first that the singularity at $e^{i\alpha_1}$ is almost isolated and that $h(z)$ is a function with an isolated essential singularity at $re^{i\alpha_1}$ such that $f(z)/h(z)$ tends to zero as z approaches $re^{i\alpha_1}$ along the appropriate paths. Then

$$\frac{f(z)}{h(z)} = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{h(w)}\cdot\frac{dw}{w-z} + \frac{1}{2\pi i}\int_{C^*}\frac{f(w)}{h(w)}\cdot\frac{dw}{w-z} = g_1(z)+g_2(z)$$

leads to the required dissection. For, as in Pólya's proof‡, we obtain

LEMMA A. *The function $f_1(z) = h(z)\,g_1(z)$ is regular and uniform everywhere outside the segment $(e^{i\alpha_1}, re^{i\alpha_1})$ and the function $f_2(z) = h(z)\,g_2(z)$ is regular inside C^* and in particular at the point $z = e^{i\alpha_1}$.*

When the point $z = e^{i\alpha_1}$ is easily approachable we get the dissection

$$\frac{f(z)}{h_1(z)\,h_2(z)} = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{h_1(w)\,h_2(w)}\cdot\frac{dw}{w-z} + \frac{1}{2\pi i}\int_{C^*}\frac{f(w)}{h_1(w)\,h_2(w)}\cdot\frac{dw}{w-z}$$

$$= g_1(z)+g_2(z).$$

Here the function $g_1(z)$ is regular save possibly at points on the contour Γ_1 and is certainly singular on this contour near $z = e^{i\alpha_1}$, while the function $g_2(z)$ is regular save possibly at points on the contour C^*, the points of dissection being branch points for both functions.

† See **5**, 740 for further references.

‡ **5**, § 73.

Thus $g_1(z)$ is regular in $|z| = r$ $(r > 1)$, outside the angle formed by the contour Γ_1 and $g_2(z)$ is regular inside the contour formed by closing C^* by an arc of $|z| = r$, assuming that the points of dissection are at a distance r from the origin. It follows that similar results hold for the functions $f_1(z) = h_1(z)\,h_2(z)\,g_1(z)$ and $f_2(z) = h_1(z)\,h_2(z)\,g_2(z)$. Hence we get

LEMMA B. *The function $f_1(z)$ is regular inside some circle $|z| = r$ with $r > 1$, except in the neighbourhood of the point $z = e^{i\alpha_1}$ while the function $f_2(z)$ is regular inside $|z| = r$ except in the neighbourhood of the points $z = e^{i\alpha_\nu}$ $(\nu = 2, 3, \ldots, k)$.*

Hence, in either case, under the decomposition $f(z) = f_1(z) + f_2(z)$, we see that $f_2(z)$ is regular at the point $z = e^{i\alpha_1}$ so that $f(z)$ and $f_1(z)$ have the same singularity at this point.

We now proceed to decompose the function $f_2(z)$ in the same way so as to separate the singularity at the point $z = e^{i\alpha_2}$, after the appropriate modification of the contour about this point as in Fig. 2.

In this way, after a finite number of operations, we decompose $f(z)$ into a set of functions each of which is singular on $|z| = 1$ at only one of the points $e^{i\alpha_\nu}$ and a residual function which is regular inside $|z| = r'$ with $r' > 1$.

The coefficients in the expansions of the separated functions have properties similar to those of the $G_\nu(z)$ in (6) and (7) and since the residual function is regular inside some circle $|z| = r'$ with $r' > 1$ formula (8) holds as before.

The case of an isolated singularity presents no difficulties unless it is critical in which case it is necessary to apply the modified treatment given above, and so we have the following result.

THEOREM A. *If $f(z) = \sum_0^\infty c_n z^n$ has on its circle of convergence $|z| = 1$ only k singularities, each either of isolated, almost isolated, or easily approachable type, then*

$$c_n = \sum_{\nu=1}^{k} e^{-i\alpha_\nu n}\, G_\nu(n) + c_n^*,$$

where the $G_\nu(z)$ are integral functions with the properties appropriate to each singularity and $\sum_0^\infty c_n^* z^n$ is regular inside some circle $|z| = r$ with $r > 1$.

3. *Two isolated essential points.* Suppose that $f(z)$ has on $|z| = 1$ two isolated essential singularities at the points $e^{i\alpha_1}$, $e^{i\alpha_2}$, and is otherwise regular

on $|z|=1$. Then†, as in Theorem A,

$$c_n = e^{-i\alpha_1 n} G_1(n) + e^{-i\alpha_2 n} G_2(n) + c_n^*, \tag{9}$$

where $G_1(z)$ and $G_2(z)$ are integral functions of order 1, minimum type, and

$$\overline{\lim_{n\to\infty}} |c_n^*|^{1/n} < 1$$

so that

$$|c_n^*| < e^{-\gamma n}$$

for some $\gamma > 0$ and $n > n_0$.

Suppose that $|c_n| < e^{-\gamma n}$ for a sequence of integers $\{n_\nu\}$ of positive upper density. Then for any given integer h, sufficiently large,

$$\overline{\lim_{\nu\to\infty}} \frac{\nu}{n_\nu} > \frac{1}{h}.$$

We thus have, with $\gamma > \gamma' > 0$,

$$|e^{-i\alpha_1 n} G_1(n) + e^{-i\alpha_2 n} G_2(n)| < e^{-\gamma' n},$$

i.e., since $|e^{i\alpha_1 n}| = 1$,

$$|G_1(n) + e^{i(\alpha_1-\alpha_2)n} G_2(n)| < e^{-\gamma' n} \tag{10}$$

for a set of integers of upper density greater than $1/h$.

We consider with each integer n the expressions

$$|G_1(n+p) + e^{i(\alpha_1-\alpha_2)(n+p)} G_2(n+p)| \quad (p = 1, 2, \ldots, h).$$

In the sequence of intervals

$$0 < n < h, \quad h < n < 2h, \quad 2h < n < 3h, \quad \ldots$$

it must happen that an increasing sequence $\{n_\mu\}$ of positive upper density exists such that the inequality (10) holds for two values of n at least in every

$$n_\mu h \leqslant n < (n_\mu + 1) h,$$

since otherwise the upper density of integers for which (10) is satisfied would be at most $1/h$.

Since the two selected values can be chosen in only $\frac{1}{2}h(h-1)$ different ways in each interval it follows that there must be a subsequence of $\{n_\mu\}$ of positive upper density such that both

$$|G_1(n) + e^{i(\alpha_1-\alpha_2)n} G_2(n)| < e^{-\gamma' n} \tag{11}$$

† See also **7**, Lemma 3.

and

$$(12) \qquad |G_1(n+p)+e^{i(\alpha_1-\alpha_2)(n+p)}G_2(n+p)| < e^{-\gamma'(n+p)} < e^{-\gamma' n}$$

hold for one of the values $p = 1, 2, \ldots, h-1$ at least.

It follows from (11) and (12) that the inequality with $\gamma' > \gamma'' > 0$

$$(13) \quad |G_1(n)\,G_2(n+p)\,e^{i(\alpha_1-\alpha_2)p} - G_1(n+p)\,G_2(n)| < e^{-\gamma' n}\{|G_2(n)|+|G_2(n+p)|\} < e^{-\gamma'' n}$$

holds for some p in a set of n of positive upper density.

Now the left-hand side of (13) with z in place of n is an integral function of at most order 1, minimum type, and is therefore subject to the theorem, due to Pólya†, given in the lemma below.

LEMMA 1. *If $G(z)$ is an integral function of at most order 1, minimum type, and if γ'' is any given positive number, then the inequality $|G(n)| > e^{-\gamma'' n}$ is satisfied by a sequence of n of density 1.*

It follows that the inequality (13) can be satisfied only for a set of integers of zero density unless

$$(14) \qquad G_1(z)\,G_2(z+p)\,e^{i(\alpha_1-\alpha_2)p} - G_1(z+p)\,G_2(z) \equiv 0.$$

The hypothesis that c_n, given by (9), is small for a sequence of n of upper density greater than $1/h$ thus ensures the existence of a positive integer $p < h$ such that

$$(15) \qquad \frac{G_1(z+p)}{G_2(z+p)} \equiv e^{i(\alpha_1-\alpha_2)p}\,\frac{G_1(z)}{G_2(z)}.$$

We now prove

LEMMA 2. *If (14) holds, then $G_1(z) = \lambda G_2(z)$, where λ is a constant, and $\alpha_1-\alpha_2 = 2m\pi/p$ for some integer m.*

From (15), if $z = a$ is a pole of $G_1(z)/G_2(z)$ then so are the points $z = a+mp$ for every integer m. But $G_1(z)/G_2(z)$ is a meromorphic function of at most order 1, minimum type, and therefore cannot possess so many poles. Hence $G_1(z)/G_2(z)$ is without poles. Similarly it has no zeros. It is thus an integral function of at most order 1 and minimum type without zeros and is therefore a constant.

† 5, Satz. V.

If λ is this constant, then

$$G_1(z) = \lambda G_2(z), \quad G_1(z+p) = \lambda G_2(z+p)$$

and so, from (15),

$$e^{i(a_1-a_2)p} = 1,$$

i.e.

$$a_1 - a_2 = 2m\pi/p.$$

Thus (9) becomes

$$c_n = G_2(n)\{\lambda + e^{i(a_1-a_2)n}\} e^{ia_1 n} + c_n^*. \tag{16}$$

Now $\lambda + e^{i(a_1-a_2)n}$ can take at most p different values and, by Lemma 1,

$$|G_2(n)| > A|c_n^*|$$

for almost all n, however large the given number A may be. Thus, from (16), the $c_n - c_n^*$ can vanish for a set of n of positive upper density only if one of the p values of $\lambda + e^{i(a_1-a_2)n}$ vanishes. For this to happen λ must have one of the p values $-e^{2m\pi i/p}$, in which case the density of the vanishing $c_n - c_n^*$ is exactly $1/q$, where s/q is m/p in its lowest terms.

Hence, if the upper density of the vanishing $c_n - c_n^*$ is positive, it follows that the non-vanishing $c_n - c_n^*$ have a density $(q-1)/q$ and that the function $f(z)$ can be dissected into the sum

$$f(z) \equiv f_1(z) + \omega f_1(z\omega') + f^*(z),$$

where ω and ω' are q-th roots of unity, $f_1(z)$ represents the expansion due to a single isolated essential singularity and $f^*(z)$ is regular inside a circle $|z| = r$ with $r > 1$.

These results lead to

THEOREM 1. *If $f(z)$ has on $|z| = 1$ only two singular points, each isolated and non-critical, the density of small coefficients in the series* (1) *is zero unless the singularities are identical in form and are placed at points which subtend an angle $2m\pi/p$ at the centre, where p is an integer greater than* 1, *and in this case the density exists and is equal to $1/q$ where s/q is m/p in its lowest terms.*

It will be noticed that the singularities are at two of the vertices of a p-sided polygon but that an infinite number of such polygons exists. If that with the smallest number of sides has q sides then the density is equal to $1/q$. In particular when the points are at the opposite ends of a diameter then $q = 2$.

4 *An extension.* If either or both of the singularities is *almost isolated but of finite exponential order* then, although the interpolating function is of mean type, the result of Lemma 1 continues to hold† so that we obtain (14) as before. In this case, however, $G_1(z)$ or $G_2(z)$, or both, are integral functions of order 1, mean type, with the following characteristics†:

$$(17) \qquad \begin{cases} \overline{\lim}_{r\to\infty} \frac{1}{r} \log|G_\nu(re^{i\theta})| = A_\nu \cos\theta & (|\pi-\theta| \leqslant \tfrac{1}{2}\pi), \\ = 0 & (|\theta| \leqslant \tfrac{1}{2}\pi). \end{cases}$$

and

$$\int_1^\infty \log^+|G_\nu(\pm ir)| \frac{dr}{r^2} < +\infty.$$

Hence, by a theorem‡ due to Miss Cartwright, for every $\delta > 0$,

$$(18) \qquad \lim_{r\to\infty} n(r, -\tfrac{1}{2}\pi+\delta, \tfrac{1}{2}\pi-\delta)/r = 0,$$

where§ $n(r, -\alpha, \alpha)$ is the number of zeros of $G_\nu(z)$ in the sector $|z| < r$, $|\arg(-z)| < \alpha$.

From (15), however, it is clear that if $G_1(z)$ has a zero at $z = a$ which does not belong to $G_2(z)$, then it has zeros at all the points $z = a+mp$ and therefore the number of its zeros in the sector $|z| < r$, $|\arg(-z)| < \frac{1}{2}\pi-\delta$ is asymptotically equivalent to r/p, in contradiction to (18). A similar remark applies to $G_2(z)$. Thus $G_1(z)$, $G_2(z)$ have the same zeros and can therefore differ only in a factor $\lambda e^{\mu z}$, so that

$$G_1(z) = \lambda e^{\mu z} G_2(z), \quad G_1(z+p) = \lambda e^{\mu(z+p)} G_2(z+p).$$

This is, however, not possible unless $\mu = 0$, since otherwise, for $|\theta| \leqslant \frac{1}{2}\pi$,

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \log|G_1(re^{i\theta})| = \mu\cos\theta + \overline{\lim_{r\to\infty}} \frac{1}{r} \log|G_2(re^{i\theta})| = \mu\cos\theta$$

in contradiction to (17).

Hence result (16) holds also in this case, and completing the argument as before, we get

THEOREM 2. *If either or both of the singularities is almost isolated but of finite exponential order then the results of Theorem* 1 *continue to hold.*

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† See **5**, Satz X. Here $|f(w)| < e^{\delta r}$, where the integral (4) is taken over the contour in the left-hand diagram in (ii). The evaluation of (4) leads to the required results.

‡ **1**, Theorem III.

§ Note that this statement is for the left-hand plane, while the theorems referred to deal with the right half-plane.

5. *A generalization.* If either or both of the singularities is almost isolated (of non-finite order) or easily approachable, then the corresponding interpolating functions $G_\nu(z)$ are of order 1, mean type in the whole plane, but of order 1, minimum type, at most in the angle $|\theta| \leqslant \frac{1}{2}\pi$ in the one case and in an angle $|\theta| \leqslant \alpha < \frac{1}{2}\pi$, in the other, so that

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \log|G_\nu(re^{i\theta})| = 0 \tag{19}$$

either in $|\theta| \leqslant \frac{1}{2}\pi$ or in $|\theta| \leqslant \alpha < \frac{1}{2}\pi$, uniformly.

In place of Lemma 1 we use two theorems† of Pólya contained in

LEMMA 3. *If $G(z)$ is an integral function of order 1, mean type, satisfying (19) for $|\theta| \leqslant \alpha \leqslant \frac{1}{2}\pi$ and if γ'' is any positive number, then the inequality $|G(n)| > e^{-\gamma'' n}$ holds for a sequence of n of upper density 1.*

It follows from Lemma 3 that the *lower* density of small $G(n)$ is zero. Thus, corresponding to the assumption in § 3 that $|c_n| < e^{-\gamma n}$ for a sequence of n of positive *upper* density, we now make the assumption that $|c_n| < e^{-\gamma n}$ for a sequence of n of positive *lower* density. Developing the argument as in § 3, but with the appropriate modifications, we find that (13) is satisfied for a sequence of n of positive lower density. Since, however, the function on the left-hand side of (14) is of the same nature as the constituent $G_\nu(z)$ in that it is of order 1, minimum type, in an angle bisected by the real axis, it follows from Lemma 3 that (13) cannot be satisfied for a sequence of n of positive lower density unless (14) holds.

From (14) we deduce, as in §§ 3 and 4, that unless $G_1(z)$, $G_2(z)$ have exactly the same zeros, they have each more than $r/(p+1)$ zeros in *every* sector $|z| < r$, $|\arg z| \leqslant \alpha' < \frac{1}{2}\pi$, for which r is sufficiently large.

Now from a theorem ‡ due to Miss Cartwright, in conjunction with (19), it follows that there exists a sequence of r, tending to infinity, such that the number of zeros of $G_\nu(z)$ in the sector $\eta r \leqslant |z| \leqslant r$, $|\arg z| \leqslant \alpha'$, for every $0 < \eta < 1$ is $o(r)$, provided that $\alpha' < \frac{1}{2}\pi$ in the one case and $\alpha' < \alpha$ in the other.

This is in contradiction to the result just obtained and we are left with the alternative that $G_1(z)$, $G_2(z)$ have the same zeros. Since they are both

† **5**, Satz VI for almost isolated and Satz III* together with Satz A* for easily approachable singularities.

‡ **1**, Theorem 2. In particular, 165. Here $A = 0 = B$ for the angle $|\arg z| \leqslant \alpha'$.

of order 1, mean type, in the whole plane, it follows that

$$G_1(z) = \lambda e^{\mu z} G_2(z)$$

and again, as in § 4, we must have $\mu = 0$ in conformity with (19).

The remainder of the proof follows that of § 3 except that density of small coefficients is replaced by lower density and density of non-zero $c_n - c_n^*$ by upper density. This leads to

THEOREM 3. *If $f(z)$ has on $|z| = 1$ only two singular points, each of almost isolated or easily approachable type*†, *then the lower density of the small coefficients in* (1) *is zero unless the singularities are identical in form and are placed at two of the vertices of a regular polygon inscribed in the unit circle.*

Of the polygons which can be so drawn let that with the smallest number of sides be q-sided. Then the upper density of non-vanishing coefficients is at least $(q-1)/q$.

6. *Three singular points: the extreme case.* In dealing with functions having more than two singularities on the circle of convergence the method used in §§ 3–5 fails because the argument which established the identity of $G_1(z)$ and $G_2(z)$ is no longer available. Instead of considering the poles and zeros of $G_1(z)/G_2(z)$ we use a theorem‡ of Whittaker and Wilson on the flat regions of integral functions. Although it is quite possible that this result is true for integral functions of minimum type, order 1, it has so far only been proved for functions of order less than 1. We now consider the case in which $f(z)$ has on $|z| = 1$ three isolated essential singularities of finite exponential order at the points $e^{i\alpha_\nu}$ $(\nu = 1, 2, 3)$. As in (2) we can express the coefficients c_n in the form

$$c_n = \sum_{\nu=1}^{3} e^{-i\alpha_\nu} G_\nu(n) + c_n^*, \tag{20}$$

where the $G_\nu(z)$ are integral functions of order less than 1 and

$$\overline{\lim_{n\to\infty}} |c_n^*|^{1/n} < 1$$

so that

$$|c_n^*|^{1/n} < e^{-\gamma n} \tag{21}$$

for some $\gamma > 0$ and $n > n_0$.

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† An isolated essential singularity is a special case of either type.

‡ 8, Lemma 2. The result is stated for positive integral values of n. The proof, however, is valid for almost all z.

Now suppose that the upper density of small coefficients c_n is greater than $\frac{1}{2}$. Then for a set of n of positive upper density one of the sets c_n, c_{n+1}, c_{n+2}; c_n, c_{n+2}, c_{n+3}; c_n, c_{n+1}, c_{n+3} must have its three terms all small. Otherwise from the four terms $c_n, c_{n+1}, c_{n+2}, c_{n+3}$ at most two would be small (except possibly for a sequence of n of zero density), and the upper density of small coefficients in (1) would then be at most $\frac{1}{2}$, contrary to the hypothesis.

Suppose then that $|c_n|, |c_{n+1}|, |c_{n+2}| < e^{-\gamma n}$ for a sequence of n of positive upper density. Then, from (20) and (21),

$$|e^{-i\alpha_1 n} G_1(n) + e^{-i\alpha_2 n} G_2(n) + e^{-i\alpha_3 n} G_3(n)| < e^{-\gamma' n}$$

for the sequence $\{n_\mu, n_\mu+1, n_\mu+2\}$. Since each $G_\nu(z)$ is of order less than 1 it follows that the modulus of the determinant

$$\begin{vmatrix} e^{-i\alpha_1 n} G_1(n) & e^{-i\alpha_2 n} G_2(n) & e^{-i\alpha_3 n} G_3(n) \\ e^{-i\alpha_1 (n+1)} G_1(n+1) & e^{-i\alpha_2 (n+1)} G_2(n+1) & e^{-i\alpha_3 (n+1)} G_3(n+1) \\ e^{-i\alpha_1 (n+2)} G_1(n+2) & e^{-i\alpha_2 (n+2)} G_2(n+2) & e^{-i\alpha_3 (n+2)} G_3(n+2) \end{vmatrix}$$

is less than $e^{-\gamma'' n}$ for the sequence $\{n_\mu\}$, where $\gamma' > \gamma'' > 0$ and n is sufficiently large, as may be proved by adding the second and third columns to the first and expanding in terms of the elements of the new first column.

On dividing the columns by $e^{-i\alpha_1 n}$, $e^{-i\alpha_2 n}$, $e^{i\alpha_3 n}$, respectively, each of which is of modulus unity, we see that the modulus of the determinant

$$\begin{vmatrix} G_1(n) & G_2(n) & G_3(n) \\ e^{-i\alpha_1} G_1(n+1) & e^{-i\alpha_2} G_2(n+1) & e^{-i\alpha_3} G_3(n+1) \\ e^{-2i\alpha_1} G_1(n+2) & e^{-2i\alpha_2} G_2(n+2) & e^{-2i\alpha_3} G_3(n+2) \end{vmatrix}$$

is less than $e^{-\gamma'' n}$ for the sequence $\{n_\mu\}$.

By the theorem of Pólya of Lemma 1 it follows that

$$\begin{vmatrix} G_1(z) & G_2(z) & G_3(z) \\ e^{-i\alpha_1} G_1(z+1) & e^{-i\alpha_2} G_2(z+1) & e^{-i\alpha_3} G_3(z+1) \\ e^{-2i\alpha_1} G_1(z+2) & e^{-2i\alpha_2} G_2(z+2) & e^{-2i\alpha_3} G_3(z+2) \end{vmatrix} \equiv 0.$$

From the Whittaker-Wilson theorem†

$$G_\nu(z+1)/G_\nu(z) \quad \text{and} \quad G_\nu(z+2)/G_\nu(z)$$

† *Loc. cit.*

each tend to unity for almost all z. It follows that

$$\begin{vmatrix} 1 & 1 & 1 \\ e^{-ia_1} & e^{-ia_2} & e^{-ia_3} \\ e^{-2ia_1} & e^{-2ia_2} & e^{-2ia_3} \end{vmatrix} = 0.$$

The value of this determinant is $\Pi(e^{-ia_2}-e^{-ia_3})$, and so it cannot vanish if the points e^{ia_1}, e^{ia_2}, e^{ia_3} are distinct.

On the supposition that c_n, c_{n+1}, c_{n+3} are small for a sequence of n of positive upper density we get in the same way

$$(22) \qquad \begin{vmatrix} 1 & 1 & 1 \\ e^{-ia_1} & e^{-ia_2} & e^{-ia_3} \\ e^{-3ia_1} & e^{-3ia_2} & e^{-3ia_3} \end{vmatrix} \equiv (e^{-ia_1}+e^{-ia_2}+e^{-ia_3})\,\Pi(e^{-ia_2}-e^{-ia_3}) = 0.$$

The vanishing of this determinant can only arise from the condition $e^{-ia_1}+e^{-ia_2}+e^{-ia_3}=0$, which in turn means that the three points e^{ia_1}, e^{ia_2}, e^{ia_3} are the vertices of an equilateral triangle.

On the assumption that c_n, c_{n+2}, c_{n+3} are small for a sequence of n of positive upper density we get in the same way

$$(22) \qquad \begin{vmatrix} 1 & 1 & 1 \\ e^{-2ia_1} & e^{-2ia_2} & e^{-2ia_3} \\ e^{-3ia_1} & e^{-3ia_2} & e^{-3ia_3} \end{vmatrix} \equiv e^{-3i(a_1+a_2+a_3)}(e^{ia_1}+e^{ia_2}+e^{ia_3})\,\Pi(e^{ia_2}-e^{ia_3}) = 0,$$

so that again the three singularities are at the vertices of an equilateral triangle.

We have thus obtained the result that, if $|c_n| < e^{-\gamma n}$ for a set of n of upper density greater than $\frac{1}{2}$, then

$$(23) \qquad c_n = e^{-ia_1 n}\{G_1(n)+\omega^n G_2(n)+\omega^{2n} G_3(n)\}+c_n^*,$$

so that two of the three inequalities

$$(24) \qquad \begin{cases} |G_1(n)+G_2(n)+G_3(n)| < e^{-\gamma' n}, \\ |G_1(n)+\omega G_2(n)+\omega^2 G_3(n)| < e^{-\gamma' n}, \\ |G_1(n)+\omega^2 G_2(n)+\omega G_3(n)| < e^{-\gamma' n} \end{cases}$$

must be satisfied for a set of n of positive upper density. A repetition of the argument of § 3 involving Pólya's theorem† shows that the two

† See Lemma 1.

corresponding integral functions out of the set

$$G_1(z)+G_2(z)+G_3(z), \quad G_1(z)+\omega G_2(z)+\omega^2 G_3(z), \quad G_1(z)+\omega^2 G_2(z)+\omega G_3(z)$$

must be identically zero and hence that $G_1(z)$, $G_2(z)$, $G_3(z)$ are proportional either to 1, ω, ω^2 or to 1, ω^2, ω or to 1, 1, 1.

Finally we deduce that the lower density of the non-small coefficients of $f(z)$ is at least $\frac{1}{2}$ unless $f(z)$ takes one of the forms

$$f_1(z)+f_1(\omega z)+f_1(\omega^2 z)+f^*(z),$$

$$f_1(z)+\omega f_1(\omega z)+\omega^2 f_1(\omega^2 z)+f^*(z),$$

$$f_1(z)+\omega^2 f_1(\omega z)+\omega f_1(\omega^2 z)+f^*(z),$$

where $f_1(z)$ has a unique singularity at $z=e^{i\alpha_1}$, and $f^*(z)$ is regular inside $|z|=r$ with $r>1$. In each of these cases the density exists and is equal to $\frac{1}{3}$.

These results are embodied in

THEOREM 4. *If $f(z)$ has on its circle of convergence only three singular points, each isolated, non-critical and of finite exponential order, then the upper density of small coefficients is at most $\frac{1}{2}$ unless the singularities are virtually identical in form and occur at the vertices of an equilateral triangle. In this case the density exists and is equal to $\frac{2}{3}$.*

It should be noted that the density of the set of vanishing coefficients of the function $(1-iz)/(1-z^2)+1/(1-iz)$ is exactly $\frac{1}{2}$, so that the upper density $\frac{1}{2}$ of small coefficients occurring in Theorem 4 is actually attained by a function of the prescribed type.

7. *Three singularities: all cases.* Under the conditions of § 6 we now make the assumption that the coefficients c_n are small for a sequence of upper density greater than $1/m$ instead of $\frac{1}{2}$. Thus, out of $2m$ consecutive terms c_n, c_{n+1}, ..., c_{n+2m-1}, at least three must be small for a sequence of n of positive upper density. In consequence of this assumption there must exist integers p, q, with $p<q<2m$, such that c_n, c_{n+p}, c_{n+q} are all three small for a sequence of n of positive upper density. An argument similar to that of § 6 leads, in place of (22), to the result

$$\begin{vmatrix} 1 & 1 & 1 \\ e^{-ip\alpha_1} & e^{-ip\alpha_2} & e^{-ip\alpha_3} \\ e^{-iq\alpha_1} & e^{-q\alpha_2} & e^{-iq\alpha_3} \end{vmatrix} = 0. \tag{25}$$

Elementary methods† suffice to show that the following are necessary and sufficient conditions for the existence of relation (25): *if s is the highest common factor of p and q, then the $\alpha_\mu - \alpha_\nu$ are all of the form $2\pi(u/p+v/s)$, or of the form $2\pi(u/q+v/s)$, or of the form $2\pi\{u/(q-p)+v/s\}$, where u and v are integers or zero.*

It is evident from this that every $e^{i(\alpha_\mu-\alpha_\nu)}$ is either a p-th root of unity or a q-th root of unity or a $(q-p)$-th root of unity. It may, in particular, be an s-th root of unity. If we write ω_2, ω_3 for $e^{i(\alpha_1-\alpha_2)}$ and $e^{i(\alpha_1-\alpha_3)}$, respectively, then the determinant in (25) is of rank 2 unless p, q have a common factor $s > 2$ and each $\alpha_\mu - \alpha_\nu$ is an s-th root of unity. For in this case the necessary conditions $\omega_2{}^p = 1 = \omega_3{}^p$ and $\omega_2{}^q = 1 = \omega_3{}^q$ are fulfilled.

Returning to the fundamental assumption that c_n is small for a sequence of positive upper density, we find, as in (23), that

$$c_n = e^{-i\alpha n}\{G_1(n)+\omega_2{}^n G_2(n)+\omega_3{}^n G_3(n)\}+c_n{}^*,$$

where $|c_n|$ is small for some sequence $\{n_\nu, n_\nu+p, n_\nu+q\}$ of positive upper density. Since, however, $\omega_2{}^n$ and $\omega_3{}^n$ can each take only a finite number of different values, it follows that, for one pair of values ω_2', ω_3' at least of $\omega_2{}^n$, $\omega_3{}^n$, the relations

$$(26) \qquad \begin{cases} |G_1(n)+\omega_2' G_2(n)+\omega_3' G_3(n)| < e^{-\gamma' n}, \\ |G_1(n)+\omega_2{}^p \omega_2' G_2(n)+\omega_3{}^p \omega_3' G_3(n)| < e^{-\gamma' n}, \\ |G_1(n)+\omega_2{}^q \omega_2' G_2(n)+\omega_3{}^q \omega_3' G_3(n)| < e^{-\gamma' n} \end{cases}$$

must be satisfied for a set of n of positive upper density.

Applying Pólya's theorem to (26), we deduce, as in (24), that each of the three integral functions

$$(27) \qquad \begin{cases} G_1(z)+\omega_2' G_2(z)+\omega_3' G_3(z), \quad G_1(z)+\omega_2{}^p \omega_2' G_2(z)+\omega_3{}^p \omega_3' G_3(z), \\ G_1(z)+\omega_2{}^q \omega_2' G_2(z)+\omega_3{}^q \omega_3' G_3(z) \end{cases}$$

is identically zero, and therefore, when the determinant (25) is of rank 2,

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† With relation (25) written in the form

$$\sin\frac{p(\alpha_3-\alpha_1)}{2}\sin\frac{q(\alpha_3-\alpha_1)}{2} = e^{-\frac{1}{2}i(p-q)(\alpha_3-\alpha_2)}\sin\frac{q(\alpha_2-\alpha_1)}{2}\sin\frac{p(\alpha_3-\alpha_1)}{2},$$

every possibility is covered by the condition that either $e^{\frac{1}{2}i(p-q)(\alpha_3-\alpha_1)}$ is real or at least one of the other factors on each side vanish simultaneously. From these alternatives we readily deduce that each $\alpha_\mu - \alpha_\nu$ is an integral multiple of either $2\pi/p$, $2\pi/q$ or $2\pi/(p-q)$.

exactly two of the relations (27) are independent and so $G_1(z)$, $G_2(z)$, $G_3(z)$ are proportional to 1, Ω_2, Ω_3, say.

It follows that the function $f(z)$ is of the form

$$f_1(z)+\Omega_2 f_1(\omega_2 z)+\Omega_3 f_1(\omega_3 z)+f^*(z), \tag{28}$$

where $f_1(z)$ has a unique singularity at $z=e^{i\alpha_1}$, and $f^*(z)$ is regular inside $|z|=r$ with $r>1$.

If the determinant (25) is of rank 1 then p and q have a common factor $s>2$ and each $\alpha_\mu-\alpha_\nu$ is an s-th root of unity, so that relations (27) each reduce to

$$G_1(z)+\omega_2' G_2(z)+\omega_3' G_3(z)=0.$$

Thus

$$c_n-c_n^*=e^{-i\alpha_1 n}\{G_1(n)+\omega_2^n G_2(n)+\omega_3^n G_3(n)\}$$

vanishes for every n for which $\omega_2^n=\omega_2'$, $\omega_3^n=\omega_3'$.

These results are set out in

THEOREM 5. *If $f(z)$ has on its circle of convergence only three singularities, each isolated, non-critical, and of finite exponential order, then the upper density of small coefficients is positive only if one of the singularities is virtually the sum of the other two and if they are placed at three of the vertices of some regular polygon inscribed in the unit circle.*

In accordance with the results following (25) and (26) the density of small coefficients exists in such cases and its value can be determined whenever the $\alpha_\mu-\alpha_\nu$ and the relations between the $G_\nu(z)$ are specified.

8. *Several singularities.* We now consider the case in which $f(z)$ of (1) has k isolated essential singularities of finite exponential order at the points $e^{i\alpha_\nu}$ ($\nu=1, 2, \ldots, k$). Then, as in (2), the coefficients c_n may be written in the form

$$c_n=\sum_{\nu=1}^{k} e^{-i\alpha_\nu} G_\nu(n)+c_n^*, \tag{29}$$

where the $G_\nu(z)$ are integral functions of order less than 1 and

$$\overline{\lim_{n\to\infty}}\,|c_n^*|^{1/n}<1$$

so that

$$|c_n^*|\geqslant e^{-\gamma n}$$

for some $\gamma>0$ and $n>n_0$.

Now suppose that the upper density of small coefficients c_n is greater than $(k-2)/(k-1)$, and consider all possible runs of $2k-2$ consecutive coefficients. In some sequence of positive upper density it must happen that $2k-3$ are small. For otherwise the upper density of small coefficients cannot exceed

$$\frac{2k-4}{2k-2} = \frac{k-2}{k-1}.$$

Of the possible arrangements of $2k-2$ consecutive coefficients of which all but one are small, at least one must occur with a frequency of positive upper density. By an argument like that preceding Theorem 4 we deduce that all the k-th order determinants from the matrix

$$(30) \qquad \begin{bmatrix} 1 & e^{-ia_1} & e^{-2ia_1} & \dots & e^{-(2k-3)ia_1} \\ 1 & e^{-ia_2} & e^{-2ia_2} & \dots & e^{-(2k-3)ia_2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & e^{-ia_k} & e^{-2ia_k} & \dots & e^{-(2k-3)ia_k} \end{bmatrix}$$

which do not involve the column corresponding to the exceptional coefficient must vanish. This column must be in one or other of the two central positions. Otherwise we could take a determinant with k consecutive columns whose modulus would be the same as that of

$$\prod_{\mu<\nu}^{k} (e^{-ia_\mu} - e^{-ia_\nu})$$

and this cannot vanish since the e^{ia_ν} are all different. We may thus suppose that the excluded column is headed by $e^{-i(k-2)a_1}$. (The other possibility gives rise to the same result.) But, as is well known†, the determinants formed from the conjugate of (30) by taking $k+1$ consecutive columns, with one omitted, are the elementary symmetric functions of the e^{ia_ν} multiplied by a factor of modulus unity. There are $k-2$ of these determinants in the conjugate of (30) and, when the column headed by $e^{i(k-2)a_1}$ is omitted, they correspond to the 2nd, 3rd, ..., $(k-1)$-st symmetric functions. Thus, writing

$$\prod_{\nu=1}^{k} (z - e^{ia_\nu}) = z^k - p_1 z^{k-1} + \dots + (-1)^\mu p_\mu z^{k-\mu} + \dots + (-1)^k p_k,$$

we get $p_\nu = 0$ for $\nu = 2, 3, \dots, k-1$.

† See, for example, **6**, 99, Ex. 10.

Now $p_{k-1}=0$ leads to the result $\sum_{\nu=1}^{k} e^{-\alpha_\nu}=0$, which is equivalent to $\sum_{\nu=1}^{k} e^{i\alpha_\nu}=0$ or $p_1=0$. If we write $p_k=e^{i\theta}$, it follows that the $e^{i\alpha_\nu}$ are the roots of $z^k=e^{i\theta}$, and it is easy to see that, if this result holds the matrix (30) with a central column missing is singular. Further, its rank is $k-1$ since otherwise the $e^{i\alpha_\nu}$ would not all be distinct†.

The remaining arguments which lead to Theorem 4 clearly apply to our present hypotheses and give rise to

THEOREM 6. *If $f(z)$ has on its circle of convergence exactly k singular points, each isolated, non-critical and of finite exponential order, then the upper density of small coefficients is at most $(k-2)/(k-1)$ unless the singularities occur at the vertices of a k-sided regular polygon inscribed in the circle of convergence. In this case the density exists and is equal to $(k-1)/k$ and the singularities are virtually identical in form.*

It is easy to construct a rational function with simple poles at k of the $(k+1)$-st roots of unity for which the density of vanishing coefficients is $(k-1)/(k+1)$, as in the example following Theorem 4, and it seems likely that $(k-1)/(k+1)$ is the best value for the upper density of small coefficients occurring in Theorem 6, except, of course, in the extreme case when the density exists and is equal to $(k-1)/k$.

The methods used in § 6 and continued in § 8 show that, if $f(z)$ of (1) has on its circle of convergence only k singular points, each isolated, non-critical and of finite exponential order, then no run of k consecutive coefficients $c_n, c_{n+1}, \ldots, c_{n+k-1}$ can vanish for a set of n of positive upper density. Thus, on writing $f(z)\equiv\Sigma c_{\lambda_n} z^{\lambda_n}$, we see that $\lambda_{n+1}-\lambda_{n-k}\leqslant k$ *save possibly for a sequence of density zero.*

In this way we obtain Whittaker's result‡ (under the restriction that the singularities are of finite exponential order) in the more precise form used by Mandelbrojt‡ and Wilson‡ when dealing with polar and algebraic-logarithmic singularities respectively.

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† The following operations on the matrix (30) are non-singular and do not affect its rank. Multiply the elements in the respective rows by $e^{i(k-2)\alpha_\nu}$ $(\nu=1, 2, \ldots, k)$, and add up the elements of each column to form a new bottom row. Noting that $p_1, p_2, \ldots, p_{k-1}$ are all zero and that the column in (30) headed by $e^{-i(k-2)\alpha_1}$ is missing, we get a row of zeros. Hence the matrix (30) is singular for these values of the $e^{i\alpha_\nu}$. If its rank were less than $(k-1)$ then the determinant formed from the elements in the right-hand $k-1$ columns and the top $(k-1)$ rows would vanish. Its modulus, however, is equal to $\left|\prod_{\mu<\nu}^{k-1}(e^{-i\alpha_\mu}-e^{-i\alpha_\nu})\right|$, and it therefore cannot vanish.

‡ See § 1 for references.

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BIRATIONAL TRANSFORMATIONS WITH A FUNDAMENTAL SURFACE

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Introduction.

The present paper is a sequel to two previous ones (**6**, **8**) which investigated the behaviour of the canonical systems of an algebraic variety under birational transformations with isolated fundamental points or fundamental curves of general character. In this paper we discuss the analogous problem for birational transformations with a non-singular fundamental surface. The results obtained suggest empirical generalizations to birational transformations with a fundamental variety of any number of dimensions (the transformation being of general character in the sense explained on p. 146 of **8**), but an attempt to generalize the arguments of this paper fails to give a complete proof without the assumption of an unproved criterion of equivalence. I have, however, thought it worth while to record these empirical results, and explain briefly the obstacle in the way of their proof.

1. *General features of birational transformations with a fundamental surface.*

We consider a birational transformation T between two non-singular algebraic varieties V' and V, each of d dimensions. There is no loss of generality in supposing that this transformation is free from fundamental elements on one of the varieties, say V (see, *e.g.* **8**, p. 145). We suppose further that the only fundamental points on V' are the points of an irreducible non-singular surface F', and that this is a fundamental variety of general character in the sense of **8** (p. 146). This means that the neighbourhood of each point P' of F' (without exception) corresponds to a variety $\phi_{P'}$ of $d-3$ dimensions on V, and that the locus of $\phi_{P'}$ as P' describes

F' is an irreducible variety Φ of $d-1$ dimensions. No two of the generating varieties $\phi_{P'}$, $\phi_{Q'}$ corresponding to different points on F' can intersect, for their point of intersection would be a fundamental point of $\mathbf{T}^{-1}$ on V, and we have supposed that there are no such points.

A differential argument exactly parallel to that on p. 147 of 8 proves that the points of $\phi_{P'}$ are in (1, 1) correspondence, without exception, with the ∞^{d-3} solids which pass through the tangent plane to F' at P' and lie in the tangent $[d]$ to V' at P'. Thus $\phi_{P'}$ is of invariant order one, and if in fact the prime sections of V correspond to the varieties of a linear system having F' as a simple base-surface each $\phi_{P'}$ is a linear space of $d-3$ dimensions.

The properties of the transformation $\mathbf{T}$ depend essentially upon the properties of F' considered as a locus on V', and not on the particular linear system on V' of which V is the projective model. For suppose that $\mathbf{T}^*$ is another birational transformation having F' as fundamental surface of general character (and no other fundamental elements), in which V' corresponds to a variety V^*, and the neighbourhood of F' to Φ^*. Then, the correspondence between V and V^* is birational *without exception*. For, if a point P of V corresponds to more than one point on V^*, it must lie on Φ, and in a definite generating variety $\phi_{P'}$ of this, and its corresponding points on V^* must lie in $\phi^*_{P'}$, the variety on Φ^* which maps the neighbourhood of P'. On the other hand, since the points of $\phi_{P'}$ and $\phi^*_{P'}$ are each in unexceptional (1, 1) correspondence with the solids lying in the tangent $[d]$ to V' at P' and containing the tangent plane to F' at P', $\phi_{P'}$ and $\phi^*_{P'}$ are themselves in birational correspondence without exception, and each point of $\phi_{P'}$ corresponds to one point, and only one, on $\phi^*_{P'}$. Thus the transformation between V and V^* has no fundamental points on V, and similarly it has none on V^*.

Accordingly we shall lose nothing in generality if we suppose, to fix the ideas, that $\phi_{P'}$ is actually a linear space of $d-3$ dimensions, for in a birational transformation (such as that between V and V^*) free from fundamental elements, canonical systems are absolutely invariant.

It is convenient at this stage to introduce some conventions in regard to notation, designed to make the formulae which follow as elegant in appearance as possible, and to save repetition of explanations. An accented capital letter (other than F' and V') will unless otherwise stated denote a V_{d-1} on V' which does not pass through F', and the same letter without the accent will denote the corresponding variety on V. The curve in which A' meets F' will be denoted by a'. In particular, X' will denote a canonical V_{d-1} on V', and x' its intersection with F'. An accented lower-case letter will represent a curve on F', and if the corresponding capital

letter does not appear in the context, this curve will not necessarily be supposed to be the intersection of F' with a V_{d-1} on V'. In particular, x_1' will denote a canonical curve on F'. Accented Greek letters (lower case) will denote sets of points on F'. We shall denote by $\phi_{c'}$ the locus of $[d-3]$'s of Φ which correspond to points of the curve c', and by $\phi_{\gamma'}$ the set of $[d-3]$'s of Φ corresponding to points of the set γ'.

If $|A'|$ is a linear system on V', whose general member does not contain F', we denote by $|A_1'|$ the sub-system of $|A'|$ which contains F' as a base. If A_1 is the variety on V corresponding to A_1', we evidently have the equivalence (on V)

$$A \equiv A_1 + \Phi. \tag{1}$$

The intersection of A and Φ is evidently $\phi_{a'}$. The intersection of A_1 and Φ is a variety of $d-2$ dimensions meeting the general $[d-3]$ of Φ in a $[d-4]$. Such a V_{d-2} may be called a *directrix variety* on Φ; more generally a V_h on Φ which meets the general $[d-3]$ in an $[h-2]$ may be called a directrix V_h. From (1) we see that (on Φ)

$$-(\Phi^2) \equiv (A_1\Phi) - \phi_{a'}; \tag{2}$$

this shows that $-(\Phi^2)$ is a directrix V_{d-2} on Φ. Hence so also are all its characteristic systems of dimension $h \geqslant 2$. That is, if we write

$$\Phi_h \equiv (\Phi^{d-h}),$$

$(-)^{d-h-1}\Phi_h$ is a directrix variety on Φ, for $h \geqslant 2$.

If c_1' and c_2' are two curves belonging to the same linear system on F', then $\phi_{c_1'}$ and $\phi_{c_2'}$ belong to the same linear system on Φ; and the converse is also true, for any directrix surface D_2 on Φ is birationally equivalent to F', containing possibly a finite number of lines which correspond to points on F', and the equivalent varieties $\phi_{c_1'}$ and $\phi_{c_2'}$ cut out equivalent curves c_1 and c_2 on D_2 which must therefore correspond to equivalent curves c_1' and c_2' on F'. In the same way, if γ_1' and γ_2' are sets of points on F', then $\phi_{\gamma_1'}$ and $\phi_{\gamma_2'}$ are equivalent on Φ if, and only if, $\gamma_1' \equiv \gamma_2'$ on F'. Further, if $h \geqslant 2$, then $(\phi_{c_1'} . \Phi_h)_\Phi$ and $(\phi_{c_2'} . \Phi_h)_\Phi$ are equivalent on Φ if, and only if, $c_1' \equiv c_2'$ on F'; this follows by considering the directrix surface in which Φ_h meets a directrix variety of dimension $d-h+1$ and the intersections of this surface with the two loci mentioned. Similarly, for $h \geqslant 2$, $(\phi_{\gamma_1'} . \Phi_h)_\Phi$ and $(\phi_{\gamma_2'} . \Phi_h)_\Phi$ are equivalent on Φ if, and only if, $\gamma_1' \equiv \gamma_2'$ on F'.

We apply these considerations to prove that any variety, Γ_h of h dimension on Φ may be expressed in the form

$$\Gamma_h \equiv a\Phi_h + (\phi_{c'} . \Phi_{h+1})_\Phi + (\phi_{\gamma'} . \Phi_{h+2})_\Phi, \tag{3}$$

and that, for $h \geqslant 2$, a is unique and c', γ' are determined to within an equivalence on F'.

The proof depends on the following lemma.

LEMMA. *Let S be a variety of k dimensions, generated by ∞^{k-1} lines of which one passes through a general point of S. Let Γ_{k-1} be a variety of $k-1$ dimensions on S, meeting a general line in n points, and let S_{k-1} denote a prime section of S. Then*

$$\Gamma_{k-1} \equiv n S_{k-1} + R_{k-1}, \tag{4}$$

where R_{k-1} is generated by lines of S.

We may suppose without loss of generality that Γ_{k-1} is effective. If $n = 0$ then Γ_{k-1} is itself generated by lines. We suppose therefore that $n > 0$. Let S lie in $[r]$ and let $(x_0, x_1, \ldots, x_r)$ be homogeneous coordinates in $[r]$. Let S_{k-1} be the section of S by $x_0 = 0$. Since only one line l of S passes through the general point P of S_{k-1}, the equations of l are determined rationally in terms of the coordinates $(x_1, \ldots, x_r)$ of P. Thus the symmetric functions of the coordinates of the intersections of l and Γ_{k-1} are rational functions of $x_1, \ldots, x_r$. Hence the rational function on l which has poles at the intersections of l and Γ_{k-1} and an n-ple zero at P, and which takes the value unity at its intersection with a fixed prime, is a rational function of $x_0, x_1, \ldots, x_r$. As P varies on S_{k-1} we have therefore a rational function on S, becoming infinite along Γ_{k-1} and zero to the n-th order on S_{k-1}. The loci of other poles and zeros of this function, if any, cannot contain points of the general line l of S, and therefore are composed entirely of generators of S. This proves the lemma.

We turn now to the proof of (3). We may suppose Γ_h to be effective.

Suppose first that $h = d-2$. When $d = 4$, Φ is generated by lines; if Γ meets each line in n points, then, by the lemma,

$$\Gamma_2 \equiv n S_2 + R_2,$$

where R_2 is the locus of ∞^1 lines of Φ, and S_2 is a prime section. Further, since $-(\Phi^2) \equiv -\Phi_2$ is a directrix we have, also by the lemma,

$$-\Phi_2 \equiv S_2 + R_2^*,$$

where R_2^* is the locus of ∞^1 lines of Φ. Hence

$$\Gamma_2 \equiv -n\Phi_2 + (R_2 - nR_2^*)$$

and $R_2 - nR_2^*$, being ruled, is a locus $\phi_{c'}$ for some c' on F'. This proves the theorem (apart from uniqueness) when $d = 4$, $h = 2$. When $d > 4$,

and $h = d-2$, the proof follows by induction, by considering the section of Γ_h by a prime and applying the criterion of linear equivalence on Φ (see, *e.g.* Zariski, **12**, p. 98).

Suppose next that $h < d-2$, and that the theorem has been proved for varieties of dimension $h+1$ on Φ. We suppose that Γ_h meets the general $[d-3]$ of Φ in a variety Γ_{h-2} of $h-2$ dimensions. (The case in which Γ_h lies entirely on a variety $\phi_{c'}$ will be considered below.) We take a directrix surface D_2 on Φ, and join each point of D_2 to the points of Γ_h which lie in the same generating space of Φ. We thus get a variety Γ_{h+1} generated by ∞^h lines, one through a general point, and on this variety Γ_h meets each generator in one point. The section of Γ_{h+1} by $-(\Phi^2)$ has the same property. Hence, by two applications of the lemma,

$$\Gamma_h \equiv (-\Phi^2 \,.\, \Gamma_{h+1}) + R_h,$$

where R_h is generated by ∞^{h-1} of the lines of Γ_{h+1}. Further, since Γ_{h+1} meets the general $[d-3]$ of Φ in the cone projecting Γ_{h-2} from a point, and $-(\Phi^2)$ meets this cone in a prime section, equivalent therefore to Γ_{h-2} on the cone, R_h has no intersection with the general $[d-3]$ of Φ and therefore lies entirely on some $\phi_{c'}$. Hence, since by the inductive hypothesis Γ_{h+1} is given by an expression of the form (3), with $h+1$ instead of h, our theorem will be proved if we can prove that it is true for varieties Γ_h on $\phi_{c'}$. A repetition of the same argument shows that it will be true for varieties Γ_h on $\phi_{c'}$ if it is true for varieties Γ_h on $\phi_{\gamma'}$. But $\phi_{\gamma'}$ is a finite set of linear spaces, and any V_h^n in $[d-3]$ is equivalent to an $[h]$ of the $[d-3]$, counted n times. The possibility of the expression (3) is thus established in all cases.

The uniqueness of the expression is easily proved (for $h \geqslant 2$). Suppose, in fact, that, on Φ,

$$a\Phi_h + (\phi_{c'} \,.\, \Phi_{h+1})_\Phi + (\phi_{\gamma'} \,.\, \Phi_{h+2})_\Phi \equiv 0.$$

Then the section of this variety by a general $[d-3]$ of Φ is equivalent to zero on the $[d-3]$. But this is simply an $[h-2]$ counted a times. Thus $a = 0$. If we now cut by a variety $\phi_{c_1'}$, we get

$$(\phi_{(c' \,.\, c_1')} \,.\, \Phi_{h+1})_\Phi \equiv 0$$

and hence $(c' \,.\, c_1') \equiv 0$ on F', which implies $c' \equiv 0$, since this must hold for all c_1'. Finally, if $a = 0$, $c' \equiv 0$, we have $(\phi_{\gamma'} \,.\, \Phi_{h+2})_\Phi \equiv 0$, and hence $\gamma' \equiv 0$.

The uniqueness of the expression (3) no longer holds when $h < 2$. In fact, since Φ_1 is a curve, it can only contain points of a single infinity of the generating $[d-3]$'s of Φ, forming a variety $\phi_{c'}$. We may therefore write

$$\Phi_1 \equiv (\phi_{\nu'} \,.\, \Phi_2)_\Phi - (\phi_{\theta'} \,.\, \Phi_3)_\Phi. \tag{5}$$

The curve v' and set of points θ' are determined to within an equivalence on F' (as may be seen by applying the argument used to deduce the uniqueness of (3) when $h \geqslant 2$). We prove that, in the notation of 5,

$$v' \equiv Y_1[F',\ V']; \quad \theta' = Y_0[F',\ V']. \tag{6}$$

We recall that the covariant system $Y_h[F',\ V']$ is defined by

$$Y_h[F',\ V'] \equiv \sum_{i=0}^{2-h} (-)^i (Y_{h+i}[F'] \,.\, X_{d-i}[V']), \tag{7}$$

where $\{X_{d-i}[V']\}$ is the canonical system of dimension $d-i$ on V', and where $Y_k[F']$ is determined in terms of the canonical systems on F' by the recurrence formulae

$$\sum_{i=k}^{2} (-)^i (Y_i[F'] \,.\, X_{2+k-i}[F'])_{F'} \equiv 0. \tag{8}$$

Thus
$$Y_1[F',\ V'] \equiv x_1' - x', \tag{9}$$

and
$$Y_0[F',\ V'] \equiv (x_1'^2)_{F'} - X_0[F'] - (x' x_1')_{F'} + (F' \,.\, X_{d-2}[V']). \tag{10}$$

Let $\mathfrak{A}'$ be the fourfold of intersection of $d-4$ varieties A_1', belonging to a linear system $|A_1'|$ having F' as base. We have supposed that F' is free from improper double points. Hence, by formula (7.03) of 5 (*cf.* also 9),

$$(F'^2)_{\mathfrak{A}} \equiv Y_0[F',\ \mathfrak{A}'].$$

Now $\mathfrak{A}'$ is the complete intersection of $d-4$ varieties of the system $|A_1'|$, and therefore we have, using the method of § 4 of 7 [*cf.* equations (8) and (10) of that paper],

$$X_3[\mathfrak{A}'] \equiv \left(\mathfrak{A}' \,.\, (d-4) A_1' + X_{d-1}[V']\right), \tag{11}$$

and

$$X_2[\mathfrak{A}'] \equiv \left(\mathfrak{A}' \,.\, \binom{d-3}{2} A_1'^2 + (d-4) A' X_{d-1}[V'] + X_{d-2}[V']\right), \tag{12}$$

so that, since $A_1' \equiv A'$ on V',

$$(F^2)_{\mathfrak{A}'} \equiv Y_0[F',\ V'] - (d-4)(A' \,.\, Y_1[F',\ V']) + \binom{d-3}{2} (F' \,.\, A'^2). \tag{13}$$

The intersection of $\mathfrak{A}'$ with two further varieties of $|A_1'|$ breaks up into F' and a residual surface G'. The curve g' of intersection of F' and G' is given [7, formula (24)] by

$$g' \equiv 2(F' A') + (F' \,.\, X_3[\mathfrak{A}'])_{\mathfrak{A}'} - X_1[F'],$$

that is, in the notation introduced above,

$$g' \equiv (d-2)\,a' + x' - x_1', \tag{14}$$

by means of (11).

To G' there corresponds a surface G on V which meets Φ in a curve having one point in each generating space of Φ corresponding to points of g'. Therefore, since $(-)^{d-1}\Phi_2$ is a directrix surface on Φ,

$$g \equiv (-)^{d-1}(\phi_{g'} \,.\, \Phi_2)_\Phi + (\phi_{\gamma'}' \Phi_3)_\Phi, \tag{15}$$

where γ' is a set of points on g'. On the other hand,

$$g \equiv (A_1^{d-2}\Phi) \equiv (\{A-\Phi\}^{d-2}\Phi)$$

$$\equiv (-)^{d-2}\left[\Phi_1 - (d-2)(A\Phi_2) + \binom{d-2}{2}(A^2\Phi_3)\right], \text{ since } (A^r\Phi) \equiv 0 \text{ for } r > 2,$$

$$\equiv (-)^{d-2}\left[\left(\phi_{v'} - (d-2)\,\phi_{a'} \,.\, \Phi_2\right)_\Phi + \left(-\phi_{g'} + \binom{d-2}{2}\phi_{(a'^2)} \,.\, \Phi_3\right)_\Phi\right],$$

by means of (5). Comparing this with (15) we see that

$$(d-2)a' - v' \equiv g'$$

and so, by (14), and (9)

$$v' \equiv x_1' - x' \equiv Y_1[F', V'];$$

this is the first of the relations (6).

The intersection of $\mathfrak{A}'$ with three further varieties A_1' is a curve c' which meets F' in a set of points a'. By (34) of 7,

$$a' \equiv (F' \,.\, 3\mathfrak{A}'\,A'^2 + 3A'\,X_3[\mathfrak{A}'] + \{X_3[\mathfrak{A}']\}^2)_{\mathfrak{A}'} - (X_1[F'] \,.\, 3\mathfrak{A}'A' + 2X_3[\mathfrak{A}'])_{\mathfrak{A}'}$$
$$- (F'^2)_{\mathfrak{A}'} + (\{X_1[F']\}^2)_{F'},$$

and by (11), (12), and (13) this reduces to

$$a' \equiv (x')^2 + (d-1)(a'\,x') + \binom{d-1}{2}(a'^2) - 2(x_1'\,x')$$
$$- (d-1)(x_1'\,a') + (x_1'^2) - Y_0[F', V']. \tag{16}$$

The curve c corresponding to c' meets Φ in a set of points α, where

$$\alpha \equiv (-)^{d-1}(\phi_{a'} \,.\, \Phi_2)_\Phi. \tag{17}$$

On the other hand,

$$\alpha \equiv (c\Phi) \equiv (A_1^{d-1}\Phi) \equiv (\{A-\Phi\}^{d-1} \,.\, \Phi)$$

$$\equiv (-)^{d-1}\left[\Phi_0 - (d-1)(A\Phi_1) + \binom{d-1}{2}(A^2\Phi_2)\right],$$

and so, using (5) and the expression obtained for Φ_0 by considering the intersection of Φ_1 and (Φ^2), we get

$$a \equiv (-)^{d-1}\left(\phi_{(v'^2)} - \phi_{\theta'} - (d-1)\,\phi_{(a'\,v')} + \binom{d-1}{2} \phi_{(a'^2)} \cdot \Phi_2 \right)_{\Phi}.$$

Comparing this with (17) we see that

$$a' \equiv (v'^2) - \theta' - (d-1)(a'\,v') + \binom{d-1}{2}(a'^2).$$

Hence from (16), using the expression for v' already found, we have

$$\theta' \equiv Y_0[F',\ V'];$$

this is the second relation (6).

2. *Transformation of the canonical systems on a* V_4.

Our chief concern in the present paper is the behaviour of the canonical systems under the birational transformation considered in the previous section. Such a transformation necessarily requires that $d \geqslant 4$. We consider the case $d = 4$ in the present section, and generalize the results by induction in the following one.

We begin by making the following remark. Since the transformation between V and V' is regular except at points of Φ and F', the jacobian variety of any linear system $|S|$ on V differs from the transform of the jacobian variety of the corresponding system $|S'|$ on V' at most by a variety lying on Φ. From the relation (*cf.* **2**, **4**) which defines the canonical systems on V, namely

$$X_h[V] \equiv J_h[S] - \sum_{r=1}^{h+2} \binom{h+2}{r} X_h[S^r], \tag{18}$$

where $J_h[S]$ is the jacobian variety of a linear ∞^{h+1} system $|S|$, it follows that the same is true for the canonical system; the variety

$$X_h[V] - \mathrm{T}\{X_h[V']\}$$

is equivalent on V to some variety lying on Φ. It is this variety that it is our task to find.

We suppose now, for the rest of this section, that $d = 4$.

Consider first a pencil $|S'|$ of threefolds on V' whose base surface meets F' only in a finite set of points. If $|S'|$ is chosen with sufficient generality

its jacobian set $J_0[S']$ consists of a finite number of points, none of which lie on F'. The pencil $|S'|$ cuts on F' a pencil of curves $|s'|$, and a finite number of the members of the pencil will touch F', the points of contact being the points of the jacobian set $J_0[s']$ of the pencil $|s'|$.

If P' is a point of the jacobian set $J_0[s']$, and S_0' is the variety of the pencil $|S'|$ which passes through P' (and touches F' there), then the corresponding variety S_0 on V touches Φ along the line $\phi_{P'}$ which corresponds to P'. The neighbourhood of P' on V' which lies in the tangent solid to S_0' at P' corresponds to a point P of $\phi_{P'}$, and S_0 has an actual double point at P. For a simple curve branch b on V, having origin at P and not touching Φ there, corresponds to a simple branch b' on V' having origin at P', whose tangent lies in the tangent solid to S_0' at P' but not in the tangent plane to F'. As a variable member of $|S'|$ approaches S_0', therefore, two of its intersections with b' tend to P', and hence two intersections of S with b tend to P as S tends to S_0. Hence, as the tangent to b at P is arbitrary (provided that it is not also tangent to Φ), P must be a double point of S_0.

It is easily seen that the points P arising in this way are the only double points of members of $|S|$ which lie on Φ. Remembering that any two points on a line $\phi_{P'}$ are equivalent on Φ we thus have

$$J_0[S] \equiv \mathbf{T}\{J_0[S']\} - (\phi_{J_0[s']} \cdot \Phi_2)_\Phi, \tag{19}$$

since $-\Phi_2$ is a directrix surface.

The transformation between S' and S has s' as a fundamental curve. Hence, by (28) of **8**,

$$X_0[S] \equiv \mathbf{T}\{X_0[S']\} - (\phi_{X_0[s']} \cdot \Phi_2)_\Phi. \tag{20}$$

The transformation between (S'^2) and (S^2) has the set of points (s'^2) as fundamental points, and so, by (12) of **6**,

$$X_0[S^2] \equiv \mathbf{T}\{X_0[S'^2]\} - (\phi_{(s'^2)} \cdot \Phi_2)_\Phi. \tag{21}$$

From (18), (19), (20), and (21) it follows that

$$X_0[V] \equiv \mathbf{T}\{X_0[V']\} - (\phi_{X_0[F']} \cdot \Phi_2)_\Phi. \tag{22}$$

Using (5) we can write this in the alternative form

$$X_0[V] \equiv \mathbf{T}\{X_0[V']\} - (\phi_{x_1'} \cdot \Phi_1)_\Phi - (\phi_{X_0[F']-(v'x_1')} \cdot \Phi_2)_\Phi, \tag{23}$$

which though less elegant than (22) is more convenient for the generalization that we have in view in the next section.

Next, let $|S'|$ be a net of threefolds on V', whose base curve does not meet F', and let $|s'|$ be the net of curves cut out by $|S'|$ on F'. Let $J_1[S']$ and $J_1[s']$ be the jacobian curves of the nets $|S'|$ and $|s'|$ respectively. We see, by using an argument similar to that used in discussing the jacobian set of a pencil, that the jacobian curve of the corresponding net $|S|$ on V breaks up into the transform of $J_1[S']$, together with a curve lying on $\phi_{J_1[s']}$ which meets each generator of this surface in one point. Hence

$$J_1[S] \equiv \mathrm{T}\{J_1[S']\} - (\phi_{J_1[s']} . \Phi_2)_\Phi - \phi_{\rho'}, \tag{24}$$

where ρ' is a set of points on $J_1[s']$.

To determine ρ', let A_1 be a threefold on V corresponding to a threefold A_1', on V' which passes through F'. The set of points in which A_1 meets the portion of $J_1[S]$ which lies on Φ is

$$\begin{aligned} -\left(A_1 . (\phi_{J_1[s']} . \Phi_2)_\Phi + \phi_{\rho'}\right) &\equiv -\left(A - \Phi . (\phi_{J_1[s']} . \Phi_2)_\Phi + \phi_{\rho'}\right) \\ &\equiv (\phi_{J_1[s']} . \Phi_1)_\Phi - (\phi_{(a' . J_1[s'])-\rho'} . \Phi_2)_\Phi \\ &\equiv (\phi_{\{(v'-a' . J_1[s'])+\rho'\}} . \Phi_2)_\Phi, \quad \text{by (5).} \end{aligned}$$

The generators of Φ which pass through these points correspond to the points β' of $J_1[s']$ at which the corresponding variety S' touches A_1', and these points are therefore the intersection of F' with the jacobian curve of the net cut by $|S'|$ on A_1'. Hence, by (18),

$$\begin{aligned} \beta' &\equiv \left(F' . X_1[A_1'] + 3X_1[S' A_1'] + 3(S'^2 A_1')\right)_{A_1'} \\ &\equiv \{X_0[F'] - (F' . X_1[F'])_{A_1'}\} + 3\{X_0[s'] - (s' . F')_{A_1'}\} + 3(s'^2), \end{aligned}$$

by the rule of adjunction. But

$$(F'^2)_{A_1'} \equiv X_1[F'] - (F' . X_2[A_1'])_{A_1'} \equiv x_1' - x' - a',$$

and hence

$$\begin{aligned} \beta' &\equiv X_0[F'] + 3X_0[s'] + 3[s'^2] - (x_1' + 3s' . x_1' - x' - a') \\ &\equiv X_0[F'] + 3X_0[s'] + 3(s'^2) - (J_1[s'] . x_1' - x' - a'). \end{aligned}$$

On the other hand, since $-\Phi_2$ is a directrix surface, we must have

$$\beta' \equiv -(v' - a' . J_1[s']) - \rho',$$

and hence, using (6) and (9), we have

$$\rho' \equiv -\{X_0[F'] + 3X_0[s'] + 3(s'^2)\}. \tag{25}$$

By (26) of 8 and (12) of 6,

$$X_1[S] \equiv \mathbf{T}\{X_1[S']\} - (\phi_{s'} \cdot \Phi_2)_\Phi + \phi_{X_0[s']},$$

$$X_1[S^2] \equiv \mathbf{T}\{X_1[S'^2]\} + \phi_{(s'^2)},$$

and hence, using (18), (24), and (25), we have

$$X_1[V] \equiv \mathbf{T}\{X_1[V']\} - (\phi_{x_1'} \cdot \Phi_2)_\Phi + \phi_{X_0[F']}. \tag{26}$$

By (5), this can also be written in the form

$$X_1[V] \equiv \mathbf{T}\{X_1[V']\} - \Phi_1 - (\phi_{x_1'-v'} \cdot \Phi_1)_\Phi + \phi_{\{X_0[F']-\theta'\}}. \tag{27}$$

To obtain the canonical surfaces on V we consider a general web $|S'|$ on V'. As before, the jacobian surface $J_2[S]$ of the corresponding web on V consists of the transform of the jacobian surface of $|S'|$ and a surface lying on Φ, and this surface is a simple directrix surface, since one member of $|S'|$ touches F' at any assigned point P', and by repeating an argument used previously we see that the corresponding member of $|S|$ touches Φ along the line $\phi_{P'}$ and has a double point P at the point of this line which corresponds to the tangent solid to S' at P'. Hence

$$J_2[S] \equiv \mathbf{T}\{J_2[S']\} - \Phi_2 - \phi_{c'}, \tag{28}$$

where c' is a curve on F'. To determine c' we consider the section by A_1. The portion of $J_2[S]$ lying on Φ meets A_1 in a curve given by

$$-(A_1 \cdot \Phi_2 + \phi_{c'}) \equiv -(A - \Phi \cdot \Phi_2 + \phi_{c'}) \equiv \Phi_1 + (\phi_{c'-a'} \cdot \Phi_2)_\Phi - \phi_{(a'c')}$$

$$\equiv (\phi_{v'+c'-a'} \cdot \Phi_2)_\Phi - \phi_{\{(a'c')+\theta'\}},$$

by (5). This curve thus lies on the scroll corresponding to the points of the curve $a'-c'-v'$ on F', and is a simple directrix curve on the scroll. Hence $a'-c'-v'$ is the curve on F' at which an S' which touches F' also touches A', so that

$$a'-c'-v' \equiv (F' \cdot X_2[A_1'] + 4A_1' S')_{A_1'} \equiv a'+x'+4s',$$

and so

$$c' \equiv -(x_1'+4s'). \tag{29}$$

By (27) of 8,

$$X_2[S] \equiv \mathbf{T}\{X_2[S']\} + \phi_{s'},$$

and hence, from (18), (28), and (29),

$$X_2[V] \equiv \mathbf{T}\{X_2[V']\} - \Phi_2 + \phi_{x_1'} \tag{30}$$

Finally, if $|S'|$ is a general linear ∞^4 system on V', a pencil of varieties of $|S'|$ touches F' at any assigned point P', and the tangent solids to the

varieties of the pencil form a pencil lying in the tangent [4] to V' at P', and having the tangent plane to F' at P' as base. Each point of $\phi_{P'}$ is thus a node of one member of the corresponding pencil on V. Hence

$$J_3[S] \equiv \mathbf{T}\{J_3[S']\} + \Phi$$

and so

$$X_3[V] \equiv \mathbf{T}\{X_3[V']\} + \Phi. \tag{31}$$

We have thus determined the relations between the canonical systems on V and V'. It may be verified, by direct calculation of the numerical invariants of V, and by substitution in formula (37) of 3, that the arithmetic genus of V is equal to that of V'.

3. *Transformation of the canonical systems on a V_d.*

Let us now consider the general case. Since $X_h[V]$ and $\mathbf{T}\{X_h[V']\}$ differ by a variety lying on Φ, we may, by (3), write

$$X_{d-1}[V] \equiv \mathbf{T}\{X_{d-1}[V']\} - a_{d-1}\Phi, \tag{32}$$

$$X_{d-2}[V] \equiv \mathbf{T}\{X_{d-2}[V']\} - a_{d-2}\Phi_{d-2} - \phi_{c'_{d-2}}, \tag{33}$$

$$X_h[V] \equiv \mathbf{T}\{X_h[V']\} - a_h\Phi_h - (\phi_{c_h'} . \Phi_{h+1})_\Phi - (\phi_{a_h'} . \dot{\Phi}_{h+2})_\Phi \quad (h < d-2). \tag{34}$$

(*Note.* The suffix in c_h', etc. is merely a distinguishing mark; c_h' in all cases represents a curve on F', a_h a number, and so on.)

We proceed to prove that the unknown coefficients on the right of (32), (33), (34) are given by

$$a_h = \lambda_{h-1}^{d-2}, \tag{35}$$

$$c_h' \equiv \lambda_h^{d-2} x_1' - \lambda_{h-1}^{d-3} v', \tag{36}$$

$$a_h' \equiv \lambda_{h+1}^{d-2} X_0[F'] - \lambda_h^{d-3}(v' x_1') + \lambda_{h-1}^{d-4}\theta', \tag{37}$$

where v' and θ' are defined by (6), and where

$$\lambda_q^p = \binom{p}{q} - \binom{p}{q-1}. \tag{38}$$

These formulae hold when $d = 4$, by (23), (27), (30), and (31). We assume them to be valid for varieties of dimension smaller than d, and deduce their validity for dimension d.

We consider first the cases in which $h > 0$. Let A_1 be a V_{d-1} on V, transform of a variety A_1' on V' which passes through F'. We proceed to obtain two expressions for $(A_1 . X_h[V])$, and prove our formulae by comparing the two expressions. As usual, A' denotes a variety on V', equivalent to A_1', but not containing F'.

Suppose first that $h < d-2$. Then $(X_h[V'] \,.\, F') \equiv 0$, and hence $(\mathbf{T}\{X_h[V']\} \,.\, \Phi) \equiv 0$.

Hence, from (34),

$$(A_1 \,.\, X_h[V])$$

$$\equiv \left(A - \Phi \,.\, \mathbf{T}\{X_h[V']\} - a_h \Phi_h - (\phi_{c_h'} \,.\, \Phi_{h+1})_\Phi - (\phi_{a_h'} \,.\, \Phi_{h+2})_\Phi \right)$$

$$\equiv \mathbf{T}\{(A' \,.\, X_h[V'])\} + a_h \Phi_{h-1} + (\phi_{c_h'} - a_h \phi_{a'} \,.\, \Phi_h)_\Phi + (\phi_{a_h'} - \phi_{(a'c_h')} \,.\, \Phi_{h+1})_\Phi. \quad (39)$$

Next, let $h = d-2$. Let us *define* α'_{d-2} by (37); we are at liberty to do this, since $X_{d-2}[V]$ is now given by (33), which does not determine any set of points on F'. We see that

$$\alpha'_{d-2} \equiv -X_0[F'] + (v' x_1') - \theta'$$

$$\equiv -(F' \,.\, X_{d-2}[V']).$$

Hence $$(\mathbf{T}\{X_{d-2}[V']\} \,.\, \Phi) \equiv -\phi_{\alpha'_{d-2}}.$$

Accordingly, from (33),

$$(A_1 \,.\, X_{d-2}[V])$$

$$\equiv (A - \Phi \,.\, \mathbf{T}\{X_{d-2}[V']\} - a_{d-2} \Phi_{d-2} - \phi_{c'_{d-2}})$$

$$\equiv \mathbf{T}\{(A' \,.\, X_{d-2}[V'])\} + a_{d-2} \Phi_{d-3} + (\phi_{c'_{d-2}} - a_{d-2} \phi_{a'} \,.\, \Phi_{d-2})_\Phi + \{\phi_{a'_{d-2}} - \phi_{(a'c'_{d-2})}\},$$

which is precisely of the form (39).

Similarly, when $h = d-1$, we define α'_{d-1} and c'_{d-1} formally by (37) and (36). We find that $\alpha'_{d-1} \equiv 0$, $c'_{d-1} \equiv -x'$; and hence

$$(\mathbf{T}\{X_h[V']\} \,.\, \Phi) \equiv -\phi_{c'_{d-1}}.$$

Thus

$$(A_1 \,.\, X_{d-1}[V]) \equiv (A - \Phi \,.\, \mathbf{T}\{X_{d-1}[V']\} - a_{d-1} \Phi)$$

$$\equiv \mathbf{T}\{(A' X_{d-1}[V'])\} + a_{d-1} \Phi_{d-2} + \{\phi_{c'_{d-1}} - a_{d-1} \phi_{a'}\},$$

which is also of the form (39), since the last term in (39) has no meaning when $h = d-1$. Thus, with the conventional definitions made, we may suppose that (39) holds for all $h > 0$.

On the other hand, by the rule of adjunction,

$$(A_1 \,.\, X_h[V]) \equiv X_{h-1}[A_1] - (A_1 \,.\, X_h[A_1]). \quad (40)$$

We apply the inductive hypothesis to express $X_h[A_1]$ in terms of $\mathbf{T}\{X_h[A_1']\}$. We have, if $h < d-3$,

$$X_h[A_1] \equiv \mathbf{T}\{X_h[A_1']\} - \lambda_{h-1}^{d-3}(A_1 \Phi_{h+1})_\Phi - (\phi_{b_h'} A_1 \Phi_{h+2})_\Phi - (\phi_{\beta_h'} \,.\, A_1 \Phi_{h+3})_\Phi, \quad (41)$$

where
$$b_h' \equiv \lambda_h^{d-3} x_1' - \lambda_{h-1}^{d-4} Y_1[F', A']$$
$$\equiv \lambda_h^{d-3} x_1' - \lambda_{h-1}^{d-4}(v' - a'), \tag{42}$$

and

$$\beta_h' \equiv \lambda_{h+1}^{d-3} X_0[F'] - \lambda_h^{d-4}(Y_1[F', A'].x_1') + \lambda_{h-1}^{d-5} Y_0[F', A']$$
$$\equiv \lambda_{h+1}^{d-3} X_0[F'] - \lambda_h^{d-4}(v' - a'.x_1') + \lambda_{h-1}^{d-5}\{\theta' - (a'x_1') + (a'x') + (a'^2)\}, \tag{43}$$

with a similar expression for $X_{h-1}[A_1]$.

If $h < d-3$, $X_h[A_1']$ does not meet F'. Hence $\mathbf{T}\{X_h[A_1']\} \equiv \mathbf{T}\{X_h[A']\}$, and so (41) becomes

$$X_h[A_1] \equiv \mathbf{T}\{X_h[A']\} - \lambda_{h-1}^{d-3}(A - \Phi.\Phi_{h+1}) - (\phi_{b_h'}.\{A-\Phi\}\Phi_{h+2})_\Phi$$
$$-(\phi_{\beta_h'}.\{A-\Phi\}\Phi_{h+3})_\Phi. \tag{44}$$

We proceed to show that (44) is valid also when $h \geqslant d-3$.

Direct substitution in (42) and (43), using (6), shows that

$$\beta_{d-3}' \equiv -(F'.X_{d-2}[V'] + A'X_{d-1}[V'] + A'^2]) \equiv -(F'.X_{d-3}[A_1'])_{A_1'},$$

$$\beta_{d-2}' \equiv 0, \quad \beta_{d-1}' \equiv 0,$$

$$b_{d-2}' \equiv -(F'.X_{d-1}[V'] + A') \equiv -(F'.X_{d-2}[A_1'])_{A_1'}, \quad b_{d-1}' \equiv 0.$$

Thus
$$\mathbf{T}\{X_{d-3}[A_1']\} \equiv \mathbf{T}\{X_{d-3}[A']\} + \phi_{\beta_{d-3}'},$$

and so

$$X_{d-3}[A_1] \equiv \mathbf{T}\{X_{d-3}[A_1']\} - \lambda_{d-4}^{d-3}(A_1\Phi_{d-2}) - (\phi_{b_{d-3}'}A_1\Phi)_\Phi$$
$$\equiv \mathbf{T}\{X_{d-3}[A']\} - \lambda_{d-4}^{d-3}(A_1\Phi_{d-2}) - (\phi_{b_{d-3}'}A_1\Phi)_\Phi + \phi_{\beta_{d-3}'}.$$

Hence, since $(\phi_{\beta_h'}.\{A-\Phi\}\Phi_{h+3}) \equiv -(\phi_{\beta_h'}\Phi_{h+2}) \equiv -\phi_{\beta_{d-3}'}$ when $h = d-3$, we see that (44) holds when $h = d-3$.

Further,
$$\mathbf{T}\{X_{d-2}[A_1']\} \equiv \mathbf{T}\{X_{d-2}[A']\} + \phi_{b_{d-2}'}$$

and so
$$X_{d-2}[A_1] \equiv \mathbf{T}\{X_{d-2}[A_1']\} - \lambda_{d-3}^{d-3}(A_1\Phi)$$
$$\equiv \mathbf{T}\{X_{d-2}[A_1']\} - \lambda_{d-3}^{d-3}(A_1\Phi) + \phi_{b_{d-2}'}.$$

But

$$(\phi_{b_h'}.\{A-\Phi\}\Phi_{h+2})_\Phi \equiv (\phi_{(a'b_h')}.\Phi_{h+2})_\Phi - (\phi_{b_h'}.\Phi_{h+1})_\Phi$$
$$\equiv -\phi_{b_{d-2}}, \quad \text{when } h = d-2.$$

Thus, since $\beta_{d-2}' \equiv 0$, (44) holds when $h = d-2$.

Finally, when $h = d-1$, (44) states that

$$A_1 \equiv \mathrm{T}\{A'\} - \lambda_{d-2}^{d-3}(A - \Phi \,.\, \Phi_d) \equiv \mathrm{T}\{A'\} + \lambda_{d-2}^{d-3}\Phi \equiv \mathrm{T}\{A'\} - \Phi,$$

which is true.

Hence (44) holds for all $h > 0$.

We now substitute from (44) in (40), and replace A_1 by $A - \Phi$. On reduction we obtain the relation

$$\begin{aligned}(A_1 \,.\, X_h[V]) \equiv \mathrm{T}\{(A' \,.\, X_h[V']\} \\ + (\lambda_{h-2}^{d-3} + \lambda_{h-1}^{d-3})\,\Phi_{h-1} + (\phi_{b_{h-1}'} + \phi_{b_h'} - \{\lambda_{h-2}^{d-3} + 2\lambda_{h-1}^{d-3}\}\,\phi_{a'} \,.\, \Phi_h)_\Phi \\ + (\phi_{\beta_{h-1}'} + \phi_{\beta_h'} - \phi_{(a'\,b_{h-1}')} - 2\phi_{(a'\,b_h')} + \lambda_{h-1}^{d-3}\,\phi_{(a'^2)} \,.\, \Phi_{h+1})_\Phi. \quad (45)\end{aligned}$$

The left-hand sides of (39) and (45) are identical. Hence the right-hand sides are equivalent. The first terms on the right of both expressions are identical. The remaining terms represent varieties of dimension $h-1$ on Φ. If $h > 2$, the expression of any V_{h-1} on Φ in the form

$$a\Phi_{h-1} + (\phi_{c'} \,.\, \Phi_h)_\Phi + (\phi_{\gamma'} \,.\, \Phi_{h+1})_\Phi$$

has been seen to be unique. Hence for $h > 2$ we may equate coefficients of Φ_{h-1}, Φ_h and Φ_{h+1} in the right-hand sides of (39) and (45). We therefore get, using (42) and (43),

$$\begin{aligned} a_h &= \lambda_{h-2}^{d-3} + \lambda_{h-1}^{d-3} = \lambda_{h-1}^{d-2}, \\ c_h' &\equiv a_h\,a' + b_{h-1}' + b_h' - (\lambda_{h-2}^{d-3} + 2\lambda_{h-1}^{d-3})\,a' \\ &\equiv -\lambda_{h-1}^{d-3}\,a' + \lambda_h^{d-2}\,x_1' - \lambda_{h-1}^{d-3}(v' - a') \\ &\equiv \lambda_h^{d-2}\,x_1' - \lambda_{h-1}^{d-3}\,v', \end{aligned}$$

and

$$\begin{aligned} a_h' &\equiv (a'\,c_h') + \beta_{h-1}' + \beta_h' - (a' \,.\, b_{h-1}' + 2b_h') + \lambda_{h-1}^{d-3}(a'^2) \\ &\equiv \lambda_{h+1}^{d-2}\,X_0[F'] - \lambda_h^{d-3}(v'\,x_1') + \lambda_{h-1}^{d-4}\,\theta', \end{aligned}$$

after a little reduction. This establishes the validity of the expressions (35), (36), (37), when $h > 2$.

Next suppose that $h = 2$. In this case we substitute for Φ_1 in (39) and (45) the expression (5). We can then equate coefficients of Φ_2 and Φ_3. Equating the coefficients of Φ_2 we find that

$$(a_2 - \lambda_1^{d-2})(v' - a') + (c_2' - \lambda_2^{d-2}\,x_1' - \lambda_1^{d-3}\,v') \equiv 0 \quad (\text{on } F').$$

Now this must be true independently of a', since $|A'|$ is an arbitrary V_{d-1} on V'. Hence we have

$$a_2 = \lambda_1^{d-1}, \quad c_2' \equiv \lambda_2^{d-2}\,x_1' - \lambda_1^{d-3}\,v'$$

and it then follows that

$$a_2' \equiv \lambda_3^{d-2} X_0[F'] - \lambda_2^{d-3}(v' x_1') + \lambda_1^{d-4}\theta'.$$

Thus (35), (36), (37) hold when $h = 2$.

When $h = 1$ the expression (34) is not unique. In fact, on account of (5), we may select a_1 arbitrarily. We choose a_1 to satisfy (35). From (5) it follows that

$$\begin{aligned}\Phi_0 &\equiv (\Phi_1 . \Phi)\\ &\equiv (\phi_{v'} . \Phi_1)_\Phi - (\phi_{\theta'} . \Phi_2)_\Phi\\ &\equiv (\phi_{(v'^2)} - \phi_{\theta'} . \Phi_2)_\Phi.\end{aligned}$$

Substituting for Φ_0 and Φ_1 in (39) and (45), and equating coefficients of Φ_2, we obtain [since a_1 by definition satisfies (35)], after a little reduction,

$$(v' . c_1' - b_1' - b_0' + a') + \{a_1' - (a' c_1') - \beta_0' - \beta_1' + (a' b_0') + 2(a' b_1') - (a'^2)\} \equiv 0.$$

Now, from (43),

$$\begin{aligned}\beta_0' + \beta_1' &\equiv \lambda_2^{d-2} X_0[F'] - \lambda_1^{d-3}(v' - a' . x_1') + \lambda_0^{d-4}\{\theta' - (a' x_1') + (a' x') + (a'^2)\}\\ &\equiv \lambda_2^{d-2} X_0[F'] - \lambda_1^{d-3}(v' x_1') + \lambda_0^{d-4}\theta' + (a' . b_1'), \quad \text{since } v' \equiv x_1' - x'.\end{aligned}$$

Further,

$$b_0' + b_1' \equiv \lambda_1^{d-2} x_1' - v' + a'.$$

Hence

$$\begin{aligned}&(v' . c_1' - \lambda_1^{d-2} x_1' + v')\\ &\quad + \{a_1' - \lambda_2^{d-2} X_0[F'] + \lambda_1^{d-3}(v' x_1') - \lambda_0^{d-4}\theta'\} - (a' . c_1' - \lambda_1^{d-2} x_1' + v') \equiv 0,\end{aligned}$$

and, since this is to hold independently of a', we must have

$$\begin{aligned}c_1' &\equiv \lambda_1^{d-2} x_1' - v' \equiv \lambda_1^{d-2} x_1' - \lambda_0^{d-3} v',\\ a_1' &\equiv \lambda_2^{d-2} X_0[F'] - \lambda_1^{d-3}(v' x_1') + \lambda_0^{d-4}\theta',\end{aligned}$$

which proves the validity of (36) and (37) when $h = 1$.

Hence, if (35), (36), (37) are valid on V_{d-1}, they are also valid on V_d if $h > 0$. To complete the induction we have therefore only the case $h = 0$ to consider.

When $h = 0$, a little calculation shows that we have to prove that

$$X_0[V] \equiv \mathrm{T}\{X_h[V']\} - (d-3)(\phi_{X_0[F']} . \Phi_2)_\Phi. \tag{46}$$

We use the method of § 2. Let $|S'|$ be a general pencil of V_{d-1}'s on V', cutting out a pencil $|s'|$ of curves on F'. The argument used in § 2 applies without modification to show that the jacobian variety of the corresponding

pencil $|S|$ on V consists (i) of the transform of the jacobian set of $|S'|$ and (ii) of a set of $[d-4]$'s on Φ, one lying in each $[d-3]$ $\phi_{P'}$ for which P' belongs to the jacobian set of the pencil $|s'|$.

We prove below that, if a particular member of a pencil $|S|$ on V possesses a double variety Γ of dimension h, then the equivalence of Γ in the virtual jacobian set of $|S|$ is $X_0[\Gamma]$. Assuming this result for the moment, and observing that the canonical series of a $[d-4]$ consists of $(-)^{d-4}(d-3)$ points, we see that

$$J_0[S] \equiv \mathbf{T}\{J_0[S']\} - (d-3)(\phi_{J_0[s']} \cdot \Phi_2)_\Phi.$$

Also, by (28) of **8** and (12) of **6**,

$$X_0[S] \equiv \mathbf{T}\{X_0[S']\} - (d-3)(\phi_{X_0[s']} \cdot \Phi_2)_\Phi,$$

$$X_0[S^2] \equiv \mathbf{T}\{x_0[S'^2]\} - (d-3)(\phi_{X_0[s'^2]} \cdot \Phi)_\Phi.$$

From these relations and (18), (46) follows at once. Our proof by induction is therefore complete.

We close this section by observing that using relations (56) and (61) of **3** we can verify the invariance of the arithmetic genus, when $d = 5$ or 6.

4. *A lemma on the canonical series.*

In this section we prove the following result*, which was assumed in the previous section:

Let $|S|$ be an irreducible linear pencil of V_{d-1}'s on a non-singular V_d having a non-singular base (S^2). Suppose that the jacobian locus of $|S|$ consists (i) *of a finite set of points δ, not lying on (S^2), each point of which is an ordinary double point of the member of $|S|$ which passes through it, and* (ii) *of a variety Γ of h dimensions consisting of a finite number of irreducible components, each of h dimensions, none of which lie on (S^2), and such that each irreducible component Γ_0 of Γ is double on one variety S_0 of $|S|$ and is such that the tangent cone to S_0 at the general point of Γ_0 is irreducible. Then the equivalence of Γ in the virtual jacobian set of $|S|$ is $X_0[\Gamma]$, that is,*

$$X_0[V] \equiv \delta + X_0[\Gamma] - 2X_0[S] - X_0[S^2]. \tag{47}$$

When $h = 0$ there is nothing to prove. We assume the result to be true when Γ has dimension less than h, and prove the theorem by induction. The method consists in following through the steps of the ordinary proof

* The special case $d = 2$, $h = 1$ of this result is included in the discussion of the Zeuthen-Segre invariant of a surface given by Jung, **1**, p. 313.

of the invariance of the canonical series on V_d, and noticing the modifications produced by the existence of Γ. We therefore follow closely the arguments of § 2 of **2**.

We take a general pencil $|T|$ on V_d, having a jacobian set δ_1.

The jacobian of the pencil cut out by $|S|$ on a general member of $|T|$ consists of the set σ of points where T is touched by a member of $|S|$, and of the intersection of T with Γ. Hence, by the inductive hypothesis,

$$X_0[T] \equiv \sigma + X_0[T\Gamma] - 2X_0[ST] - X_0[S^2 T]. \tag{48}$$

Similarly,

$$X_0[T^2] \equiv \rho + X_0[T^2\Gamma] - 2X_0[ST^2] - X_0[S^2 T^2], \tag{49}$$

where ρ is the set of points in which (T^2) is touched by a member of $|S|$ (at a simple point of the corresponding S).

Let J denote the curve of proper contacts of members of $|S|$ and $|T|$, and let τ denote the group of points of J at which the contact is stationary.

Then
$$(JT) \equiv \sigma + \rho, \tag{50}$$

and, by considering the linear series cut on J by $|T|$, we get

$$X_0[J] + 2(JT) \equiv \delta_1 + 2\rho + \rho_1 + \tau, \tag{51}$$

where ρ_1 is the set of contacts of members of $|T|$ with (S^2).

Hence

$$\begin{aligned} X_0[V_d] &\equiv \delta_1 - 2X_0[T] - X_0[T^2] \\ &\equiv X_0[J] - \tau - \rho - \rho_1 - 2X_0[T\Gamma] - X_0[T^2\Gamma] + 4X_0[ST] \\ &\qquad + 2X_0[S^2 T] + 2X_0[ST^2] + X_0[S^2 T^2] \end{aligned} \tag{52}$$

by (48), (49), (50), and (51).

Let σ_1 denote the set of double points cut by $|T|$ on a general S. Then

$$X_0[S] \equiv \sigma_1 - 2X_0[ST] - X_0[ST^2], \tag{53}$$

$$X_0[S^2] \equiv \rho_1 - 2X_0[S^2 T] - X_0[S^2 T^2], \tag{54}$$

$$(JS) \equiv \sigma_1 + \rho_1. \tag{55}$$

If ζ denotes the points of contact of Γ with members of the pencil $|T|$, the set ζ lies on J and belongs to the jacobian set of the linear series cut out by $|S|$ on J. Hence

$$X_0[J] + 2[JS] \equiv \delta + \zeta + 2\rho_1 + \rho + \tau. \tag{56}$$

From (53), (54), (55), and (56), it follows that

$$\delta-2X_0[S]-X_0[S^2]\equiv X_0[J]-\rho-\rho_1-\tau-\zeta+4X_0[ST]+2X_0[S^2T]$$
$$+2X_0[ST^2]+X_0[S^2T^2]. \quad (57)$$

Comparing (52) and (57), we see that

$$X_0[V_d]\equiv\delta-2X_0[S]-X_0[S^2]+\zeta-2X_0[T\Gamma]-X_0[T^2\Gamma]$$
$$\equiv\delta+X_0[\Gamma]-2X_0[S]-X_0[S^2],$$

which proves (47).

5. *An empirical extension of the foregoing results.*

The results of § 3 suggest an empirical extension to birational transformations with a fundamental variety F' of k dimensions, of general character. To the neighbourhoods of points of F' then correspond ∞^k varieties (which, as in § 1, may be taken without loss of generality to be linear spaces) of $d-k-1$ dimensions, generating a locus Φ of $d-1$ dimensions. The result suggested by the results of § 3 is that the canonical systems of V and V' are related by the equivalences

$$X_h[V]\equiv\mathbf{T}\{X_h[V']\}-\sum_{r=t}^{k}(\phi_{\Gamma'_{(r,h)}}.\Phi_{h+k-r})_\Phi, \quad (58)$$

where
$$t=\max(0,\ h+k-d+1) \quad (59)$$

and
$$\Gamma'_{(r,h)}\equiv\sum_{s=0}^{k-r}(-)^s\lambda_{h+1-r-s}^{d-k-s}(Y_{k-s}[F',\ V'].X_{r+s}[F'])_{F'}. \quad (60)$$

Here $\phi_{\Gamma'}$ denotes the locus of the $[d-k-1]$'s of Φ corresponding to points of Γ' on F', and Φ_l is the characteristic system of Φ on V, of dimension l.

We are unable to present a complete proof of this result. We may, however, outline the evidence provided for it, and explain where the obstacle to the proof arises.

The results hold when $k=2$, as we have proved in §§ 2 and 3. They also agree with the results of **6** and **8** when $k=0,\ 1$.

Let us suppose for the moment that the results have been established for birational transformations with fundamental varieties of dimension less than k. The methods of § 2 of the present paper can be generalized to prove that the proposed results are valid for transformations of a V_{k+2} with a fundamental V_k. We seek to generalize the argument of § 3. The case in which $h=0$ is disposed of exactly as in § 3, making use of the lemma of § 4. When $h>0$ we assume the result proposed to hold for transformations of V_{d-1}, and seek to obtain two expressions for $(A_1.X_h[V])$,

where A_1 is the transformation of a variety A_1' on V', of $d-1$ dimensions, passing through F'. After some rather complicated algebraical work we obtain expressions analogous to (39) and (45), from which we can deduce that

$$\sum_{r=t}^{h} (\phi_{\Gamma'_{(r,h)}} - \phi_{\Delta'_{(r,h)}} . A_1 \Phi_{h+k-r})_\Phi \equiv 0, \tag{61}$$

where $\Gamma'_{(r,h)}$ is the unknown coefficient appearing on the right of (58), and $\Delta'_{(r,h)}$ stands for the right-hand side of (60).

Now, by an obvious extension of the results of § 1, any variety of h dimensions on Φ can be expressed in the form

$$a\Phi_h + \sum_{s=1}^{t} (\phi_{\Gamma'_{k-s}} . \Phi_{h+s})_\Phi,$$

where $t = \min(k, d-1-h)$, and this expression is unique provided that $h \geqslant k$. Hence, putting $A_1 \equiv A - \Phi$ in (61), we can deduce, as in § 3, that $\Gamma'_{(r,h)} \equiv \Delta'_{(r,h)}$ provided that $h > k$, and by using the criterion of linear equivalence on F' we can deal, as in § 3, with the cases $h = k$, $k-1$. When we attempt to use the same method to deal with the cases $1 \leqslant h \leqslant k-1$, we find that we can deduce the theorem we require provided that, from the relation

$$(A' . \Gamma'_{(r,h)} - \Delta'_{(r,h)}) \equiv 0 \quad (\text{on } F')$$

(with A' arbitrary) we can deduce that

$$\Gamma'_{(r,h)} - \Delta'_{(r,h)} \equiv 0.$$

However, as no proof of this is at present available, the proof of the main theorem remains incomplete.

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MULTIPLE INTERSECTIONS ON AN ALGEBRAIC V_4

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In a recent paper (1) about the geometry on an algebraic V_4, J. A. Todd has investigated the virtual characters of the intersections of threefolds which pass simply through a given curve or surface. His results are here extended to the case where the passage through the given curve or surface is multiple; and additional topics, arising from this multiplicity, are discussed. I employ a technique which includes the degeneration method of B. Segre (2), who dealt with such problems in regard to a V_3; and I especially acknowledge the use of ideas contained in recent papers by Todd.

Usually we denote a set of points, a curve, a surface, a threefold respectively by small Greek, capital German, capital Latin, capital Clarendon letters; lower case Latin letters denote numbers. V_n denotes a variety of dimension n and Σ indicates summation. $\mathbf{K}$ is a canonical threefold on V_4, K a canonical surface, $\mathfrak{K}$ a canonical curve, and κ a canonical set (3). If $\mathbf{A}$ is a threefold on V_4, we denote by $K_{\mathbf{A}}$, $\mathfrak{K}_{\mathbf{A}}$, $\kappa_{\mathbf{A}}$ respectively, a canonical surface, a canonical curve, and a canonical set on $\mathbf{A}$; in a similar way we use $\mathfrak{K}_F$, κ_F in regard to a surface F, and $\kappa_{\mathfrak{C}}$ in regard to a curve $\mathfrak{C}$.

Threefolds passing multiply through a given curve $\mathfrak{C}$ on a given surface F: residual intersections with F.

1. We suppose that $\mathfrak{C}$ is simple on F and that both curve and surface are prescribed free from base elements. $|\mathbf{A}|$ and $|\mathbf{B}|$ are two linear systems of threefolds on V_4 which are free from prescribed base elements on F. $\mathbf{A}_0$ is a member of $|\mathbf{A}|$ prescribed to have $\mathfrak{C}$ to virtual multiplicity a (so that, if $\bar{\mathfrak{C}}$ is the infinitesimal threefold on V_4 constituted by the first neighbourhood of $\mathfrak{C}$, $\mathbf{A} \equiv \mathbf{A}_0 + a\bar{\mathfrak{C}}$); and $\mathbf{B}_0$ is a member of $|\mathbf{B}|$ prescribed

to have $\mathfrak{C}$ to multiplicity b. The residual intersection $\mathfrak{D}$ of F with $\mathbf{A}_0$ is given by

$$(F,\ \mathbf{A}) \equiv a\mathfrak{C}+\mathfrak{D}.$$

We seek expressions for $(\mathfrak{C},\ \mathfrak{D})_F$, $(\mathfrak{D}^2)_F$, $\kappa_{\mathfrak{D}}$ and for β, the set of intersections of $\mathbf{B}_0$ and $\mathfrak{D}$ which do not lie on $\mathfrak{C}$.

We have

$$(\mathfrak{C},\ \mathfrak{D})_F \equiv (\mathfrak{C},\ a\mathfrak{C}+\mathfrak{D})_F - a(\mathfrak{C}^2)_F,$$

$$(\mathfrak{C}^2)_F \equiv -(\mathfrak{C},\ \mathfrak{K}_F)_F + \kappa_{\mathfrak{C}};$$

hence

$$(1) \qquad (\mathfrak{C},\ \mathfrak{D})_F \equiv (\mathfrak{C},\ \mathbf{A}) + a(\mathfrak{C},\ \mathfrak{K}_F)_F - a\kappa_{\mathfrak{C}}.$$

Therefore, since

$$(\mathfrak{D}^2)_F \equiv (F,\ \mathbf{A}^2) - 2a(\mathfrak{C},\ \mathfrak{D})_F - a^2(\mathfrak{C}^2)_F,$$

we have

$$(2) \qquad (\mathfrak{D}^2)_F \equiv (F,\ \mathbf{A}^2) - 2a(\mathfrak{C},\ \mathbf{A}) - a^2(\mathfrak{C},\ \mathfrak{K}_F)_F + a^2\kappa_{\mathfrak{C}};$$

and

$$\kappa_{\mathfrak{D}} \equiv (\mathfrak{D},\ \mathfrak{K}_F)_F + (\mathfrak{D}^2)_F$$

$$\equiv (\mathfrak{K}_F,\ \mathbf{A}) - a(\mathfrak{C},\ \mathfrak{K}_F)_F + (\mathfrak{D}^2)_F,$$

$$(3) \qquad \kappa_{\mathfrak{D}} \equiv (F,\ \mathbf{A}^2) + (\mathfrak{K}_F,\ \mathbf{A}) - 2a(\mathfrak{C},\ \mathbf{A}) - a(a+1)(\mathfrak{C},\ \mathfrak{K}_F)_F + a^2\kappa_{\mathfrak{C}}.$$

Lastly,

$$\beta \equiv (\mathfrak{D},\ \mathbf{B}) - b(\mathfrak{C},\ \mathfrak{D})_F,$$

$$(4) \qquad \beta \equiv (F,\ \mathbf{AB}) - a(\mathfrak{C},\ \mathbf{B}) - b(\mathfrak{C},\ \mathbf{A}) - ab(\mathfrak{C},\ \mathfrak{K}_F)_F + ab\kappa_{\mathfrak{C}}.$$

Threefolds passing multiply through a given curve.

2. The given curve $\mathfrak{C}_0$ is supposed to be virtually free from multiple points. $|\mathbf{A}|$, $|\mathbf{B}|$, $|\mathbf{C}|$, $|\mathbf{D}|$ are four linear systems of threefolds on V_4 with no prescribed base element on $\mathfrak{C}_0$. $\mathbf{A}_0$ is a member of $|\mathbf{A}|$ which has $\mathfrak{C}_0$ to virtual multiplicity a (so that $\mathbf{A} \equiv \mathbf{A}_0 + a\overline{\mathfrak{C}}_0$); $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{D}_0$, defined similarly, have $\mathfrak{C}_0$ to respective virtual multiplicities b, c, d.

$\mathbf{A}_0$, $\mathbf{B}_0$ intersect in a surface $F_0 = (\mathbf{A}_0\mathbf{B}_0)$ which has $\mathfrak{C}_0$ to virtual multiplicity ab. This surface coincides in position with a surface $F \equiv (\mathbf{AB})$. $\mathfrak{C}$ denotes the curve on F which is in $(ab,\ 1)$ correspondence with $\mathfrak{C}_0$; evidently $\mathfrak{C} \equiv ab\mathfrak{C}_0$ on V_4.

$\mathbf{C}_0$ meets F_0, apart from $\mathfrak{C}_0$, in a curve $\mathfrak{D}$, such that $(F, \mathbf{C}) \equiv c\mathfrak{C}+\mathfrak{D}$. α denotes the virtual intersection set $(\mathfrak{C}, \mathfrak{D}) = (\mathfrak{C}_0, \mathfrak{D})$. $\mathbf{D}_0$ meets $\mathfrak{D}$, apart from α, in a set β such that $(\mathbf{D}, \mathfrak{D}) \equiv d\alpha+\beta$.

We put

$$\mathbf{A} \equiv \mathbf{A}_1+\ldots+\mathbf{A}_a, \quad \mathbf{A}_0 \equiv \mathbf{A}_{0,1}+\ldots+\mathbf{A}_{0,a},$$

$$\mathbf{B} \equiv \mathbf{B}_1+\ldots+\mathbf{B}_b, \quad \mathbf{B}_0 \equiv \mathbf{B}_{0,1}+\ldots+\mathbf{B}_{0,b},$$

$$\mathbf{C} \equiv \mathbf{C}_1+\ldots+\mathbf{C}_c, \quad \mathbf{C}_0 \equiv \mathbf{C}_{0,1}+\ldots+\mathbf{C}_{0,c},$$

$$\mathbf{D} \equiv \mathbf{D}_1+\ldots+\mathbf{D}_d, \quad \mathbf{D}_0 \equiv \mathbf{D}_{0,1}+\ldots+\mathbf{D}_{0,d},$$

where $\mathbf{A}_{0,i}$, $\mathbf{B}_{0,j}$, $\mathbf{C}_{0,k}$, $\mathbf{D}_{0,l}$ are virtual threefolds each having $\mathfrak{C}_0$ to virtual multiplicity 1 and $\mathbf{A}_i \equiv \mathbf{A}_{0,i}+\overline{\mathfrak{C}}_0$, and so on. We define F_{ij}, $\mathfrak{D}_{ijk}$, α_{ijk}, β_{ijkl} in an obvious manner.

By considering variation in the restricted linear systems $|\mathbf{A}_0|$, $|\mathbf{B}_0|$, $|\mathbf{C}_0|$, $|\mathbf{D}_0|$, which have $\mathfrak{C}_0$ as a base curve to respective multiplicities a, b, c, d, we see that

$$F \equiv \sum_{i,j} F_{ij}, \qquad \mathfrak{D} \equiv \sum_{i,j,k} \mathfrak{D}_{ijk},$$

$$\alpha \equiv \sum_{i,j,k} \alpha_{ijk}, \qquad \beta \equiv \sum_{i,j,k,l} \beta_{ijkl}.$$

Todd [1, formulae (16), (18)] has proved that

$$F_{ij} \equiv (\mathbf{A}_i \mathbf{B}_j), \quad \mathfrak{D}_{ijk} \equiv (\mathbf{A}_i \mathbf{B}_j \mathbf{C}_k)-\mathfrak{C}_0,$$

$$\alpha_{ijk} \equiv (\mathfrak{C}_0, \ \mathbf{A}_i+\mathbf{B}_j+\mathbf{C}_k+\mathbf{K})-\kappa_{\mathfrak{C}_0},$$

$$\beta_{ijkl} \equiv (\mathbf{A}_i \mathbf{B}_j \mathbf{C}_k \mathbf{D}_l)-(\mathfrak{C}_0, \ \mathbf{A}_i+\mathbf{B}_j+\mathbf{C}_k+\mathbf{D}_l+\mathbf{K})+\kappa_{\mathfrak{C}_0};$$

hence

$$F \equiv (\mathbf{AB}), \quad \mathfrak{D} \equiv (\mathbf{ABC})-abc\mathfrak{C}_0,$$

$$\alpha \equiv (\mathfrak{C}_0, \ bc\mathbf{A}+ca\mathbf{B}+ab\mathbf{C}+abc\mathbf{K})-abc\kappa_{\mathfrak{C}_0}, \tag{5}$$

$$\beta \equiv (\mathbf{ABCD})-(\mathfrak{C}_0, bcd\mathbf{A}+acd\mathbf{B}+abd\mathbf{C}+abc\mathbf{D}+abcd\mathbf{K})+abcd\kappa_{\mathfrak{C}_0}. \tag{6}$$

3. We next find an expression for γ, the set of pinch-points which F_0 has on $\mathfrak{C}_0$, these being the branch points in the correspondence between $\mathfrak{C}_0$ and $\mathfrak{C}$.

First we show that, if A_0, $\mathbf{B}_0$ vary in the restricted linear systems of §2, then γ varies in a series of equivalence on $\mathfrak{C}_0$. Since such a variation results from first varying A_0 while $\mathbf{B}_0$ is kept fixed and then varying $\mathbf{B}_0$

while $\mathbf{A}_0$ is kept fixed, and since these two operations have the same geometrical character, it is sufficient to prove the lemma for the case where $\mathbf{A}_0$ varies and $\mathbf{B}_0$ remains fixed.

By means of primals of high enough order which pass through $\mathfrak{C}_0$, we can transform V_4 birationally into a V_4^* so that to each point ϵ of $\mathfrak{C}_0$ corresponds a plane E^* on V_4^* and to the whole of $\mathfrak{C}_0$ corresponds a three-fold $\mathbf{E}^*$, the locus of E^*. Let $\mathbf{A}_0^*$ be the reduced transform of $\mathbf{A}_0$ and, generally, let a star denote a homologue on V_4^* (then, for example, $\mathbf{A}^* \equiv \mathbf{A}_0^* + a\mathbf{E}^*$). $\mathbf{A}_0^*$ meets E^* in a curve $\mathfrak{A}^*$ of order a and $\mathbf{B}_0^*$ meets E^* in a curve of $\mathfrak{B}^*$ of order b. $\mathfrak{A}^*$ meets $\mathfrak{B}^*$ in the ab points common to E^* and F_0^*; and the locus of these ab points is a curve $\mathfrak{C}^*$ in (1, 1) correspondence with $\mathfrak{C}$. To the branch points on $\mathfrak{C}_0$ correspond the planes E^* which touch $\mathfrak{C}^*$: let ϑ^* be the set of points of contact.

Let $\mathbf{A}_0'$, $\mathbf{A}_0''$ be two positions of $\mathbf{A}_0$ and let dashes indicate the corresponding positions of the other loci involved. As $\mathbf{A}_0^*$ varies in the pencil determined by $\mathbf{A}_0'^*$ and $\mathbf{A}_0''^*$, $\mathfrak{C}^*$ varies in a pencil on a surface C^*. Let $\mathfrak{E}^*$ be the locus of the corresponding sets ϑ^*. These sets form on $\mathfrak{E}^*$ an ∞^1 rational involutory series and are therefore equivalent on $\mathfrak{E}^*$. In the correspondence between $\mathfrak{C}_0$ and $\mathfrak{E}^*$, in which with each point of $\mathfrak{E}^*$ is associated the point on $\mathfrak{C}_0$ giving rise to the plane E^* through the point on $\mathfrak{E}^*$, γ' corresponds to ϑ'^* and γ'' to ϑ''^*. Since equivalence is preserved under this correspondence, $\gamma' \equiv \gamma''$; this proves the lemma.

Consider now the decomposition of § 2,

$$A_0 \equiv A_{0,1} + \ldots + \mathbf{A}_{0,a}, \quad \mathbf{B}_0 \equiv \mathbf{B}_{0,1} + \ldots + \mathbf{B}_{0,b}.$$

One position for $\mathfrak{C}^*$ is then a set of ab curves each unisecant to the planes E^*; and the set ϑ^* (on $\mathfrak{E}^*$) becomes twice the set of virtual intersections of these components. Thus, if

$$(\mathbf{A}_l \mathbf{A}_m \mathbf{B}_p) = \mathfrak{C}_0 + \mathfrak{D}_{lmp}, \quad (\mathbf{A}_l \mathbf{B}_p \mathbf{B}_q) = \mathfrak{C}_0 + \mathfrak{D}_{lpq},$$

where

$$l \neq m; \ l, m = 1, 2, \ldots, a \quad \text{and} \quad p \neq q; \ p, q = 1, 2, \ldots, b,$$

we have

$$\gamma \equiv 2 \sum_{l,m,p} (\mathfrak{C}_0, \mathfrak{D}_{lmp}) + 2 \sum_{l,p,q} (\mathfrak{C}_0, \mathfrak{D}_{lpq}).$$

We have seen that

$$(\mathfrak{C}_0, \mathfrak{D}_{lmp}) \equiv (\mathfrak{C}_0, \mathbf{A}_l + \mathbf{A}_m + \mathbf{B}_p + \mathbf{K}) - \kappa_{\mathfrak{C}_0},$$

$$(\mathfrak{C}_0, \mathfrak{D}_{lpq}) \equiv (\mathfrak{C}_0, \mathbf{A}_l + \mathbf{B}_p + \mathbf{B}_q + \mathbf{K}) - \kappa_{\mathfrak{C}_0},$$

and hence

$$\gamma \equiv 2b(a-1)(\mathfrak{C}_0,\ \mathbf{A})+a(a-1)(\mathfrak{C}_0,\ \mathbf{B})+ab(a-1)\{(\mathfrak{C}_0,\ \mathbf{K})-\kappa_{\mathfrak{C}_0}\}$$
$$+2a(b-1)(\mathfrak{C}_0,\ \mathbf{B})+b(b-1)(\mathfrak{C}_0,\ \mathbf{A})+ab(b-1)\{(\mathfrak{C}_0,\ \mathbf{K})-\kappa_{\mathfrak{C}_0}\},$$

that is

$$(7)\quad \gamma \equiv b(2a+b-3)(\mathfrak{C}_0,\ \mathbf{A})+a(a+2b-3)(\mathfrak{C}_0,\ \mathbf{B})$$
$$+ab(a+b-2)\{(\mathfrak{C}_0,\ \mathbf{K})-\kappa_{\mathfrak{C}_0}\}.$$

4. We now determine expressions for $\kappa_{\mathfrak{C}}$, $\kappa_{\mathfrak{D}}$, $(\mathfrak{C}^2)_F$, $(\mathfrak{D}^2)_F$. By applying Zeuthen's formula to the correspondence between $\mathfrak{C}$ and $\mathfrak{C}_0$, we have at once

$$\kappa_{\mathfrak{C}} \equiv ab\kappa_{\mathfrak{C}_0}+\gamma,$$

whence

$$(8)\quad \kappa_{\mathfrak{C}} \equiv b(2a+b-3)(\mathfrak{C}_0,\ \mathbf{A})+a(a+2b-3)(\mathfrak{C}_0,\ \mathbf{B})$$
$$+ab(a+b-2)(\mathfrak{C}_0,\ \mathbf{K})-ab(a+b-3)\kappa_{\mathfrak{C}_0}.$$

Next we observe that, since the canonical threefolds on $V_4{}^*$ are equivalent to $\mathbf{K}^*+2\mathbf{E}^*$, a formula due to Todd [1, formula (8)] gives

$$\mathfrak{K}_{F_0{}^*} \equiv (F_0{}^*,\ \mathbf{A}_0{}^*+\mathbf{B}_0{}^*+\mathbf{K}^*+2\mathbf{E}^*)$$
$$\equiv \left(F_0{}^*,\ \mathbf{A}^*+\mathbf{B}^*+\mathbf{K}^*-(a+b-2)\,\mathbf{E}^*\right).$$

There is a birational correspondence between F_0 and $F_0{}^*$ which introduces no exceptional curve on either surface; hence

$$(9)\qquad \mathfrak{K}_{F_0} \equiv (F_0,\ \mathbf{A}+\mathbf{B}+\mathbf{K})-ab(a+b-2)\,\mathfrak{C}_0.$$

Thus canonical curves are cut on F_0 by threefolds equivalent to $\mathbf{A}+\mathbf{B}+\mathbf{K}$ which are made to have $\mathfrak{C}_0$ to multiplicity $a+b-2$.

Since F, F_0 also correspond birationally without the introduction of any exceptional curves, these same threefolds cut canonical curves on F; that is

$$(10)\qquad \mathfrak{K}_F \equiv (F,\ \mathbf{A}+\mathbf{B}+\mathbf{K})-(a+b-2)\,\mathfrak{C}.$$

Putting $\mathbf{C}=\mathbf{D}=\mathbf{A}+\mathbf{B}+\mathbf{K}$ and $c=d=a+b-2$ in formula (6), we have

$$(11)\quad (\mathfrak{K}_{F_0}^2)_{F_0} \equiv (\mathbf{AB},\ \{\mathbf{A}+\mathbf{B}+\mathbf{K}\}^2)+ab(a+b-2)^2\kappa_{\mathfrak{C}_0}$$
$$-(a+b-2)\left(\mathfrak{C}_0,\ b(3a+b-2)\,\mathbf{A}+a(a+3b-2)\,\mathbf{B}+ab(a+b)\,\mathbf{K}\right).$$

Again, putting $C = \mathbf{A}+\mathbf{B}+\mathbf{K}$ and $c = a+b-2$ in formula (5), we have

$$(12)\quad (\mathfrak{C},\ \mathfrak{K}_F)_F \equiv \big(\mathfrak{C}_0,\ b(2a+b-2)\,\mathbf{A}+a(a+2b-2)\,\mathbf{B}+ab(a+b-1)\,\mathbf{K}\big) - ab(a+b-2)\kappa_{\mathfrak{C}_0}.$$

Since $$\kappa_{\mathfrak{C}} \equiv (\mathfrak{C}^2)_F+(\mathfrak{C},\ \mathfrak{K}_F)_F,$$

formulae (8) and (12) give

$$(13)\quad (\mathfrak{C}^2)_F \equiv -(\mathfrak{C}_0,\ b\mathbf{A}+a\mathbf{B}+ab\mathbf{K})+ab\kappa_{\mathfrak{C}_0}.$$

If in formula (6) we replace $\mathbf{D}$ and d first by $\mathbf{C}$ and c respectively and then by $\mathbf{A}+\mathbf{B}+\mathbf{K}$ and $a+b-2$ respectively, we obtain

$$(14)\quad (\mathfrak{D}^2)_F \equiv (\mathbf{ABC}^2)-(\mathfrak{C}_0,\ bc^2\mathbf{A}+ac^2\mathbf{B}+2abc\mathbf{C}+abc^2\mathbf{K})+abc^2\kappa_{\mathfrak{C}_0},$$

$$(15)\quad (\mathfrak{D},\ \mathfrak{K}_F)_F \equiv (\mathbf{ABC},\ \mathbf{A}+\mathbf{B}+\mathbf{K})+abc(a+b-2)\kappa_{\mathfrak{C}_0} - \big(\mathfrak{C}_0,\ bc(a+b-2)\,\mathbf{A}+ca(a+b-2)\,\mathbf{B}+ab(a+b-2)\,\mathbf{C} +abc(\mathbf{A}+\mathbf{B}+\mathbf{K})+abc(a+b-2)\,\mathbf{K}\big).$$

Hence, since

$$\kappa_{\mathfrak{D}} \equiv (\mathfrak{D}^2)_F+(\mathfrak{D},\ \mathfrak{K}_F)_F,$$

we have

$$(16)\quad \kappa_{\mathfrak{D}} \equiv (\mathbf{ABC},\ \mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{K})+abc(a+b+c-2)\kappa_{\mathfrak{C}_0} - \big(\mathfrak{C}_0,\ bc(2a+b+c-2)\,\mathbf{A}+ca(a+2b+c-2)\,\mathbf{B} +ab(a+b+2c-2)\,\mathbf{C}-abc(a+b+c-1)\,\mathbf{K}\big).$$

An alternative form for this expression is obtained by replacing $(\mathbf{ABC})$ by the equivalent $abc\mathfrak{C}_0+\mathfrak{D}$, so that

$$(17)\quad \kappa_{\mathfrak{D}} \equiv (\mathfrak{D},\ \mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{K})-(a+b+c-2)(\mathfrak{C}_0,\ \mathfrak{D}).$$

Thus canonical sets are cut on $\mathfrak{D}$ by threefolds equivalent to $\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{K}$ made to pass $a+b+c-2$ times through $\mathfrak{C}_0$.

Threefolds passing multiply through a surface.

5. To begin with we suppose that the given surface F is free from improper nodes, and this much simplifies the work. The results are then extended, in §§ 15 and 16, to the case where improper nodes are present.

$|\mathbf{A}|$ is a linear system of threefolds with no prescribed base element on F. $\mathbf{A}_0$ is a member of $|\mathbf{A}|$ which has F to virtual multiplicity a, so

that $\mathbf{A} \equiv \mathbf{A}_0 + a\bar{F}$, where $\bar{F}$ is the infinitesimal neighbourhood of F on V_4. $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{D}_0$, similarly defined, have F to respective virtual multiplicities b, c, d. $\mathbf{A}_0$, $\mathbf{B}_0$ intersect, apart from F, in a surface $G \equiv (\mathbf{AB}) - abF$. $\mathfrak{C}$ denotes the curve in which F meets G. $\mathbf{C}_0$ meets G, apart from $\mathfrak{C}$, in a curve $\mathfrak{D} \equiv (\mathbf{C}, G) - c\mathfrak{C}$; α is the set $(\mathfrak{C}, \mathfrak{D})_G$. $\mathbf{D}_0$ meets $\mathfrak{D}$, apart from α, in a set $\beta \equiv (\mathbf{D}, \mathfrak{D}) - d\alpha$.

We put

$$\mathbf{A}_0 \equiv \mathbf{A}_{0,1} + \ldots + \mathbf{A}_{0,a}, \quad \mathbf{A} \equiv \mathbf{A}_1 + \ldots + \mathbf{A}_a,$$

$$\mathbf{B}_0 \equiv \mathbf{B}_{0,1} + \ldots + \mathbf{B}_{0,b}, \quad \mathbf{B} \equiv \mathbf{B}_1 + \ldots + \mathbf{B}_b,$$

$$\mathbf{C}_0 \equiv \mathbf{C}_{0,1} + \ldots + \mathbf{C}_{0,c}, \quad \mathbf{C} \equiv \mathbf{C}_1 + \ldots + \mathbf{C}_c,$$

$$\mathbf{D}_0 \equiv \mathbf{D}_{0,1} + \ldots + \mathbf{D}_{0,d}, \quad \mathbf{D} \equiv \mathbf{D}_1 + \ldots + \mathbf{D}_d,$$

where $\mathbf{A}_{0,i}$ is a virtual threefold having F to virtual multiplicity 1 and $\mathbf{A}_i \equiv \mathbf{A}_{0,i} + \bar{F}$, and so on. In general, $\mathbf{A}_{0,i}$ has a set $\nu_{\mathbf{A}_{0,i}}$ of double points on F.

We define G_{ij}, $\mathfrak{C}_{ij}$, $\mathfrak{D}_{ijk}$, α_{ijk}, β_{ijkl} in an obvious manner. Evidently

$$G \equiv \sum_{i,j} G_{ij}, \quad \mathfrak{C} \equiv \sum_{i,j} \mathfrak{C}_{ij}, \quad \mathfrak{D} \equiv \sum_{i,j,k} \mathfrak{D}_{ijk},$$

$$\alpha \equiv \sum_{i,j,k} \alpha_{ijk}, \quad \beta \equiv \sum_{i,j,k,l} \beta_{ijkl}.$$

Todd has shown [**1**, formula (24)] that

$$\mathfrak{C}_{ij} \equiv (F,\ \mathbf{A}_i + \mathbf{B}_j + \mathbf{K}) - \mathfrak{K}_F,$$

whence

$$\mathfrak{C} \equiv (F,\ b\mathbf{A} + a\mathbf{B} + ab\mathbf{K}) - ab\mathbf{K}_F, \tag{18}$$

and so

$$(\mathfrak{C}^2)_F \equiv (F,\ \{b\mathbf{A} + a\mathbf{B} + ab\mathbf{K}\}^2) - 2ab(\mathfrak{K}_F,\ b\mathbf{A} + a\mathbf{B} + ab\mathbf{K}) + a^2 b^2 (\mathfrak{K}_F^2)_F. \tag{19}$$

Since

$$\begin{aligned} \kappa_{\mathfrak{C}} &\equiv (\mathfrak{C}^2)_F + (\mathfrak{C},\ \mathfrak{K}_F)_F \\ &\equiv (\{\mathfrak{C} + ab\mathfrak{K}_F\}^2)_F - (2ab-1)(\mathfrak{C} + ab\mathfrak{K}_F,\ \mathfrak{K}_F)_F + ab(ab-1)(\mathfrak{K}_F^2)_F, \end{aligned}$$

we have

$$\kappa_{\mathfrak{C}} \equiv (F,\ \{b\mathbf{A} + a\mathbf{B} + ab\mathbf{K}\}^2) - (2ab-1)(\mathfrak{K}_F,\ b\mathbf{A} + a\mathbf{B} + ab\mathbf{K}) + ab(ab-1)(\mathfrak{K}_F^2)_F. \tag{20}$$

Again [Todd, **1**, formulae (33), (34), (36)],

$$\mathfrak{D}_{ijk} \equiv (\mathbf{A}_i \mathbf{B}_j \mathbf{C}_k) - (F,\ \mathbf{A}_i + \mathbf{B}_j + \mathbf{C}_k + \mathbf{K}) + \mathfrak{K}_F,$$

$$\alpha_{ijk} \equiv (F,\ \mathbf{B}_j \mathbf{C}_k + \mathbf{C}_k \mathbf{A}_i + \mathbf{A}_i \mathbf{B}_j + \{\mathbf{A}_i + \mathbf{B}_j + \mathbf{C}_k,\ \mathbf{K}\} + \mathbf{K}^2) - (\mathfrak{K}_F,\ \mathbf{A}_i + \mathbf{B}_j + \mathbf{C}_k + 2\mathbf{K}) - (F^2) + (\mathfrak{K}_F{}^2)_F,$$

$$\beta_{ijkl} \equiv (\mathbf{A}_i \mathbf{B}_j \mathbf{C}_k \mathbf{D}_l) + (\mathfrak{K}_F,\ \mathbf{A}_i + \mathbf{B}_j + \mathbf{C}_k + \mathbf{D}_l + 2\mathbf{K}) + (F^2) - (\mathfrak{K}_F{}^2)_F - (F,\ \mathbf{B}_j \mathbf{C}_k + \mathbf{C}_k \mathbf{A}_i + \mathbf{A}_i \mathbf{B}_i + \mathbf{A}_i \mathbf{D}_l + \mathbf{B}_i \mathbf{D}_l + \mathbf{C}_i \mathbf{D}_l + \{\mathbf{A}_i + \mathbf{B}_j + \mathbf{C}_k + \mathbf{D}_l,\ \mathbf{K}\} + \mathbf{K}^2),$$

and hence

$$(21) \quad \mathfrak{D} \equiv (\mathbf{ABC}) - (F,\ bc\mathbf{A} + ca\mathbf{B} + ab\mathbf{C} + abc\mathbf{K}) + abc\mathfrak{K}_F,$$

$$(22) \quad \alpha \equiv (F,\ a\mathbf{BC} + b\mathbf{CA} + c\mathbf{AB} + \{bc\mathbf{A} + ca\mathbf{B} + ab\mathbf{C},\ \mathbf{K}\} + abc\mathbf{K}^2) - (\mathfrak{K}_F,\ bc\mathbf{A} + ca\mathbf{B} + ab\mathbf{C} + 2abc\mathbf{K}) - abc(F^2) + abc(\mathfrak{K}_F{}^2)_F.$$

$$(23) \quad \beta \equiv (\mathbf{ABCD}) + (\mathfrak{K}_F,\ bcd\mathbf{A} + acd\mathbf{B} + abd\mathbf{C} + abc\mathbf{D} + 2abcd\mathbf{K}) - (F,\ ad\mathbf{BC} + bd\mathbf{CA} + cd\mathbf{AB} + bc\mathbf{AD} + ca\mathbf{BD} + ab\mathbf{CD}) - (F,\ \{bcd\mathbf{A} + acd\mathbf{B} + abd\mathbf{C} + abc\mathbf{D},\ \mathbf{K}\} + abcd\mathbf{K}^2) + abcd(F^2) - abcd(\mathfrak{K}_F{}^2)_F.$$

6. To find the canonical curves on G, we transform V_4 birationally, by means of primals of high enough order which pass through F, into a $V_4{}^*$; then to each point λ of F corresponds a line $\mathfrak{l}^*$ and to the whole of F corresponds a threefold $\mathbf{L}^*$, the locus of $\mathfrak{l}^*$. Let $\mathbf{A}_0{}^*$ be the reduced transform of $\mathbf{A}_0$ and, generally, let a star denote a homologue on $V_4{}^*$; then, for example, $\mathbf{A}^* \equiv \mathbf{A}_0{}^* + a\mathbf{L}^*$.

The canonical threefolds on $V_4{}^*$ are equivalent to $\mathbf{K}^* + \mathbf{L}^*$; hence

$$\mathfrak{K}_{G^*} \equiv (G^*,\ \mathbf{A}_0{}^* + \mathbf{B}_0{}^* + \mathbf{K}^* + \mathbf{L}^*) \equiv \left(G^*,\ \mathbf{A}^* + \mathbf{B}^* + \mathbf{K}^* - (a+b-1)\,\mathbf{L}^*\right).$$

The birational correspondence between G and G^* does not give rise to any exceptional curve on either surface, so

$$(24) \qquad \mathfrak{K}_G \equiv (G,\ \mathbf{A} + \mathbf{B} + \mathbf{K}) - (a+b-1)\,\mathfrak{C};$$

that is to say, canonical curves are cut on G by threefolds equivalent to $\mathbf{A}+\mathbf{B}+\mathbf{K}$ which are made to pass $a+b-1$ times through $\mathfrak{C}$ (or F).

From (18) we see that an alternative form to (24) is

$$(25)\quad \mathfrak{K}_G \equiv (\mathbf{AB},\ \mathbf{A}+\mathbf{B}+\mathbf{K}) - b(2a+b-1)(F,\ \mathbf{A}) - a(a+2b-1)(F,\ \mathbf{B}) - ab(a+b)(F,\ \mathbf{K}) + ab(a+b-1)\,\mathfrak{K}_F.$$

7. By putting $\mathbf{D} = \mathbf{C}$ and $d = c$ in (23) we get

$$(26)\quad (\mathfrak{D}^2)_G \equiv (\mathbf{ABC^2}) + (\mathfrak{K}_F,\ bc^2\,\mathbf{A} + ac^2\,\mathbf{B} + 2abc\mathbf{C} + 2abc^2\,\mathbf{K}) - (F,\ ac\mathbf{BC} + bc\mathbf{CA} + c^2\mathbf{AB} + bc\mathbf{AC} + ca\mathbf{BC} + ab\mathbf{C^2}) - (F,\ \{bc^2\mathbf{A} + ac^2\mathbf{B} + 2abc\mathbf{C},\ \mathbf{K}\} + abc^2\mathbf{K}) + abc^2(F^2) - abc^2(\mathfrak{K}_F^2)_F.$$

Again, putting $\mathbf{D} = \mathbf{A}+\mathbf{B}+\mathbf{K}$ and $d = a+b-1$ in (23), we get

$$(\mathfrak{D},\ \mathfrak{K}_G)_G \equiv (\mathbf{ABC},\ \mathbf{A}+\mathbf{B}+\mathbf{K}) + \big(\mathfrak{K}_F,\ bc(a+b-1)\,\mathbf{A} + ac(a+b-1)\,\mathbf{B} + ab(a+b-1)\,\mathbf{C} + abc(\mathbf{A}+\mathbf{B}+\mathbf{K}) + 2abc(a+b-1)\,\mathbf{K}\big) - (F,\ \{a\mathbf{BC} + b\mathbf{CA} + c\mathbf{AB}\}(a+b-1) + \{bc\mathbf{A} + ca\mathbf{B} + ab\mathbf{C},\ \mathbf{A}+\mathbf{B}+\mathbf{K}\}) - \big(F,\ \{\mathbf{K},\ (a+b-1)(bc\mathbf{A} + ca\mathbf{B} + ab\mathbf{C}) + abc(\mathbf{A}+\mathbf{B}+\mathbf{K})\} + abc(a+b-1)\,\mathbf{K}^2\big) + abc(a+b-1)(F^2) - abc(a+b-1)(\mathfrak{K}_F^2)_F.$$

Hence, from $\qquad \kappa_{\mathfrak{D}} \equiv (\mathfrak{D}^2)_G + (\mathfrak{D},\ \mathfrak{K}_G)_G,$

we have

$$(27)\quad \mathfrak{K}_{\mathfrak{D}} \equiv (\mathbf{ABC},\ \mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{K}) + \big(\mathfrak{K}_F,\ bc(2a+b+c-1)\,\mathbf{A} + ca(a+2b+c-1)\,\mathbf{B} + ab(a+b+2c-1)\,\mathbf{C} + abc(2a+2b+2c-1)\,\mathbf{K}\big) - \big(F,\ a(a+2b+2c-1)\,\mathbf{BC} + b(2a+b+2c-1)\,\mathbf{CA} + c(2a+2b+c-1)\,\mathbf{AB} + bc\mathbf{A^2} + ca\mathbf{B^2} + ab\mathbf{C^2}\big) - \big(F,\ \{\mathbf{K},\ bc(2a+b+c)\,\mathbf{A} + ca(a+2b+c)\,\mathbf{B} + ab(a+b+2c)\,\mathbf{C} + abc(a+b+c)\,\mathbf{K}\}\big) + abc(a+b+c-1)\{(F^2) - (\mathfrak{K}_F^2)_F\}.$$

By means of (21) and (22) we can rewrite (27) in the more expressive form

$$\kappa_{\mathfrak{D}} \equiv (\mathfrak{D},\ \mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{K})-(a+b+c-1)\,\alpha. \tag{28}$$

Thus canonical sets are cut on $\mathfrak{D}$ by threefolds equivalent to $\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{K}$ which are made to pass $a+b+c-1$ times through α (or F).

8. To find κ_G, we deal first with the case $a=b=1$. Todd gives a formula [**1**, formula (32)] for this case but it is not in a form suitable for our purpose. His work can be adapted as follows:

We have $$\kappa_G \equiv (G,\ \mathfrak{K}_{\mathbf{A}_0}+\mathfrak{K}_G)_{\mathbf{A}_0}.$$

Since $$\mathfrak{K}_{\mathbf{A}_0} \equiv (K+K_{\mathbf{A}_0},\ \mathbf{A}_0) \equiv (K,\ \mathbf{A}_0)+(\mathbf{K}\mathbf{A}_0^2)+(\mathbf{A}_0^3),$$

$$(G,\ \mathfrak{K}_{\mathbf{A}_0})_{\mathbf{A}_0} \equiv (G,\ K+\mathbf{A}\mathbf{K}+\mathbf{A}^2)-\nu_{\mathbf{A}_0}.$$

Next $$(G,\ \mathfrak{K}_G)_{\mathbf{A}_0} \equiv (F+G,\ \mathfrak{K}_G)_{\mathbf{A}_0}-(F,\ \mathfrak{K}_G)_{\mathbf{A}_0}$$

$$\equiv (\mathbf{B},\ \mathfrak{K}_G)-(\mathfrak{C},\ \mathfrak{K}_G)_G.$$

Using (24) we get

$$\kappa_G \equiv (G,\ \mathbf{A}^2+\mathbf{A}\mathbf{B}+\mathbf{B}^2+\mathbf{A}\mathbf{K}+\mathbf{B}\mathbf{K}+K)-\nu_{\mathbf{A}_0}-(\mathfrak{C},\ \mathbf{B})-(\mathfrak{C},\ \mathfrak{K}_G)_G.$$

Now [Todd, **1**, formula (29)]

$$\nu_{\mathbf{A}_0} \equiv (\mathfrak{C},\ \mathbf{A})-(F,\ G).$$

Thus

$$\kappa_G \equiv (G,\ \mathbf{A}^2+\mathbf{A}\mathbf{B}+\mathbf{B}^2+\mathbf{A}\mathbf{K}+\mathbf{B}\mathbf{K}+K)+(F,\ G)-(\mathfrak{C},\ \mathbf{A}+\mathbf{B})-(\mathfrak{C},\ \mathfrak{K}_G)_G. \tag{29}$$

Again [Todd, **1**, formula (23)]

$$(F,\ G)+(\mathfrak{C},\ K)-(\mathfrak{C},\ \mathfrak{K}_F)_F-(\mathfrak{C},\ \mathfrak{K}_G)_G+\kappa_{\mathfrak{C}} \equiv 0,$$

since F, G have no intersection outside $\mathfrak{C}$. Using this to eliminate the term $(F,\ G)-(\mathfrak{C},\ \mathfrak{K}_G)_G$ from (29), and simplifying the result by means of (18), (20), we get the expression sought:

$$\kappa_G \equiv (G,\ \mathbf{A}^2+\mathbf{A}\mathbf{B}+\mathbf{B}^2+\mathbf{A}\mathbf{K}+\mathbf{B}\mathbf{K}+K)-2(\mathfrak{C},\ \mathbf{A}+\mathbf{B}+\mathbf{K})+(\mathfrak{C},\ \mathfrak{K}_F)_F. \tag{30}$$

9. We must now show that on V_4^* the canonical surfaces are equivalent to $K^*-(\mathbf{L}^{*2})+H^*$, where H^* is a canonical system of generators on $\mathbf{L}^*$ (*i.e.* the surface of generators arising from the points of a canonical curve on F).

The correspondence between G and G^* introduces no exceptional curve on either surface, so [Todd, 4, formula (12)] κ_G corresponds to a set κ_{G^*}, and by (30)

$$(31)\quad \kappa_{G^*} \equiv (G^*,\ \mathbf{A}^{*2}+\mathbf{A}^*\mathbf{B}^*+\mathbf{B}^{*2}+\mathbf{A}^*\mathbf{K}^*+\mathbf{B}^*\mathbf{K}^*+K^*) \\ -2(G^*,\ \{\mathbf{A}^*+\mathbf{B}^*+\mathbf{K}^*,\ \mathbf{L}^*\})+(G^*,\ H^*).$$

Let Z^* be a canonical surface on V_4^*; then we can apply (30) to G^* by considering this surface as the total intersection $\mathbf{A}_0^*$, $\mathbf{B}_0^*$. Since $\mathbf{A}_0^* \equiv \mathbf{A}^*-\mathbf{L}^*$, $\mathbf{B}_0^* \equiv \mathbf{B}^*-\mathbf{L}^*$, we have

$$(32)\quad \kappa_{G^*} \equiv \big(G^*,\ (\mathbf{A}^*-\mathbf{L}^*)^2+(\mathbf{A}^*-\mathbf{L}^*)(\mathbf{B}^*-\mathbf{L}^*)+(\mathbf{B}^*-\mathbf{L}^*)^2 \\ +(\mathbf{A}^*-\mathbf{L}^*)(\mathbf{K}^*+\mathbf{L}^*)+(\mathbf{B}^*-\mathbf{L}^*)(\mathbf{K}^*+\mathbf{L}^*)+Z^*\big).$$

Comparing (31), (32) we find that

$$(33)\qquad (G^*,\ Z^*) \equiv \big(G^*,\ K^*-(\mathbf{L}^{*2})+H^*\big).$$

Now (5) Z^* differs from K^* by a surface, say U^*, lying on $\mathbf{L}^*$. Since $(\mathbf{L}^{*2}) \equiv (\mathbf{L}^*,\ \mathbf{A}^*-\mathbf{A}_0^*)$ and since $(\mathbf{L}^*,\ \mathbf{A}^*)$ is ruled by generators of $\mathbf{L}^*$ while $(\mathbf{L}^*,\ \mathbf{A}_0^*)$ is unisecant to the generators of $\mathbf{L}^*$ because $\mathbf{A}_0$ passes simply through F, $-(\mathbf{L}^{*2})$ is a directrix surface on $\mathbf{L}^*$. Therefore, if U^* meets a general generator of $\mathbf{L}^*$ in u points, by an equivalence criterion of Severi (6) we have

$$U^* \equiv -u(\mathbf{L}^{*2})+W^*,\ \text{on}\ \mathbf{L}^*,$$

where W^* is ruled by generators of $\mathbf{L}^*$; and so

$$Z^* \equiv K^*+U^* \\ \equiv K^*-u(\mathbf{L}^{*2})+W^*,$$

whence

$$(34)\qquad (G^*,\ Z^*) \equiv \big(G^*,\ K^*-u(\mathbf{L}^{*2})+W^*\big).$$

By (33), (34), we have then for all surfaces G^*,

$$\big(G^*,\ H^*-(\mathbf{L}^{*2})\big) \equiv \big(G^*,\ W^*-u(\mathbf{L}^{*2})\big).$$

From the enumerative content of this formula or from the fact that $-(\mathbf{L}^{*2})$ is independent of any surface ruled on $\mathbf{L}^*$, we have $u = 1$ and, for all G^*,

$$(G^*,\ H^*) \equiv (G^*,\ W^*).$$

Let $\mathfrak{W}$ be the curve on F corresponding to the ruled surface W^*; then the last equivalence shows that, for all curves $\mathfrak{C}$,

$$(\mathfrak{C},\ \mathfrak{K}_F)_F \equiv (\mathfrak{C},\ \mathfrak{W})_F.$$

For given systems $|\mathbf{A}|$, $|\mathbf{B}|$, the curves $\mathfrak{C}$ have no fundamental curves; hence by another equivalence criterion of Severi (7),

$$\mathfrak{K}_F \equiv \mathfrak{W} \text{ on } F;$$

this implies that $$H^* \equiv W^* \text{ on } \mathbf{L}^*.$$

Thus

$$Z^* \equiv K^* - (\mathbf{L}^{*2}) + H^*. \tag{35}$$

10. Having reached this stage we can discard the restriction $a = b = 1$, made in § 8, and calculate κ_G in the general case.

G^* is now the total intersection on V_4^* of $\mathbf{A}_0^* \equiv \mathbf{A}^* - a\mathbf{L}^*$ and $\mathbf{B}_0^* \equiv \mathbf{B}^* - b\mathbf{L}^*$. By (30),

$$\begin{aligned}\kappa_{G^*} \equiv &\left(G^*,\ (\mathbf{A}^*-a\mathbf{L}^*)^2+(\mathbf{A}^*-a\mathbf{L}^*)(\mathbf{B}^*-b\mathbf{L}^*)+(\mathbf{B}^*-b\mathbf{L}^*)^2\right)\\ &+\left(G^*,\ (\mathbf{A}^*-a\mathbf{L}^*)(\mathbf{K}^*+\mathbf{L}^*)+(\mathbf{B}^*-b\mathbf{L}^*)(\mathbf{K}^*+\mathbf{L}^*)\right.\\ &\qquad\left.+K^*-(\mathbf{L}^{*2})+H^*\right)\\ \equiv &\,(G^*,\ \mathbf{A}^{*2}+\mathbf{A}^*\mathbf{B}^*+\mathbf{B}^{*2}+\mathbf{A}^*\mathbf{K}^*+\mathbf{B}^*\mathbf{K}^*+K^*)+(G^*,\ H^*)\\ &-\left(G^*,\ \{(2a+b-1)\mathbf{A}^*+(a+2b-1)\mathbf{B}^*+(a+b)\mathbf{K}^*,\ \mathbf{L}^*\}\right)\\ &\qquad+(a^2+ab+b^2-a-b-1)(G^*,\ \mathbf{L}^{*2}).\end{aligned}$$

Therefore

$$\begin{aligned}\kappa_G \equiv &\,(G,\ \mathbf{A}^2+\mathbf{AB}+\mathbf{B}^2+\mathbf{AK}+\mathbf{BK}+K)+(\mathfrak{C},\ \mathfrak{K}_F)_F\\ &+(a^2+ab+b^2-a-b-1)(\mathfrak{C}^2)_G\\ &-\left(\mathfrak{C},\ (2a+b-1)\mathbf{A}+(a+2b-1)\mathbf{B}+(a+b)\mathbf{K}\right).\end{aligned} \tag{36}$$

For the evaluation of $(\mathfrak{K}_G^2)_G$ and of κ_G we need an expression for $(\mathfrak{C}^2)_G$. This is obtained as follows: on V_4^*,

$$\begin{aligned}ab(G^*,\ \mathbf{L}^{*2}) &\equiv (G^*,\ \{\mathbf{A}^*-a\mathbf{L}^*,\ \mathbf{B}^*-b\mathbf{L}^*\}-\mathbf{A}^*\mathbf{B}^*+\{b\mathbf{A}^*+a\mathbf{B}^*,\ \mathbf{L}^*\})\\ &\equiv (G^*,\ \mathbf{A}_0^*\mathbf{B}_0^*)-(G^*,\ \mathbf{A}^*\mathbf{B}^*)+(G,\ \{b\mathbf{A}^*+a\mathbf{B}^*,\ \mathbf{L}^*\}).\end{aligned}$$

Hence, on V_4,

$$ab(\mathfrak{C}^2)_G \equiv (G,\ \mathbf{A}_0\mathbf{B}_0)-(G,\ \mathbf{AB})+(\mathfrak{C},\ b\mathbf{A}+a\mathbf{B}).$$

$(G,\ \mathbf{A}_0\mathbf{B}_0)$ is the value taken by β in (23) when we replace $\mathbf{C}$, $\mathbf{D}$, c, d respectively by $\mathbf{A}$, $\mathbf{B}$, a, b; hence

$$(37)\quad (G,\ \mathbf{A}_0\mathbf{B}_0)\equiv(\mathbf{A}^2\mathbf{B}^2)+2ab(\mathfrak{K}_F,\ b\mathbf{A}+a\mathbf{B}+ab\mathbf{K})+a^2b^2\{(F^2)-(\mathfrak{K}_F{}^2)_F\}$$
$$-(F,\ b^2\mathbf{A}^2+4ab\mathbf{A}\mathbf{B}+a^2\mathbf{B}^2+2ab\{b\mathbf{A}+\mathbf{B},\ \mathbf{K}\}+a^2b^2\mathbf{K}^2),$$

$$(38)\quad -(G,\ \mathbf{A}\mathbf{B})\equiv-(\mathbf{A}^2\mathbf{B}^2)+ab(F,\ \mathbf{A}\mathbf{B}).$$

From (18),

$$(\mathfrak{C},\ b\mathbf{A}+a\mathbf{B})\equiv(F,\ b^2\mathbf{A}^2+2ab\mathbf{A}\mathbf{B}+a^2\mathbf{B}^2+ab\{b\mathbf{A}+a\mathbf{B},\ \mathbf{K}\})$$
$$-ab(\mathfrak{K}_F,\ b\mathbf{A}+a\mathbf{B}).$$

Therefore

$$(39)\quad ab(\mathfrak{C}^2)_G\equiv-ab(F,\ \mathbf{A}\mathbf{B}+b\mathbf{A}\mathbf{K}+a\mathbf{B}\mathbf{K}+ab\mathbf{K}^2)+ab(\mathfrak{K}_F,\ b\mathbf{A}+a\mathbf{B}+2ab\mathbf{K})$$
$$+a^2b^2\{(F^2)-(\mathfrak{K}_F{}^2)_F\}.$$

It will be noticed that the factor ab is common to all the terms in this equivalence. The equivalence will not necessarily persist if we divide by ab: but the factor can be dropped if only an enumerative result is wanted.

From (24) we have

$$(40)\quad (\mathfrak{K}_G{}^2)_G\equiv(G,\ \{\mathbf{A}+\mathbf{B}+\mathbf{K}\}^2)-2(a+b-1)(\mathfrak{C},\ \mathbf{A}+\mathbf{B}+\mathbf{K})$$
$$+(a+b-1)^2(\mathfrak{C}^2)_G.$$

It is now a simple, if lengthy, matter to evaluate $ab(\mathfrak{K}_G{}^2)_G$ and $ab\kappa_G$. The results are

$$(41)\quad ab(\mathfrak{K}_G{}^2)_G\equiv ab\Big[(\mathbf{A}\mathbf{B},\ \{\mathbf{A}+\mathbf{B}+\mathbf{K}\}^2)-\big((2a+2b-1)^2-a^2-b^2\big)(F,\ \mathbf{A}\mathbf{B})$$
$$-b(3a+2b-2)(F,\ \mathbf{A}^2)-a(2a+3b-2)(F,\ \mathbf{B}^2)$$
$$-b\big((2a+b)^2-a^2-1\big)(F,\ \mathbf{A}\mathbf{K})-a\big((a+2b)^2-b^2-1\big)(F,\ \mathbf{B}\mathbf{K})$$
$$-ab(a+b)^2(F,\ \mathbf{K}^2)+ab(a+b-1)^2\{(F^2)-(\mathfrak{K}_F{}^2)_F\}$$
$$+(a+b-1)\big(\mathfrak{K}_F,\ b(3a+b-1)\mathbf{A}+a(a+3b-1)\mathbf{B}+2ab(a+b)\mathbf{K}\big)\Big],$$

$$(42)\quad ab\kappa_G\equiv ab\Big[(\mathbf{A}\mathbf{B},\ \mathbf{A}^2+\mathbf{A}\mathbf{B}+\mathbf{B}^2+\mathbf{A}\mathbf{K}+\mathbf{B}\mathbf{K}+K)-\big((a+b)^2+1\big)(F,\ \mathbf{A}\mathbf{B})$$
$$-b(3a+b-1)(F,\ \mathbf{A}^2)-a(a+3b-1)(F,\ \mathbf{B}^2)$$
$$-b\big((a+1)^2-(b-1)^2+1\big)(F,\ \mathbf{A}\mathbf{K})-a\big(-(a-1)^2+(b+1)^2+1\big)(F,\ \mathbf{B}\mathbf{K})$$
$$-ab(-a^2-ab-b^2+2a+2b+1)(F,\ \mathbf{K}^2)-ab(F,\ K)$$
$$+ab\big(\mathfrak{K}_F,\ (2a+b-1)\mathbf{A}+(a+2b-1)\mathbf{B}+(a+b)\mathbf{K}\big)+(\mathfrak{K}_F,\ b\mathbf{A}+a\mathbf{B}+ab\mathbf{K})$$
$$+ab(a^2+ab+b^2-a-b-1)(F^2)-ab(a^2+ab+b^2-a-b)(\mathfrak{K}_F{}^2)_F\Big].$$

The surface F considered on $\mathbf{A}_0$.

11. We denote by F_0 the surface *on* $\mathbf{A}_0$ which is in $(a, 1)$ correspondence with F. There is a curve $\mathfrak{B}$ on F along which two of the tangent solids to $\mathbf{A}_0$ coincide; and on $\mathfrak{B}$ a set η of points at which three tangent solids coincide. In regard to the correspondence between F and F_0, $\mathfrak{B}$ is the branch curve and η is the set of triple branch points. We now find equivalences for $\mathfrak{B}$ and η.

First we need to show that, if $\mathbf{A}_0$ varies in the restricted linear system $|\mathbf{A}_0|$, which has F as base surface to virtual multiplicity a, then both $\mathfrak{B}$ and η vary in systems of equivalence on F. The method follows the lines of § 3, and we refer to the V_4^* already mentioned in § 6.

$\mathbf{A}_0^*$ meets $\mathfrak{L}^*$ in a set of a points and the locus of these, as $\mathfrak{L}^*$ varies in $\mathbf{L}^*$, is a surface L^* in $(1, 1)$ correspondence with F_0. In general, for $a > 1$, $\mathbf{A}_0$ has no point on F to multiplicity greater than a, so that L^* then contains no line $\mathfrak{L}^*$; but, when $a = 1$, $\mathbf{A}_0$ has a set $\nu_{\mathbf{A}_0}$ of double points on F and each of these gives rise to a line $\mathfrak{L}^*$ on L^*; the effect of this possibility is noted in § 12.

To $\mathfrak{B}$ corresponds the ruled surface B^* whose generators are the lines $\mathfrak{L}^*$ touching L^*; and to η corresponds the set of generators $\{\mathfrak{L}^*\}$ which have three point contact with L^*. Let $\mathfrak{M}^*$ be the curve of contact of B^* with L^*, and let μ^* be the set of points on $\mathfrak{M}^*$ at which the set of lines $\{\mathfrak{L}^*\}$ touch L^*.

Let $\mathbf{A}_0'$, $\mathbf{A}_0''$ be two positions of $\mathbf{A}_0$ and let dashes indicate the corresponding positions of the other loci involved. As $\mathbf{A}_0^*$ varies in the pencil determined by $\mathbf{A}_0'^*$ and $\mathbf{A}_0''^*$, the surface L^* varies in a pencil; let M^* be the locus of all the corresponding curves $\mathfrak{M}^*$. These curves form on M^* a rational ∞^1 system and are therefore (8) equivalent on M^*. In the correspondence between F and M^*, in which with each point of M^* is associated the line $\mathfrak{L}^*$ touching a surface L^* there and so is associated a point of F, $\mathfrak{B}'$ corresponds to $\mathfrak{M}'^*$ and $\mathfrak{B}''$ to $\mathfrak{M}''^*$. In this correspondence equivalence is preserved (9), so that $\mathfrak{B}' \equiv \mathfrak{B}''$. Hence the curves $\mathfrak{B}$ vary in a system of equivalence.

Again, as $\mathbf{A}_0$ varies in the pencil mentioned, the sets μ^* describe a curve $\mathfrak{H}^*$ on M^*. These sets form on $\mathfrak{H}^*$ an ∞^1 rational involutory series and are therefore equivalent. Consequently the corresponding sets η are equivalent on the curve $\mathfrak{H}$ on F whose points arise from the lines $\mathfrak{L}^*$ which have three point contact with an L^* at a point of $\mathfrak{H}^*$; these sets η are therefore also equivalent on F.

12. Now consider the decomposition

$$\mathbf{A}_0 \equiv \mathbf{A}_{0,1} + \ldots + \mathbf{A}_{0,a}.$$

One position for L^* is then a set of a surfaces each unisecant to the lines $\mathfrak{l}^*$ and each containing a finite number of these lines, arising from the double points of the $\mathbf{A}_{0,i}$ on F. B^* becomes twice the aggregate of lines $\mathfrak{l}^*$ which pass through the curves common to these component surfaces; and $\{\mathfrak{l}^*\}$ becomes six times the set of lines $\mathfrak{l}^*$ which pass through the sets of points common to these component surfaces, taken in threes, plus the set of lines $\mathfrak{l}^*$ lying on these components, counted a certain number h of times. We shall show in a moment that $h = 3(a-2)$; and, as a matter of interest, we give an independent analytical treatment in §14 to show that these entities should be counted as often as stated.

It follows that, if

$$(\mathbf{A}_{0,l},\ \mathbf{A}_{0,m}) = F+G_{lm}, \qquad (F,\ G_{lm}) = \mathfrak{C}_{lm},$$

$$(\mathbf{A}_{0,l}\mathbf{A}_{0,m}\mathbf{A}_{0,n}) = F+\mathfrak{D}_{lmn}, \quad (F,\ \mathfrak{D}_{lmn}) = \alpha_{lmn},$$

where $l, m, n = 1, 2, \ldots, a$ and $(m-n)(n-l)(l-m) \neq 0$, we have

$$\mathfrak{B} \equiv 2 \sum_{l,m} \mathfrak{C}_{lm}, \quad \eta \equiv 6 \sum_{l,m,n} \alpha_{lmn} + h \sum_{l} \nu_{A0,l}. \tag{43}$$

By (18), (22),

$$\mathfrak{C}_{lm} \equiv (F,\ \mathbf{A}_l+\mathbf{A}_m+\mathbf{K})-\mathfrak{K}_F,$$

$$\alpha_{lmn} \equiv (F,\ \mathbf{A}_m\mathbf{A}_n+\mathbf{A}_n\mathbf{A}_l+\mathbf{A}_l\mathbf{A}_m+\{\mathbf{A}_l+\mathbf{A}_m+\mathbf{A}_n,\ \mathbf{K}\}+\mathbf{K}^2)$$
$$-(\mathfrak{K}_F,\ \mathbf{A}_l+\mathbf{A}_m+\mathbf{A}_n+2\mathbf{K})-(F^2)+(\mathfrak{K}_F{}^2)_F;$$

also [Todd, **1**, formula (30)]

$$\nu_{A0,l} \equiv (F,\ \mathbf{A}_l{}^2+\mathbf{A}_l\mathbf{K})+(F^2)-(\mathfrak{K}_F,\ \mathbf{A}_l),$$

whence

$$\mathfrak{B} \equiv (a-1)\{2(F,\ \mathbf{A})+a(F,\ \mathbf{K})-a\mathfrak{K}_F\} \tag{44}$$

and

$$\eta \equiv 3(a-2)(F,\ \mathbf{A}^2-\mathbf{A}_1{}^2-\mathbf{A}_2{}^2-\ldots-\mathbf{A}_a{}^2)$$
$$+3(a-1)(a-2)(F,\ \mathbf{AK})+a(a-1)(a-2)(F,\ \mathbf{K}^2)$$
$$-(a-1)(a-2)(\mathfrak{K}_F,\ 3\mathbf{A}+2a\mathbf{K})-a(a-1)(a-2)\{(F^2)-(\mathfrak{K}_F{}^2)_F\}$$
$$+h\{(F,\ \mathbf{A}_1{}^2+\mathbf{A}_2{}^2+\ldots+\mathbf{A}_a{}^2+\mathbf{AK})+a(F^2)-(\mathfrak{K}_F,\ \mathbf{A})\}.$$

This expression for η must be independent of the particular decomposition chosen and this is the case only when $h = 3(a-2)$, which is then the value

to be taken for h. Thus

$$(45)\quad \eta \equiv (a-2)\{(F,\ 3\mathbf{A}^2+3a\mathbf{K}\mathbf{A}+a(a-1)\mathbf{K}^2)-(\mathfrak{K}_F,\ 3\mathbf{A}+2a(a-1)\mathbf{K}) -a(a-4)(F^2)+a(a-1)(\mathfrak{K}_F{}^2)_F\}.$$

13. From the correspondence between F_0 and F, Severi's analogue of Zeuthen's formula gives

$$\mathfrak{K}_{F_0} \equiv a\mathfrak{K}_F+\mathfrak{B},$$

so that

$$(46)\qquad \mathfrak{K}_{F_0} \equiv (a-1)(F,\ 2\mathbf{A}+a\mathbf{K})-a(a-2)\,\mathfrak{K}_F.$$

And we have (10)

$$\kappa_{F_0} \equiv a\kappa_F+\kappa_{\mathfrak{B}}-\eta.$$

Now

$$\kappa_{\mathfrak{B}} \equiv (\mathfrak{B}^2)_F+(\mathfrak{B},\ \mathfrak{K}_F)_F$$

$$\equiv(\{\mathfrak{B}+a(a-1)\,\mathfrak{K}_F\}^2)_F-(2a^2-2a-1)(\mathfrak{B}+a(a-1)\,\mathfrak{K}_F,\ \mathfrak{K}_F)_F +a(a-1)(a^2-a-1)(\mathfrak{K}_F{}^2)_F,$$

$$(47)\quad \kappa_{\mathfrak{B}} \equiv (a-1)^2(F,\ \{2\mathbf{A}+a\mathbf{K}\}^2)-(2a^2-2a-1)(a-1)(\mathfrak{K}_F,\ 2\mathbf{A}+a\mathbf{K}) +a(a-1)(a^2-a-1)(\mathfrak{K}_F{}^2)_F.$$

Hence

$$(48)\quad \kappa_{F_0} \equiv a\kappa_F+(4a^2-11a+10)(F,\ \mathbf{A}^2)+a(4a^2-11a+10)(F,\ \mathbf{A}\mathbf{K}) +a(a-1)(a^2-2a+2)(F,\ \mathbf{K}^2)-(a-1)(2a^2-2a-1)(\mathfrak{K}_F,\ 2\mathbf{A}+a\mathbf{K}) +(a-2)(\mathfrak{K}_F,\ 3\mathbf{A}+2a(a-1)\mathbf{K})+a(a-1)^3(\mathfrak{K}_F{}^2)_F+a(a-2)(a-4)(F^2).$$

14. We now give an analytical investigation of the statement in §12 that

$$\mathfrak{B} \equiv p\sum_{l,m}\mathfrak{C}_{lm},\quad \eta = q\sum_{l,m,n}a_{lmn}+r\sum_{l}\nu_{A_0,l},$$

where $p=2$, $q=6$, $r=3(a-2)$.

The values of p, q, r do not depend on the nature of V_4 and the surface F, so we may suppose, for our purpose, that V_4 is a linear space, in which x, y, z, u, v, are the coordinates, and that F is a plane, which we may take to be $u=v=0$.

The equation of a threefold A_0 with F to multiplicity a and of order n is of the form

$$(u,\ v)_a+(u,\ v)_{a+1}+\ldots+(u,\ v)_n=0,$$

where $(u,\ v)_m$ is a homogeneous polynomial of order m in u, v, with coefficients which are homogeneous polynomials in x, y, z, each of order $n-m$. Let

$$(u,\ v)_a\equiv c_0(x,\ y,\ z)\,u^a+c_1(x,\ y,\ z)\,u^{a-1}v+\ldots+c_a(x,\ y,\ z)\,v^a.$$

Then the tangent solids to A_0 at a point $(x_1,\ y_1,\ z_1,\ 0,\ 0)$ of F are given by

$$0=c_0(x_1,\ y_1,\ z_1)\,u^a+c_1(x_1,\ y_1,\ z_1)\,u^{a-1}v+\ldots+c_a(x_1,\ y_1,\ z_1)\,v^a.$$

The points of $\mathfrak{B}$ are those whose coordinates x, y, z make equal two roots of the equation in θ:

$$0=c_0\theta^a+c_1\theta^{a-1}+\ldots+c_a; \tag{49}$$

and the equation of $\mathfrak{B}$ is therefore obtained by eliminating θ between (49) and

$$0=ac_0\theta^{a-1}+(a-1)\,c_1\theta^{a-2}+\ldots+c_{a-1}. \tag{50}$$

The resultant is an equation of order $2a-2$ in the c_i; hence $\mathfrak{B}$ is a curve of order $(2a-2)(n-a)$.

From another point of view, the order of $\mathfrak{B}$ is $k_1(n-a)$, where k_1 is the class of the system of primes which touch a fixed rational normal curve $\mathfrak{Y}$ in a space $[a]$. Primes through an $[a-2]$ cut a g_a^1 on $\mathfrak{Y}$, so that $2a-2$ of them touch $\mathfrak{Y}$, *i.e.* $k_1=2a-2$.

The points of η are those for which (49) has three equal roots. The number of points in η is therefore $k_2(n-a)^2$, where k_2 is the class of the system of primes which have three-point contact with $\mathfrak{Y}$. Primes through an $[a-3]$ cut a g_a^2 on $\mathfrak{Y}$, so that $3(a-2)$ of them have three point contact with $\mathfrak{Y}$, *i.e.* $k_2=3(a-2)$. Thus η is a set of $3(a-2)(n-a)^2$ points.

We have next to find the composition of $\mathfrak{B}$ and η when $(u,\ v)_a$ takes the form

$$(s_1u+t_1v)(s_2u+t_2v)\ldots(s_au+t_av), \tag{51}$$

where s_i, t_i are homogeneous polynomials in x, y, z of order n_i-1, and $n=n_1+n_2+\ldots+n_a$; these factors correspond to the threefolds $A_{0,1}$, $A_{0,2}$, ..., $A_{0,a}$.

The points of $\mathfrak{B}$ are now those whose coordinates make two factors of (51) proportional or one factor identically zero. Their locus is thus the set of curves $\mathfrak{C}_{lm}$ given by

$$s_l t_m - s_m t_l = 0; \quad l \neq m; \quad l, m = 1, \ldots, a.$$

The aggregate of these curves $\mathfrak{C}_{lm}$ is of order

$$\sum_{l,m} (n_l + n_m - 2) = (a-1)(n-a).$$

Each curve contributes an equal number of times to $\mathfrak{B}$, which we found to have the order $(2a-2)(n-a)$; hence $p = 2$.

The double points of $A_{0,l}$ on F are given by $s_l = t_l = 0$ and are therefore $(n_l-1)^2$ in number. The coordinates of each of these points make every c_i zero and so, for such points, equation (49) and its first two derivatives with regard to θ are all satisfied identically. Thus each set $\nu_{A_{0,l}}$ contributes (an equal number r of times) to η.

The remaining points of η are those whose coordinates make three factors of (51) proportional. They are given by the sets of simultaneous equations

$$\left.\begin{aligned} s_m t_n - s_n t_m &= 0 \\ s_n t_l - s_l t_n &= 0 \\ s_l t_m - s_m t_l &= 0 \end{aligned}\right\} \quad \begin{aligned} &(m-n)(n-l)(l-m) \neq 0, \\ &l, m, n = 1, \ldots, a. \end{aligned}$$

The number of points in such a set a_{lmn} is

$$(n_l+n_m-2)(n_l+n_n-2)-(n_l-1)^2$$
$$= (n_m n_n + n_n n_l + n_l n_m) - 2(n_l+n_m+n_n) + 3.$$

Every point arising in this way contributes an equal number q of times to η.

Hence, for every partition of n into $n_1+\ldots+n_a$,

$$3(a-2)(n-a)^2$$
$$\equiv q\left\{\sum_{l,m,n}(n_m n_n + n_n n_l + n_l n_m) - 2\sum_{l,m,n}(n_l+n_m+n_n) + \tfrac{1}{2}a(a-1)(a-2)\right\}$$
$$+ r\left\{\sum_l n_l^2 - 2\sum n_l + a\right\}$$
$$\equiv q\left\{\tfrac{1}{2}(a-2)\left(n^2 - \sum_l n_l^2\right) - (a-1)(a-2)n + \tfrac{1}{2}a(a-1)(a-2)\right\}$$
$$+ r\left\{\sum_l n_l^2 - 2n + a\right\}.$$

Therefore $$q = 6, \quad r = 3(a-2).$$

The case where F has improper nodes.

15. Suppose now that F has improper nodes at the set δ on V_4, and denote by σ the set of places on F at δ. On V_4, $\sigma \equiv 2\delta$.

We transform V_4 birationally into a V_4' by means of primals of sufficiently high order which pass simply through δ. To each point ϵ_i of δ corresponds on V_4' a solid $\mathbf{E}_i'$, these solids being mutually skew. If dashes denote homologues on V_4', the canonical threefolds on V_4' are equivalent to $\mathbf{K}'+3\sum_i \mathbf{E}_i'$, and the canonical surfaces to $K'+2\sum_i (\mathbf{E}_i'^2)$ [Todd, 4, formula (12)].

The reduced surface F', corresponding to F, is free from improper double points and meets $\mathbf{E}_i'$ in a pair of skew lines $\mathfrak{C}_{i,1}'$, $\mathfrak{C}_{i,2}'$ which arise from the two places on F at ϵ_i. The canonical curves on F are equivalent to $\mathfrak{K}_F'+\sum_i(\mathfrak{C}_{i,1}'+\mathfrak{C}_{i,2}')$.

A threefold $\mathbf{A}_0$ has a $2a$-ple point at each ϵ_i and is transformed into a reducible threefold consisting of each $\mathbf{E}_i'$ counted $2a$ times together with threefold from the system $\left|\mathbf{A}'-2a\sum_i \mathbf{E}_i'\right|$ prescribed to pass a times through F'.

It will simplify the expressions to follow if we put

$$\mathbf{E}' \equiv \sum_i \mathbf{E}_i', \quad \mathfrak{C}' \equiv \sum_i (\mathfrak{C}_{i,1}'+\mathfrak{C}_{i,2}').$$

Then it is easily verified that the sets

$$-(\mathbf{E}'^4), \quad -(F', \mathbf{E}'^2) = -(\mathfrak{C}', \mathbf{E}'), \quad -(\mathfrak{C}'^2)_{F'}, \quad (F'^2)$$

on V_4' transform back into the sets

$$\delta, \quad 2\delta, \quad 2\delta, \quad (F^2)-4\delta$$

on V_4 respectively.

16. We are now in a position to extend the formulae obtained for the case of a surface F without improper nodes.

Take (20) first. A canonical set on $\mathfrak{C}$ transforms into a canonical set

on the reduced curve $\mathfrak{C}'$. Applying (20) to $\mathfrak{C}'$, we have

$$\begin{aligned}\kappa_{\mathfrak{G}'} \equiv & \Big(F',\ \{b(\mathbf{A}'-2a\mathbf{E}')+a(\mathbf{B}'-2b\mathbf{E}')+ab(\mathbf{K}'+3\mathbf{E}')\}^2\Big)\\ & -(2ab-1)\Big(\mathfrak{R}_{F'}+\mathfrak{E}',\ b(\mathbf{A}'-2a\mathbf{E}')+a(\mathbf{B}'-2b\mathbf{E}')+ab(\mathbf{K}'+3\mathbf{E}')\Big)\\ & +ab(ab-1)(\{\mathfrak{R}_{F'}+\mathfrak{E}'\}^2)_{F'}\\ \equiv & (F',\ \{b\mathbf{A}'+a\mathbf{B}'+ab\mathbf{K}'\}^2)-(2ab-1)(\mathfrak{R}_{F'},\ b\mathbf{A}'+a\mathbf{B}'+ab\mathbf{K}')\\ & +ab(ab-1)(\mathfrak{R}_{F'}^2)_{F'}+a^2b^2(F',\ \mathbf{E}'^2)+ab(2ab-1)(\mathfrak{E}',\ \mathbf{E}')\\ & +ab(ab-1)(\mathfrak{C}'^2)_{F'}.\end{aligned}$$

Therefore, on V_4,

$$\begin{aligned}(20')\quad \kappa_{\mathfrak{G}} \equiv & (F,\ \{b\mathbf{A}+a\mathbf{B}+ab\mathbf{K}\}^2)-(2ab-1)(\mathfrak{R}_F,\ b\mathbf{A}+a\mathbf{B}+ab\mathbf{K})\\ & +ab(ab-1)(\mathfrak{R}_F{}^2)_F-4ab(2ab-1)\delta.\end{aligned}$$

This example shows clearly enough how the other formulae may be extended. We need to bear in mind that G has $2ab$ places at ϵ_i so that the reduced transform G' of G meets E_i' in $2ab$ skew lines $\mathfrak{G}_{i,1}', \ldots, \mathfrak{G}_{i,2ab}'$ which are exceptional on G'. The canonical curves on G' are equivalent to $\mathfrak{R}_G'+\sum_{i,j}\mathfrak{G}_{i,j}'$ and the canonical sets to $\kappa_G'-\sum_{i,j}(\mathfrak{G}_{i,j}'^2)_{G'}$. The main results for the general case are easily found to be

$$\begin{aligned}(23')\quad \beta \equiv & (\mathbf{ABCD})+(\mathfrak{R}_F,\ bcd\mathbf{A}+acd\mathbf{B}+abd\mathbf{C}+abc\mathbf{D}+2abcd\mathbf{K})\\ & -(F,\ ad\mathbf{BC}+bd\mathbf{CA}+cd\mathbf{AB}+bc\mathbf{AD}+ca\mathbf{BD}+ab\mathbf{CD})\\ & -(F,\ \{bcd\mathbf{A}+acd\mathbf{B}+abd\mathbf{C}+abc\mathbf{D},\ \mathbf{K}\}+abcd\mathbf{K}^2)\\ & +abcd\{(F^2)-(\mathfrak{R}_F{}^2)_F+4\delta\},\end{aligned}$$

$$\begin{aligned}(27')\quad \kappa_{\mathfrak{D}} \equiv & (\mathbf{ABC},\ \mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{K})\\ & +\Big(\mathfrak{R}_F,\ bc(2a+b+c-1)\,\mathbf{A}+ca(a+2b+c-1)\,\mathbf{B}\\ & \qquad +ab(a+b+2c-1)\,\mathbf{C}+abc(2a+2b+2c-1)\,\mathbf{K}\Big)\\ & -\Big(F,\ a(a+2b+2c-1)\,\mathbf{BC}+b(2a+b+2c-1)\,\mathbf{CA}\\ & \qquad +c(2a+2b+c-1)\,\mathbf{AB}+bc\mathbf{A}^2+ca\mathbf{B}^2+ab\mathbf{C}^2\Big)\\ & -\Big(F,\ \{\mathbf{K},\ bc(2a+b+c)\,\mathbf{A}+ca(a+2b+c)\,\mathbf{B}\\ & \qquad +ab(a+b+2c)\,\mathbf{C}+abc(a+b+c)\,\mathbf{K}\}\Big)\\ & +abc(a+b+c-1)\{(F^2)-(\mathfrak{R}_F{}^2)_F+4\delta\},\end{aligned}$$

$$
\begin{aligned}
(41') \quad & ab(\mathfrak{K}_G{}^2)_G \\
& \equiv ab\Big[(\mathbf{AB},\ \{\mathbf{A}+\mathbf{B}+\mathbf{K}\}^2)-\big((2a+2b-1)^2-a^2-b^2\big)(F,\ \mathbf{AB}) \\
& \quad -b(3a+2b-2)(F,\ \mathbf{A}^2)-a(2a+3b-2)(F,\ \mathbf{B}^2) \\
& \quad -b\big((2a+b)^2-a^2-1\big)(F,\ \mathbf{AK})-a\big((a+2b)^2-b^2-1\big)(F,\ \mathbf{BK}) \\
& \quad -ab(a+b)^2(F,\ \mathbf{K}^2) \\
& \quad +(a+b-1)\big(\mathfrak{K}_F,\ b(3a+b-1)\,\mathbf{A}+a(a+3b-1)\,\mathbf{B}+2ab(a+b)\,\mathbf{K}\big) \\
& \quad +ab(a+b-1)^2\{(F^2)-(\mathfrak{K}_F{}^2)_F+4\delta\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
(42') \quad & ab\kappa_G \\
& \equiv ab\Big[(\mathbf{AB},\ \mathbf{A}^2+\mathbf{AB}+\mathbf{B}^2+\mathbf{AK}+\mathbf{BK}+K)-\big((a+b)^2+1\big)(F,\ \mathbf{AB}) \\
& \quad -b(3a+b-1)(F,\ \mathbf{A}^2)-a(a+3b-1)(F,\ \mathbf{B}^2) \\
& \quad -b\big((a+1)^2-(b-1)^2+1\big)(F,\ \mathbf{AK}) \\
& \quad -a\big(-(a-1)^2+(b+1)^2+1\big)(F,\ \mathbf{BK}) \\
& \quad +ab(a^2+ab+b^2-2a-2b-1)(F,\ \mathbf{K}^2)-ab(F,\ K) \\
& \quad +ab\big(\mathfrak{K}_F,\ (2a+b-1)\,\mathbf{A}+(a+2b-1)\,\mathbf{B}+(a+b)\,\mathbf{K}\big) \\
& \qquad\qquad +(\mathfrak{K}_F,\ b\mathbf{A}+a\mathbf{B}+ab\mathbf{K}) \\
& \quad +ab(a^2+ab+b^2-a-b-1)\{(F^2)-(\mathfrak{K}_F{}^2)_F+4\delta\}-ab\{(\mathfrak{K}_F{}^2)_F-6\delta\}\Big].
\end{aligned}
$$

Note on the invariants of F.

17. It follows immediately from the formulae given above, although we need invoke only the case $a=b=1$, that

$$
(52) \qquad (\mathfrak{K}_F{}^2)_F+(F,\ K)+2\delta\equiv(F^2)+(\mathfrak{K}_F,\ \mathbf{K})+\kappa_F.
$$

To see this we interchange the roles of F, G and evaluate the main terms of the equivalence by the formulae.

When V_4 is a linear space, K is a surface of order 10 and $-\mathbf{K}$ a threefold of order 5 (Todd, **1**, § 12). If, with the usual notation, F has order μ_0, then $\mathfrak{K}_F$ has order $\mu_1-3\mu_0$, and we find that

$$
(53) \qquad p^{(2)}-(I+4)=\mu_0{}^2+5\mu_0-5\mu_1-2d.
$$

References.

1. J. A. Todd, "Intersections of loci on an algebraic V_4", *Proc. Cambridge Phil. Soc* 33 (1937), 425.
2. B. Segre, "Nuovi contributi alla geometria sulle varietà algebriche", *Mem. R. Acc. d'Italia*, 5 (1934), 479. The method by which we obtain formula (16) here provides an alternative derivation of Segre's formula [39].
3. For the definitions and properties of these canonical loci see, *inter alia*, J. A. Todd, "The geometrical invariants of algebraic loci", *Proc. London Math. Soc.* (2), 43 (1937), 127, and 45 (1939), 410.
4. J. A. Todd, "Birational transformations with isolated fundamental points", *Proc. Edinburgh Math. Soc.* (2), 5 (1937), 117.
5. The argument is given, *mutatis mutandis*, by J. A. Todd, "Birational transformations possessing fundamental curves", *Proc. Cambridge Phil. Soc.* 34 (1938), 144.
6. F. Severi, "Osservazioni varie di geometria sopra una superficie algebrica e sopra una varietà", *Atti Veneto*, 65 (1906), 633; and "Alcune relazioni di equivalenza tra gruppi di punti d'una curva algebrica o tra curve di una superficie", *Atti Veneto*, 70 (1911), 373.
7. Reference **6**; for equivalence criteria on surfaces see also F. Enriques, *Teoria delle superficie algebriche, I* (Padova, 1932), §41.
8. F. Enriques *loc. cit.* **7**, §8.
9. F. Severi, "Sulle relazione che legano i caratteri invarianti di due superficie in corrispondenza algebrica", *Rend. Ist. Lombardo* (2), 36 (1903), 498.
10. F. Severi, *loc. cit.* **9**; and B. Segre, "Quelques résultats nouveaux dans la géométrie sur une V_3 algébrique", *Mém. Acad. Roy. Belgique*, 14 (1936), 35.

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NOTES ON A NET OF QUADRIC SURFACES: (IV) COMBINANTAL COVARIANTS OF LOW ORDER

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1. Some of the combinantal covariants of a net $\mathfrak{N}$ of quadric surfaces have already been discussed in this series of Notes*; they are all of order eight or more, those actually of order eight being

R^8: the scroll of trisecants of the Jacobian curve;

E^8: the locus of equianharmonic base curves of pencils of quadrics belonging to $\mathfrak{N}$;

and ρ^8: a scroll generated by quadritangent lines of R^8.

To these we may add, from an earlier paper†, a fourth combinant,

Π^8: the product of the eight tritangent planes of R^8.

There seems to be no record, for a general net of quadrics, of any combinantal covariant whose order is less than eight; in the present Note we obtain combinantal covariants of orders four and six. It is not intended to give a complete list of combinants, but just to register those which present themselves most readily.

The Note falls into four sections, or rather into three sections to which an appendix (§§ 19–23) has been added; in this appendix the opportunity has been taken of applying some of the considerations of the

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* The three preceding Notes appeared as follows: (I) *Proc. London Math. Soc.* (2), 43 (1937), 302–315; (II) *Journal London Math. Soc.*, 12 (1937), 276–280; (III) *Proc. London Math. Soc.* (2), 44 (1938), 466–480.

† *Proc. London Math. Soc.* (2), 34 (1932), 492–525.

Note proper to the particular net of quadrics whose eight base points form a pair of Möbius tetrads. Most of the results for the sake of which the Note was written appear in the third section (§§ 14–18). Before these can be obtained, however, the ground has to be prepared; hence the first section (§§ 2–6) is, except for § 4, mostly concerned with explaining and co-ordinating those results obtained by other writers which are essential for the prosecution of the work. § 4 itself is in the nature of a digression, in the course of which some remarks are made about combinantal invariants of $\mathfrak{N}$.

One of the quartic surfaces which arises as a combinant of $\mathfrak{N}$ has the property of having all the twenty-eight lines of intersection of the pairs of eight associated planes as bitangents. The second section of the Note is concerned with establishing this property, the recognition of which is surely overdue.

I.

2. No systematic search for the combinants of three quadrics has yet been made. Turnbull* and, after him, Williamson†, have studied the complete system of concomitants of three quadrics; but the combinantal forms cannot be selected forthwith from the lists of concomitants that they obtain, because the algebraic standpoint of these authors gives an utterly different view of the system from that given by the geometrical standpoint adopted here‡. We obtain combinantal covariants as surfaces defined by some geometrical relation to the net of quadrics as a whole; for us the order to which the coefficients in the equation of a quadric of the net enter in the equation of such a surface is not so important, though often enough it could easily be found. Yet, even when we do know the degrees of the equation of such a surface both in the variables and in these coefficients, we cannot identify the corresponding combinant with a concomitant found by Turnbull or Williamson for which these degrees are the same. In their work, concomitants which differ only by reducible terms (reducible, that is, in the usual sense of the theory of algebraic forms) are considered as equivalent; whereas for us the combinantal

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* *Proc. London Math. Soc.* (2), 20 (1921), 465–489; and 23 (1924), 423–427.

† *Proc. Edin. Math. Soc.*, 43 (1925), 2–16.

‡ The remarks of Grace and Young in § 229 of their *Algebra of invariants* (Cambridge, 1903) are pertinent

property is vital, and the rejection of reducible terms may well destroy the combinantal character of a concomitant.

3. There is no doubt that, from a geometrical standpoint, it is the combinants which are the more important; we wish to consider surfaces related to the net of quadrics as a whole, and not to some particular set of three quadrics on which the net happens to be based. To borrow Sylvester's phrase, used when he first wrote about and invented the name of combinants*, we consider surfaces which are related to the net *in its corporate capacity*.

This idea of regarding the net as a whole, though advocated by Sylvester so long ago, has not been used to sufficient purpose since; and but for this neglect of combinants it is probable that the net of quadrics would have been studied to much better advantage than it has been hitherto. The neglect is doubtless partly due to the manner in which pencils of conics and quadrics are treated in the text-books on two- and three-dimensional analytical geometry. These pencils are simple enough to allow of a large number of concomitants being obtained, and their geometrical interpretation found, without regard to any combinantal forms. The influence of this treatment of the concomitants of two conics is seen, for example, in Salmon's treatise where†, when he obtains a set of invariants of three conics, he does not stop to enquire which of them, or what functions of them, are combinants; indeed the term "combinant" does not appear in the index of the book! This influence has persisted; though, whatever may have been the outlook of those who followed him, Salmon himself did realise that combinants deserved special attention; indeed he gave‡, and interpreted geometrically, three of the combinantal invariants of a net of quadrics.

4. Before proceeding, let us add two more combinantal invariants to the three given by Salmon. The quadrics of $\mathfrak{N}$ have equations of the form

$$\xi Q_0+\eta Q_1+\zeta Q_2=0,$$

and we denote by I_w a combinantal invariant which is of degree w in the coefficients of Q_i; the three invariants given by Salmon may then be called I_2, I_{10}, I_{16}.

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* *Cambridge and Dublin Mathematical Journal*, 8 (1853), 256–269; *Mathematical Papers* I, 411–422.

† *Conic sections* (Dublin, 1879), §389b.

‡ *Analytical geometry of three dimensions* (Fourth Edition, Dublin, 1882), 208–212.

We now do for $\mathfrak{N}$ what was done by W. S. Burnside* for a net of conics; but, whereas Burnside only had to express the combinants which he found in terms of the two known combinants T and M (see § 5 below), the combinants which are found by the analogous procedure for quadrics are new. The discriminant of the quadratic form $\xi Q_0+\eta Q_1+\zeta Q_2$ is a ternary quartic in ξ, η, ζ whose coefficients are all invariants, but not combinants, of the three quadrics. But the invariants of this ternary quartic, except for being multiplied by a power of the determinant of the transformation, are unchanged when ξ, η, ζ are subjected to a non-singular linear transformation, and so *are* combinants; the degree of such a combinant in the coefficients of Q_i is the weight of the invariant of the ternary quartic. Now two invariants of a ternary quartic are well known. The first, denoted by $(abc)^4$ in Aronhold's symbolism and written out *in extenso* in Salmon's *Higher plane curves*, is of weight four, and so furnishes a combinant I_4. The second is the determinant of the coefficients in the six second polars of the ternary quartic, its vanishing being the condition that the quartic should be expressible as a sum of five fourth powers; it is of weight eight, and furnishes a combinant I_8.

We may therefore add I_4 and I_8 to the three combinants given by Salmon.

Let us now take the discriminant of the ternary quartic, not because it furnishes a new invariant but, on the contrary, because of its relation to the known invariants; it is of weight thirty-six, and so gives a combinant I_{36}. If I_{36} vanishes, $\mathfrak{N}$ must be such that the ternary quartic, when equated to zero, gives a quartic curve with a node; this node corresponds to a cone Q of $\mathfrak{N}$ which counts for two among the four cones of any pencil of quadrics which includes Q and belongs to $\mathfrak{N}$. Now this can happen in two ways.

(*a*) $\mathfrak{N}$ includes a plane-pair Q. For such a net, $I_{10}=0$.

(*b*) $\mathfrak{N}$ has two coincident base points; it then includes a cone Q with this base point for vertex. For such a net, $I_{16}=0$.

Thus I_{36} contains both I_{10} and I_{16} as factors†; and indeed I_{36} is a numerical multiple of the product $I_{10}^2 I_{16}$.

This factorisation of I_{36} may also be seen by the following argument,

* *Quart. J. of Math.*, 10 (1870), 242.

† For the special case when the net has self-polar pentahedra, see Salmon, *Analytical geometry of three dimensions* (Fourth Edition, Dublin, 1882), 212–3.

which is, in a manner, the converse of Clebsch's principle and was often used by Salmon.

Consider not a net but a web of quadrics

$$\xi S_0+\eta S_1+\zeta S_2+\tau S_3=0.$$

The discriminant of the quadratic form on the left of this equation is a quaternary quartic in ξ, η, ζ, τ which, when equated to zero, gives the quartic surface known as the Cayley symmetroid. This surface has ten nodes, corresponding to the ten plane-pairs of the web. Any net belonging to the web is given by a linear relation between ξ, η, ζ, τ, and so corresponds to a plane. If the net is such that $I_{36}=0$, the curve in which the plane meets the symmetroid has a node. Thus the nets for which $I_{36}=0$ are found simply by taking the tangential equation of the symmetroid, and it is well known that this equation contains the *squares* of the equations of the nodes. The degree of the residual factor is the "proper" class of the symmetroid, namely 16.

5. A plane cuts the net $\mathfrak{N}$ of quadrics in a net of conics; if the plane is such that this net of conics has some special property, expressed by some single relation between its invariants, then, by Clebsch's well-known principle of transference*, the envelope of the plane is a corresponding contravariant of $\mathfrak{N}$.

It was shown by Gundelfinger† that all combinantal invariants of a net of conics are expressible as polynomials in two of them. The first of these two invariants we denote by T, and call Sylvester's invariant; it was first obtained by him, in the paper already cited, as a commutant of the line-equation of a general conic of the net. If T vanishes the net of conics has the property, given by Salmon and Gundelfinger, that its Jacobian curve is apolar to the envelope of lines which are cut in involution by the net. The second invariant we call M; its vanishing is a necessary and sufficient condition for the net to be such that one of its members is a repeated line.

The actual expressions for T and M are found in Salmon's *Conic sections*; the coefficients in the equation of a conic of the net occur to

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* *Journal für Math.*, 59 (1861), 28. The principle is also described by Turnbull, *The theory of determinants, matrices and invariants* (Blackie, 1929), 287–289, as well as by Grace and Young, *Algebra of invariants* (Cambridge, 1903), 265 and 271.

† *Journal für Math.*, 80 (1875), 73–85.

degree two in T and to degree four in M. It follows that, when T is expressed in Aronhold's symbolic notation, it must be a sum of products each of *four* determinantal factors, since each determinant is of degree three in the symbols, and the total number of symbols in a term must be twelve, to account for the second degree in the coefficients of each of three conics on which the net is based. Hence the envelope of those planes which cut $\mathfrak{N}$ in nets of conics for which T vanishes is a contravariant of class *four*. This will be denoted by ϕ^4 and called Gundelfinger's contravariant, since its existence was first established by him*; it is important for our work.

Similarly it follows that the envelope of those planes which cut $\mathfrak{N}$ in nets of conics for which M vanishes is a contravariant of class eight. From the geometrical interpretation of the vanishing of M which has already been given it follows, by Clebsch's principle, that this contravariant is simply the envelope of the cones which belong to $\mathfrak{N}$; its existence was first pointed out by Sturm†, who defined it as the envelope of the cones and obtained its class. It will be denoted here by σ^8.

6. The resultant of three conics is a linear combination of T^2 and M; it was indeed by obtaining the resultant in this form that Sylvester discovered these invariants‡. Gundelfinger, by applying Clebsch's principle, deduced that the resultant of three quadrics and a plane is a linear combination of $(\phi^4)^2$ and σ^8. This resultant, however, consists of the eight base points of the net determined by the three quadrics, so that there is an identity

$$\pi^8 \equiv (\phi^4)^2 + \lambda\sigma^8, \qquad \text{(G)}$$

where $\pi^8 = 0$ is the equation of the eight base points.

This identity was given, possibly in consequence of Gundelfinger's paper, by Fiedler in his German translation§ of Salmon's *Analytic geometry of three dimensions*; it is to be regretted that, through some misplaced

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* *Loc. cit.*, 83.

† *Journal für Math.*, 70 (1869), 236.

‡ Sylvester, *loc. cit.* The first explicit method of finding the resultant of three conics was given earlier by Sylvester, who applied his dialytic process of elimination to the three conics and their Jacobian, *Cambridge and Dublin Mathematical Journal*, 2 (1841), 232–6; *Mathematical Papers* I, 64–65.

§ Salmon-Fiedler, *Analytische Geometrie des Raumes* I (*Dritte Auflage*), (Leipzig, 1879), 319.

expurgatory zeal, it has been omitted from later German editions of this work.

II.

7. Consider now those planes, eight in all if they are finite in number, which belong* to the envelope σ^8 and which pass through the join $B_i B_j$ of two of the eight base points B_i and B_j of the net $\mathfrak{N}$. Since every cone of $\mathfrak{N}$ passes through both B_i and B_j, no tangent plane of such a cone can pass through $B_i B_j$ unless $B_i B_j$ lies entirely on the cone. Now $B_i B_j$ is a chord of ϑ, the Jacobian curve of $\mathfrak{N}$, and is known to lie on two of the cones; each of these two cones has one of its tangent planes passing through $B_i B_j$, and these are the only planes through $B_i B_j$ which can belong to σ^8. Wherefore the eight planes of σ^8 which pass through $B_i B_j$ must consist of these two, each counted four times. Now every plane through $B_i B_j$ belongs (doubly in fact) to the envelope π^8; it follows from (G) that any plane of ϕ^4 which passes through $B_i B_j$ must, since its coordinates cause π^8 and ϕ^4 both to vanish, also be a plane of σ^8. Hence the four planes of ϕ^4 which pass through $B_i B_j$ must consist of the above two planes, each counted twice. Every one of the twenty-eight joins of pairs of base points has this property.

Just as we speak of a line, two of whose intersections with a surface coincide, as a *tangent* of a surface, so we may speak, in the dual sense, of a line which is so related to an envelope that two of the planes of the envelope which pass through it coincide as a *tangent* of the envelope; and, dual to a bitangent or *double tangent* of a surface, we have a *double tangent* of an envelope. *Gundelfinger's contravariant ϕ^4 then has the property of having the twenty-eight joins of the pairs of base points all as double tangents.* This property of ϕ^4 does not appear to have been noticed before; indeed no instance seems yet to have been met with of a quartic envelope having as double tangents the twenty-eight joins of the pairs of eight associated points, nor has even the *a priori* possibility of this happening been considered.

8. This property of ϕ^4 followed from the result that those planes of ϕ^4 which pass through $B_i B_j$ are also planes of σ^8; this result was itself

* Instead of saying in full "a plane which belongs to the envelope whose equation is $\sigma^8 = 0$" we shall, when there is no possibility of ambiguity or misunderstanding, say, more briefly, "a plane which belongs to the envelope σ^8", or "a plane of the envelope σ^8", or "a plane of σ^8". A corresponding procedure will be adopted, as the occasion serves, for other envelopes and, dually, for surfaces.

obtained as an immediate corollary of (G). Let us now give an alternative way of obtaining it, without appealing to (G). We first use the fact that ϕ^4 is the envelope of those planes which meet $\mathfrak{N}$ in nets of conics for which T vanishes, and then deduce the result by using the form of T given by Salmon in § 389 of his *Conic sections*.

We desire, then, to obtain those planes which belong to ϕ^4 and pass through $B_i B_j$. Such a plane meets $\mathfrak{N}$ in a net of conics which has two base points and for which Sylvester's invariant vanishes. Now, when a net of conics has two base points it may, by taking these as two of the three vertices of the triangle of reference, be based on three conics whose equations have the forms

$$a_1 x^2 + 2f_1 yz + 2g_1 zx + 2h_1 xy = 0,$$

$$a_2 x^2 + 2f_2 yz + 2g_2 zx + 2h_2 xy = 0,$$

$$a_3 x^2 + 2f_3 yx + 2g_3 zx + 2h_3 xy = 0.$$

For such a net of conics, all the terms in Salmon's expression for T vanish except the last; and the condition $T = 0$ reduces, for this special net, to the vanishing of the determinant $(f_1 g_2 h_3)$. This, however, is precisely the condition for the net to include the conic $x^2 = 0$, which is the repeated line joining the base points. Hence the plane must touch a cone of $\mathfrak{N}$ along $B_i B_j$, and we have again the result that any plane through $B_i B_j$ which belongs to ϕ^4 must belong also to σ^8. It may be remarked in passing that the invariant T for the above net of conics is a numerical multiple of the *square* of $(f_1 g_2 h_3)$, and this implies that the two planes of σ^8 which pass through $B_i B_j$ count *doubly* as planes of ϕ^4.

9. There are, of course, properties dual to the above for a net of quadric envelopes, and a corresponding identity

$$\Pi^8 \equiv (F^4)^2 + \lambda S^8, \qquad (\Gamma)$$

where Π^8 is the product of the eight base planes,

S^8 is the surface generated by the conics—quadric envelopes with vanishing discriminants—belonging to the net,

F^4 is a quartic surface, dual to Gundelfinger's contravariant.

This quartic surface has the twenty-eight lines of intersection of the pairs of base planes all as double tangents.

An immediate consequence of (Γ) is that S^8 touches each of the eight base planes along the quartic curve in which this plane meets F^4; these quartic curves are the loci of the points of contact of the respective base planes with the conics belonging to the net of quadric envelopes.

10. The above derivation of the property of ϕ^4 and F^4 of having twenty-eight bitangent lines which are, for ϕ^4, joins of pairs of eight associated points, and, for F^4, intersections of pairs of eight associated planes, rests primarily on the work of Sylvester and Gundelfinger. This work is algebraic; we now give an alternative derivation of the property, which rests on some work, of a more geometrical character, of Aronhold and Frobenius. It is a matter of choice whether we prove the property for ϕ^4 or for F^4; once it is proved for either, the principle of duality establishes it for the other. We shall, however, prove the property for F^4, as we are thereby able to appeal more directly to known results.

11. The six points of contact of three bitangents of a non-singular plane quartic curve may lie on a conic, in which case the three bitangents are said to be syzygetic; or they may not. If any set of eight, or more, of the twenty-eight bitangents is taken, it always includes at least one syzygetic triplet; the maximum number of bitangents which can be chosen in such a way that it does *not* include a syzygetic triplet is seven. Such a set of seven bitangents is called an *Aronhold set*. If seven lines of general position are given in a plane, it is known that there is a unique quartic curve for which these seven lines are an Aronhold set of bitangents*.

Take now eight associated planes $\Pi_1, \Pi_2, \ldots, \Pi_8$; there is a net of quadric envelopes to which they belong, and this net includes ∞^1 envelopes which degenerate into conics. We quote, concerning this configuration, two results due to Frobenius†:

(i) The locus of the points of contact of the conics with the plane Π_i is a quartic curve C_i. This is the curve which has the seven lines in which Π_i is met by the remaining seven planes as an Aronhold set of bitangents.

(ii) C_i and C_j touch π_{ij}, the line common to Π_i and Π_j, in the same two points.

* Aronhold, *Berliner Monatsberichte* (1864), 499.

† *Journal für Math.*, 99 (1886), 278–280.

12. We now prove that *the eight quartic curves C_i lie on a quartic surface.*

One way of proving this would be to take a general quartic surface, and so restrict the coefficients in its equation by linear conditions that it contains first one, then another, of the curves C_i. Proceeding in this way we should find that, at a certain stage, all those curves C_i through which the surface has not already been made to pass would lie on the surface automatically. We will, however, marshal the argument somewhat differently; we take four of the curves and prove that they lie on a quartic surface *other* than that formed by the four planes in which they lie. From this the desired result easily follows.

Take the four curves C_1, C_2, C_3, C_4 lying in the planes Π_1, Π_2, Π_3, Π_4. The line π_{12} is touched by C_1 and C_2 in the same two points; in order that a quartic surface should touch π_{12} at these points it must be subjected to four linear conditions. Similarly four linear conditions must be imposed to ensure the two proper contacts with π_{23}, and four more to ensure them with π_{31}. The quartic surface, now subjected to twelve linear conditions, has the three concurrent bitangents π_{12}, π_{23}, π_{31}; we next proceed to make the surface touch π_{14}, π_{24}, π_{34} in the requisite pairs of points. A new feature here appears in the argument; for, if a plane quartic has each of three lines for bitangents, any quartic curve which passes through the six points of contact and touches the given quartic at five of them must also touch it at the sixth—this being a particular instance of the theorem that the twelve points in which a quartic meets a cubic curve are determined when eleven of them are assigned. Hence the quartic surface, already having π_{12}, π_{23}, π_{31} as bitangents, need only be subjected to three further linear conditions in order that it should touch, say, π_{34} at the two requisite points; similarly three further linear conditions are sufficient to ensure the proper contacts with π_{14}, and three more to ensure them with π_{24}.

The general quartic surface has now, by the imposition of twenty-one linear conditions, been caused to have as bitangents the six edges of the tetrahedron whose faces are Π_1, Π_2, Π_3, Π_4; the two points of contact with each edge are the points where this edge is touched by two of the curves C_i. We now impose further linear conditions on the surface so as to make it contain the curves C_1, C_2, C_3, C_4 entirely, and three conditions are required for each of the four curves. For, when a quartic curve passes through the twelve intersections of a second quartic with a cubic, its four remaining intersections with this second quartic are collinear; therefore, if it is constrained to pass through three arbitrary points of this second quartic it must contain it completely. Hence a general quartic surface can, by the imposition of thirty-three linear conditions, be made to contain

the four curves C_1, C_2, C_3, C_4. This leaves one free constant in the equation of the surface, and so a pencil of quartic surfaces through the four curves, if the thirty-three conditions are independent. Were they not independent, there would be a linear system, of freedom greater than one, of quartic surfaces containing the four curves; but the total order of the four curves is 16, so that this latter contingency cannot arise. Thus C_1, C_2, C_3, C_4 together constitute the base of a pencil of quartic surfaces*, one surface of the pencil of course being the product of the four planes Π_1, Π_2, Π_3, Π_4.

There is one quartic surface belonging to this pencil which contains the four remaining curves C_5, C_6, C_7, C_8. For the base curve of the pencil meets C_5, say, in sixteen points, namely the contacts with its four bitangents π_{15}, π_{25}, π_{35}, π_{45}; through any further point of C_5 there passes one surface of the pencil, and this contains C_5 entirely. This particular surface also contains C_6, C_7, C_8; for it already has twenty intersections with each of them at their points of contact with five of their bitangents.

13. We have now obtained a quartic surface F^4 which contains the eight curves C_i. It has therefore the twenty-eight lines π_{ij} for bitangents. It is the same surface F^4 as that which appears in the identity (Γ); for both these quartic surfaces F^4 contain the eight curves C_i, and so must be one and the same surface. We are not, however, going to appeal to the identity (Γ), but to obtain it anew by continuing the present argument.

The surface S^8 which is generated by the conics which belong to the net of quadric envelopes touches, by Frobenius' result, Π_i along C_i. Hence, if F^4 is the surface, just obtained, on which all the curves C_i lie, every octavic surface which belongs to the pencil $(F^4)^2+\lambda S^8$ touches each of the planes Π_i along the corresponding curve C_i. There is one surface of this pencil which contains the whole of the plane Π_i; this surface not only touches any other, Π_j say, of the eight planes along C_j, but it meets it further in the line π_{ij}; hence it contains the whole of the plane Π_j. It must therefore be the product of the eight planes, so that we have an identity

$$\Pi^8 \equiv (F^4)^2+\lambda S^8. \qquad (\Gamma)$$

* The same argument, with the few necessary verbal alterations, establishes the same result for four plane quartics which are such that any two meet the line of intersection of their planes in the same four (distinct) points.

III.

14. It has, then, been known, since the publication of Gundelfinger's paper, that a net of quadric surfaces possesses a combinantal contravariant of class four. It will now be shown that it also possesses a combinantal covariant of order four. The existence of such a covariant, indeed the existence of several, follows immediately from the theory of algebraic forms; this will be explained below. But let us obtain one geometrically first.

The net $\mathfrak{N}$ has a Jacobian curve ϑ, whose trisecants generate a scroll R^8; this scroll has eight tritangent planes. It was shown by W. P. Milne* that *these eight planes are associated.* Denote by ν the net of quadric envelopes to which these eight planes belong. Then any combinantal concomitant of either of the nets $\mathfrak{N}$ or ν must be a combinantal concomitant of $\mathfrak{N}$; in particular, any covariant† of ν must also be be a covariant of $\mathfrak{N}$. But ν certainly has a combinantal covariant F^4, namely the dual of Gundelfinger's contravariant. So we have obtained, in the surface F^4, a quartic surface which is a combinantal covariant of $\mathfrak{N}$.

15. Milne, starting from a canonical form for $\mathfrak{N}$, obtains equations for ν and observes that this "net of quadric envelopes is obviously combinantal in character" with respect to $\mathfrak{N}$. Obvious geometrically it certainly is; but algebraically it is perhaps not quite so obvious as Milne appears to believe. The coefficients in any combinant of three quadrics must be functions of the 120 determinants, of three rows and columns, of the 3×10 matrix formed by the coefficients in the equations of the three quadrics; the coefficients in Milne's equations (19) have not this property as they stand‡. They do, however, acquire the property if they are multiplied throughout, in Milne's notation, by k^3. This, at first sight small, criticism is important algebraically; for it means that the equation

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* *Journal London Math. Soc.*, 8 (1933), 211–216.

† The quadrics of $\mathfrak{N}$ are quaternary forms in the *point* coordinates, those of ν quaternary forms in the *plane* coordinates. Here the term *covariant* will denote always a concomitant which is a quaternary form in the *point* coordinates. Thus we speak below of forms as covariants of ϕ^4 although the variables (namely the point coordinates) which they contain are contragredient to those in ϕ^4.

‡ Incidentally, in the third of these three equations, the first term should have a *minus* sign and, in the last term, n is a misprint for m.

of a quadric envelope of ν contains the coefficients of the equations of the quadrics of $\mathfrak{N}$ each to degree *two*, not merely to degree one.

16. We now have, as combinantal concomitants of $\mathfrak{N}$, the covariant F^4, which is the dual of Gundelfinger's contravariant for ν, and the contravariant ϕ^4, which is Gundelfinger's contravariant for $\mathfrak{N}$. The whole of the simultaneous system of concomitants of F^4 and ϕ^4 must consist of combinantal concomitants of $\mathfrak{N}$. This system includes all the covariants of ϕ^4 and so, when once the existence of ϕ^4 is known, we can obtain a quartic covariant of $\mathfrak{N}$ forthwith. For every covariant of ϕ^4 is necessarily a covariant of $\mathfrak{N}$, and ϕ^4 certainly has a quartic covariant, namely the form which, when ϕ^4 is written symbolically as $a_u{}^4$, is given by $(\alpha\beta\gamma x)^4$. This is all that needed to be said, at any moment since the publication of Gundelfinger's paper sixty-five years ago, to establish the existence of a quartic surface combinantally covariant for a net of quadric surfaces.

But there is more to be said; for ϕ^4 has other covariants of the fourth order. The best known of these is obtained as follows*. Take the ten second polars of ϕ^4, which are quadric functions of the four plane coordinates, and the matrix of their coefficients. The determinant of this matrix is an invariant of ϕ^4, whose vanishing is a necessary and sufficient condition for ϕ^4 to have an outpolar quadric, or to be expressible as a sum of nine fourth powers. [If ϕ^4 was expressible as a sum of $k \leqslant 9$ fourth powers, it would have a linear system of ∞^{9-k} outpolar quadrics, and the determinant would be of rank k.] If, now, this matrix is bordered by a row and column whose elements are the squares and products of the point coordinates, the determinant of this bordered matrix is the quartic covariant in question. It is of degree nine in the coefficients of ϕ^4, and would vanish identically if ϕ^4 happened to be expressible as the sum of eight (or less) fourth powers.

We have thus found three quartic surfaces which are combinantal covariants of $\mathfrak{N}$; namely F^4 and the two quartic covariants that we have obtained from ϕ^4. We also have, by dual reasoning, three quartic envelopes which are combinantal contravariants; namely ϕ^4 and the two quartic contravariants that are correspondingly derived from F^4.

17. The processes by which the two quartic covariants of ϕ^4 were derived also furnish, when generalized, for any form of even degree $2q$, two concomitants, also of degree $2q$, in the contragredient variables; if

* Reye, *Journal für Math.*, 78 (1874), 123–129.

there are p variables*, the degrees of these concomitants in the coefficients of the original form are $p-1$ and $\{(q+p-1)!/q!\,(p-1)!\}-1$ respectively. In the special case when $q=1$, however, the two concomitants coincide, and we have the well-known process of obtaining the tangential equation of a quadric by bordering the matrix of its discriminant with a row and column of prime coordinates. Another known example occurs when the original form is a ternary quartic in the point co-ordinates, for the second concomitant is then Clebsch's contravariant Ω. Though Clebsch did not actually obtain Ω by bordering the invariant determinant, the expression which he gives for it† shows clearly that it is obtainable in this way. Geometrical interpretations of Ω have been given by other writers‡; by dualizing these, and generalizing them so that they refer to quaternary instead of to ternary forms, geometrical interpretations of the corresponding covariant of ϕ^4 can be found.

18. When a combinantal covariant C^p, of order p, of $\mathfrak{N}$ has been found, another one, of order $p+2$, is at once obtained as the Jacobian of C^p and three linearly independent quadrics of $\mathfrak{N}$—at least unless this Jacobian happens to vanish identically, as it would if C^p was a polynomial

* Binary forms ($p=2$) are exceptional, in that the variables contragredient to (x_1, x_2) are $(-x_2, x_1)$. The two processes do, of course, furnish covariants, but the first of them is merely a multiple of the binary form itself. The second, however, is a genuine covariant, of degree q in the coefficients, whenever $q>1$, and is the penultimate member of a scale of covariants found by Sylvester, *Mathematical Papers* I, 297. For the binary quartic $(a, b, c, d, e\mathcal{X}x_1, x_2)^4$ it is

$$\begin{vmatrix} a & b & c & x_2^2 \\ b & c & d & -x_1x_2 \\ c & d & e & x_1^2 \\ x_2^2 & -x_1x_2 & x_1^2 & 0 \end{vmatrix},$$

the Hessian; for the binary sextic it is, with us,

$$\begin{vmatrix} a & b & c & d & -x_2^3 \\ b & c & d & e & x_2^2x_1 \\ c & d & e & f & -x_2x_1^2 \\ d & e & f & g & x_1^3 \\ -x_2^3 & x_2^2x_1 & -x_2x_1^2 & x_1^3 & 0 \end{vmatrix},$$

and, with Sylvester,

$$\begin{vmatrix} ax_1^2+2bx_1x_2+cx_2^2 & bx_1^2+2cx_1x_2+dx_2^2 & cx_1^2+2dx_1x_2+ex_2^2 \\ bx_1^2+2cx_1x_2+dx_2^2 & cx_1^2+2dx_1x_2+ex_2^2 & dx_1^2+2ex_1x_2+fx_2^2 \\ cx_1^2+2dx_1x_2+ex_2^2 & dx_1^2+2ex_1x_2+fx_2^2 & ex_1^2+2fx_1x_2+gx_2^2 \end{vmatrix}.$$

† *Journal für Math*, 59 (1861), 139.

‡ *Encyklopädie der Math. Wissenschaften*, III, C 5, 524.

of degree $\frac{1}{2}p$ in the three quadrics. The covariant C^{p+2} so obtained will be spoken of simply as the Jacobian of C^p. In virtue of its character as a Jacobian, the surface C^{p+2} always passes through ϑ, the Jacobian curve of $\mathfrak{N}$.

Thus there arises a sextic covariant F^6, combinantal for $\mathfrak{N}$, which is the Jacobian of F^4 and which contains ϑ. This surface is the locus of points whose polar planes with respect to F^4 and all the quadrics of $\mathfrak{N}$ are concurrent; or, expressing the same statement slightly differently, it is the locus of a point O which is such that its polar plane with respect to F^4 passes through O', the point which, in the manner fully described in Note I, is conjugate to O with respect to $\mathfrak{N}$. The curve of intersection of F^4 and F^6 is the locus of points where quadrics of $\mathfrak{N}$ touch F^4.

Similarly, the other two quartic covariants give rise to sextic covariants as their Jacobians.

Consider, next, combinantal covariants of order eight. It was mentioned at the outset that these include R^8, E^8, ρ^8 and Π^8; we may add to these S^8, the locus of conics which belong to ν. Further, we also have the Hessians of the three quartic covariants and the Jacobians of the three sextic covariants, making eleven octavic covariants in all. These may not all be distinct; and it would certainly be of interest to investigate what linear relations connect them with one another and with the squares of the quartic covariants. One such relation is given by the identity (Γ).

For covariants of order ten we may take the Jacobians of covariants of order eight; though the Jacobian of E^8 vanishes identically. Having found all the non-vanishing Jacobians, we may add those surfaces which are conjugate, in the sense of Note I, to the three sextic covariants; these three surfaces of order ten have ϑ for a nodal curve. And then it may be investigated what linear relations exist between these covariants and between the nine products, such as F^4F^6, of a quartic and a sextic covariant. And so on.

IV.—*The Möbius Net as an example.*

19. When properties of a general net of quadrics have been found, it is of interest to verify them for some particular net that has been studied previously; though it must be borne in mind that a special net will have, merely in consequence of the specialization, properties which are not possessed by the general net. Suppose, then, that $\mathfrak{N}$ is a net whose eight base points are a pair of Möbius tetrads T, T'; it may fittingly be called a *Möbius Net*. We obtain the corresponding net ν of quadric envelopes, and the identity (Γ). The Möbius Net has been described in detail in a

previous paper*; we shall refer to this paper as M.T., and use here the same symbols as there.

If T is taken as tetrahedron of reference for a system of homogeneous co-ordinates x, y, z, t, the faces of T' are found (M.T. p. 338) by equating to zero the linear forms x', y', z', t' given by

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} . & -n & m & -l \\ n & . & -l & -m \\ -m & l & . & -n \\ l & m & n & . \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

Here l, m, n are any three numbers, subject only to the condition that the above square matrix is non-singular; thus $l^2+m^2+n^2$ must not be zero. We describe the eight vertices of the two tetrahedra as the *base points*, and of the eight faces of the two tetrahedra as the *base planes* of the Möbius configuration. The base points of the configuration are also the base points of $\mathfrak{N}$; it will be seen that the base planes of the configuration are also the base planes of ν.

A general net of quadrics may be based on any three of its members which are linearly independent; save for this condition of linear independence it is immaterial which three are chosen, the net having no peculiarity that enables us to pick out one set of quadrics rather than another. But when the net is specialised this may no longer be so, as the specialization may well throw into prominence certain particular quadrics of the net. This happens for the Möbius Net; and, moreover, it happens very conveniently for the algebraical treatment, for the net can be based symmetrically on three particular quadrics. These are (M.T., p. 341):

$$Q_1 \equiv yy'+zz' \equiv -xx'-tt' = 0,$$

$$Q_2 \equiv zz'+xx' \equiv -yy'-tt' = 0,$$

$$Q_3 \equiv xx'+yy' \equiv -zz'-tt' = 0.$$

20. In order to obtain the net ν of quadric envelopes which corresponds to the Möbius Net $\mathfrak{N}$, we have to discover its eight base planes; these are, as was said in § 14, tritangent planes of R^8, the scroll generated by the trisecants of the Jacobian curve of $\mathfrak{N}$. The Möbius Net, however, includes the four plane-pairs

$$xx' = 0, \quad yy' = 0, \quad zz' = 0, \quad tt' = 0,$$

* "The net of quadric surfaces associated with a pair of Möbius tetrads", *Proc. London Math. Soc.* (2), 41 (1936), 337–360.

whose vertices are, respectively, four skew lines a, b, c, d; the Jacobian curve of the net consists (*cf.* M.T., p. 343) of these four lines and their two transversals e, f. Hence, if the regulus generated by the transversals of a, b, c is denoted by $\{abc\}$, the scroll R^8 now consists of the four quadrics on which lie, respectively, $\{abc\}$, $\{bcd\}$, $\{cda\}$ and $\{dab\}$.

Any plane through a meets b, c, d in points β, γ, δ; the lines $\gamma\delta, \delta\beta, \beta\gamma$ belong respectively to $\{cda\}$, $\{dab\}$, $\{abc\}$; hence the plane contains three generators of R^8. Similarly, any plane through b, or c, or d contains three generators of R^8. Thus, for the Möbius Net, R^8 has not merely eight but an infinite number of tritangent planes, and so the base planes of ν cannot be found merely by using their property of being tritangent planes of R^8. Some further property which they possess must be invoked in order to identify them. Such a property is known*: those trisecants of of the Jacobian curve of a general net of quadrics which lie in tritangent planes of R^8 are also generators of cones belonging to the net. Now none of $\gamma\delta, \delta\beta, \beta\gamma$ can meet either e or f; hence they cannot possibly be generators of cones of the net with point-vertices. The only possibility of their being generators of cones of the net with line-vertices is for their plane to be one of the two planes which make up the plane-pair with vertex a. So it is seen that the eight base planes of ν are precisely the eight base planes of the Möbius configuration. Incidentally, Milne's result that, for a general net of quadrics, the eight tritangent planes of R^8 are associated, includes as a special case the well-known result that the eight base planes of a Möbius configuration are associated.

21. Consider now the identity (Γ); it contains three terms. Of these Π^8 is the product of the eight base planes of ν, and so is a multiple of $16\,xx'yy'zz'tt'$, *i.e.* of

$$Q_1^4+Q_2^4+Q_3^4-2Q_2^2Q_3^2-2Q_3^2Q_1^2-2Q_1^2Q_2^2.$$

As for S^8, it was explained in M.T. (pp. 344–5) that the locus of the conics which touch the eight base planes of a Möbius configuration consists of a pair of Plücker surfaces U and V; the equation of U was given (p. 343) and that of V is obtained from it by changing the sign of p. From this it is found that S^8 is a multiple of

$$\Sigma\{(l^2+m^2)^2(l^2+n^2)^2Q_1^4+2(m^2+n^2)^2(m^2n^2+l^2p^2)Q_2^2Q_3^2\},$$

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* *Proc. London Math. Soc.* (2), 34 (1932), 513–514.

where Σ denotes the sum of the three terms obtained by permuting cyclically, and simultaneously, the letters l, m, n and the suffixes 1, 2, 3; and

$$l^2+m^2+n^2+p^2=0.$$

The expressions for both Π^8 and S^8 being now known, that for F^4 is found from the fact that its square must be of the form $\Pi^8-\lambda S^8$. Since Π^8 and S^8 are both homogeneous quartic polynomials in Q_1, Q_2, Q_3 it is already clear, before the actual expression for F^4 is found, that it must be a homogeneous quadratic polynomial in Q_1, Q_2, Q_3. Hence, for the Möbius Net, F^4 is an octadic surface with nodes at the eight base points; this property of F^4 is due to the specialization of the net, and does not hold for a general net of quadrics. An incidental consequence is that, for the Möbius Net, the Jacobian F^6 of F^4 vanishes identically. The actual identity (Γ) is found to be

$$\begin{aligned}4l^2m^2n^2p^2\{&Q_1^4+Q_2^4+Q_3^4-2Q_2^2Q_3^2-2Q_3^2Q_1^2-2Q_1^2Q_2^2\}\\ &\equiv\{(m^2n^2+l^2p^2)Q_1^2+(n^2l^2+m^2p^2)Q_2^2+(l^2m^2+n^2p^2)Q_3^2\}^2\\ &\quad-\Sigma\{(l^2+m^2)^2(l^2+n^2)^2Q_1^4+2(m^2+n^2)^2(m^2n^2+l^2p^2)Q_2^2Q_3^2\}.\end{aligned}$$

22. The equation of the quartic surface F^4 is therefore

$$(m^2n^2+l^2p^2)Q_1^2+(n^2l^2+m^2p^2)Q_2^2+(l^2m^2+n^2p^2)Q_3^2=0;$$

like any other octadic surface it can be generated by base curves of pencils of quadrics belonging to the net, and a geometrical interpretation of the surface can be given. For the conic whose line equation is

$$(m^2n^2+l^2p^2)\lambda_1^2+(n^2l^2+m^2p^2)\lambda_2^2+(l^2m^2+n^2p^2)\lambda_3^2=0$$

is seen to be the Φ-conic of the pair of conics u and v whose point equations (*cf.* M.T. p. 342) are

$$(m^2+n^2)^2k_1^2+(lm+np)^2k_2^2+(nl-mp)^2k_3^2=0$$

and

$$(m^2+n^2)^2k_1^2+(lm-np)^2k_2^2+(nl+mp)^2k_3^2=0.$$

Hence we have the following. Any pencil of quadrics belonging to the net includes two pairs of cones; the cones of one pair have their vertices on e and are enveloping cones of U, those of the other pair have their vertices on f and are enveloping cones of V. If these two pairs of cones

are harmonically conjugate to one another within the pencil, then the base curve of the pencil lies on F^4, and F^4 is generated by such base curves*.

23. We conclude by verifying, for the Möbius Net, that F^4 has all the lines of intersection of pairs of base planes of ν as bitangents. It is a known property of the Möbius configuration that, of the 28 intersections of pairs of base planes, 24 are also joins of pairs of base points; but the base points are all nodes of F^4, so that these 24 lines may certainly be reckoned as bitangents. It only remains to verify that those four lines are bitangents of F^4 which are intersections of pairs of base planes but *not* joins of pairs of base points. These are the lines denoted above by a, b, c, d; and it is easily shown, using the equation of F^4 and the algebra of the first three pages of M.T., that each of these four lines touches F^4 at a point on e and at a point on f.

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* It was shown in Note II that the locus of harmonic base curves for a general net of quadrics is a surface of order 12 with sextuple points at the base points. For the Möbius Net this surface consists of F^4 and a surface F^8 having quadruple points at the base points.

THE GENERALIZED LIMIT OF A FUNCTION

By B. KUTTNER

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1.1. In a recent paper*, I considered the properties of certain conventional limits of the function $A(u)$ as $u \to \infty$. In connection with this paper Prof. Hardy drew my attention to a paper by M. Riesz† containing results in some respects analogous to those of C.L.F. These results were obtained by Riesz as particular cases of more general theorems, and Prof. Hardy suggested that I should investigate the analogues of these general theorems. The present paper is the outcome of this suggestion.

In order to exhibit clearly the analogy between the results of Riesz and those of the present paper, I first state Riesz's results. Riesz investigated the conventional limit of the sequence $\{s_n\}$ as $n \to \infty$ defined by

$$\lim_{n\to\infty} \frac{1}{A_n} \{a_n s_0 + a_{n-1} s_1 + \ldots + a_0 s_n\}, \tag{1}$$

where

$$A_n = a_0 + a_1 + \ldots + a_n, \tag{2}$$

the a's being any set of positive numbers satisfying the condition

$$a_n = o(A_n) \quad \text{as} \quad n \to \infty; \tag{3}$$

if the limit (1) exists, and is equal to s, we say that the sequence $\{s_n\}$ is summable (a) to s as $n \to \infty$. It is evident that (3) ensures that the consistency condition should be satisfied; if $s_n \to s$ as $n \to \infty$, then the sequence $\{s_n\}$ is summable (a) to s as $n \to \infty$.

We may now define summability (b) in a similar manner, but with b's replacing the a's, and consider the relation between summability (a) and summability (b). Riesz's main object is to obtain necessary and

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* B. Kuttner, "Note on the Cesàro limit of a function", *Journal London Math. Soc.*, 14 (1939), 132–137. This will be referred to as C.L.F.

† M. Riesz, "Sur l'équivalence de certaines méthodes de sommation", *Proc. London Math. Soc.* (2), 22 (1924), 412–424.

sufficient conditions for the equivalence of summability (a) and summability (b). He does obtain certain results on the conditions under which summability (a) implies summability (b) without the converse being necessarily true, but he lays no stress on these results, regarding them merely as lemmas which are required for the proof of his main theorem. Since, however, the latter problem would appear also to be of some interest, I state these preliminary lemmas as well as Riesz's main result. Riesz's results are:

THEOREM A. *Let p_n and q_n be defined by**

$$\frac{\Sigma b_n x^n}{\Sigma a_n x^n} = \frac{\Sigma B_n x^n}{\Sigma A_n x^n} = \Sigma p_n x^n; \quad \frac{\Sigma a_n x^n}{\Sigma b_n x^n} = \frac{\Sigma A_n x^n}{\Sigma B_n x^n} = \Sigma q_n x^n. \tag{4}$$

Then in order that summability (a) should imply summability (b) it is necessary and sufficient that

$$\text{(i)} \quad |p_0|A_n + |p_1|A_{n-1} + \cdots + |p_n|A_0 = O(B_n) \quad as \quad n \to \infty,$$

and that $$\text{(ii)} \quad p_n = o(B_n) \quad as \quad n \to \infty.$$

THEOREM B. (a) *Suppose that*

$$\text{(i)} \quad \Sigma p_n \text{ is absolutely convergent,}$$

and that $$\text{(ii)} \quad A_n = O(B_n) \quad as \quad n \to \infty.$$

Then summability (a) implies summability (b).

(b) *If summability (a) implies summability (b), then $A_n = O(B_n)$. If, also, $B_n = O(A_n)$, then Σp_n is absolutely convergent.*

Since
$$A_n = \sum_{r=0}^{n} q_r B_{n-r},$$

we see that, if Σq_n is absolutely convergent, then $A_n = O(B_n)$ as $n \to \infty$. We therefore have, as an immediate corollary of Theorem B,

THEOREM C. *In order that summability (a) and summability (b) should be equivalent, it is necessary and sufficient that Σp_n and Σq_n should both be absolutely convergent.*

This is Riesz's main theorem.

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* In this paragraph, sums are to be taken from $n = 0$ to $n = \infty$ wherever no limits of summation are given.

1.2. In this paper, $A(u)$ is any measurable function bounded in any finite interval. We define summability (f) of $A(u)$ as $u \to \infty$ by means of the limit

$$\lim_{x\to\infty} \frac{1}{f_1(x)} \int_0^x A(t)\, f(x-t)\, dt, \tag{5}$$

if it exists, where

$$f_1(x) = \int_0^x f(t)\, dt, \tag{6}$$

and where $f(x)$ is a function defined for $x > 0$ which is everywhere positive or zero, which is integrable in $(0, A)$ for any positive A, and which is such that

$$f(x) = o[f_1(x)] \quad \text{as} \quad x \to \infty. \tag{7}$$

It follows from (7) that the consistency condition is satisfied. We define summability (g) in a similar manner, and investigate the relation between summability (f) and summability (g).

The constants p_n defined by equation (4) evidently satisfy the equation

$$b_n = \sum_{r=0}^{n} p_{n-r}\, a_r,$$

and it is for this reason that the relation between summability (a) and summability (b) depends on the values of these constants. In a similar way, if we could find a function $p(x)$ such that

$$g(x) = \int_0^x p(x-t)\, f(t)\, dt, \tag{8}$$

we should expect the relation between summability (f) and summability (g) to depend on the properties of $p(x)$. Such a function may be obtained formally as follows. If we suppose that (8) is satisfied, then, writing

$$F(s) = \int_0^\infty e^{-xs} f(x)\, dx, \quad G(s) = \int_0^\infty e^{-xs} g(x)\, dx, \quad P(s) = \int_0^\infty e^{-xs} p(x)\, dx, \tag{9}$$

we have

$$G(s) = P(s)\, F(s).$$

Hence, by a trivial transformation of Mellin's inversion formula,

$$p(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs}\, P(s)\, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{xs}\, \frac{G(s)}{F(s)}\, ds. \tag{10}$$

But these formal calculations cannot always be justified, and, even when "reasonable" restrictions are imposed on $f(x)$ and $g(x)$, a function $p(x)$ which satisfies (8) does not necessarily exist; for example, no such function exists whenever $f(x)$ and $g(x)$ are identical. We shall deal with this difficulty as follows. It is clear that, if we are to obtain any general

theorems, we must impose some restrictions on $f(x)$ and $g(x)$ beyond those already indicated. We therefore suppose that $f(x)$, $g(x)$ are such that*

$$f(x) \sim kx^{\lambda}, \quad g(x) \sim kx^{\mu} \quad \text{as} \quad x \to 0,$$

and that they satisfy certain other restrictions. We then show that a necessary condition that summability (f) should imply summability (g) is that $\mu \geqslant \lambda$. We may therefore exclude the case when $\mu < \lambda$. If $\mu > \lambda$, we shall see that the above formal calculations can be justified, and thus only the case $\mu = \lambda$ is left to be dealt with. In this case, we cannot determine a function $p(x)$ which satisfies (8), but we shall see that we can determine a function $p(x)$ such that

$$g(x) = cf(x) + \int_0^x p(x-t) f(t)\, dt, \tag{11}$$

and this serves our purpose equally well.

1.3. Throughout the rest of the paper, we suppose that $f(x)$ and $g(x)$ are any functions satisfying the following conditions†:

(i) $f(x) \geqslant 0$,

(ii) $f(x) = o[f_1(x)]$ as $x \to \infty$, where $f_1(x)$ is defined by equation (6).

(iii) $f(x) = kx^{\lambda} + cx^{\lambda+1} + cx^{\lambda+2} + cx^{\lambda+3} + cx^{\lambda+4} + r(x)$, where $\lambda > -1$, and where $r(x)$ is, in $(0, \infty)$ (including the origin itself), an $n+4$-th indefinite integral of a function which is‡ $O(e^{\epsilon x})$ as $x \to \infty$ for any positive ϵ, n denoting the least integer greater than or equal to λ. Further, $r(x)$ and its first $n+3$ derivatives, are to be equal to 0 when $x = 0$.

(iv) $g(x)$ satisfies conditions similar to those imposed on $f(x)$, but with λ replaced by μ $(\mu > -1)$, so that

$$g(x) \sim kx^{\mu} \quad \text{as} \quad x \to 0.$$

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* Throughout this paper k denotes a constant different from zero, and c denotes any constant. c and k may be different constants at each occurrence.

† It is easily seen that the theorems of this paper continue to hold when (iii) is replaced by the more general condition

$$f(x) = kx^{\lambda} + \sum_{\nu=1}^{m} cx^{\lambda_\nu} + r(x),$$

where $r(x)$ satisfies the same conditions as before, and where $\lambda_1, \lambda_2, \ldots, \lambda_m$, are any real constants greater than λ. But condition (iii) is satisfied in the most interesting applications and we have therefore obtained our results in the less general form for the sake of simplicity.

‡ It is easy to prove that condition (ii) implies that $f(x)$ [and hence $r(x)$] is $O(e^{\epsilon x})$ as $x \to \infty$. The condition that the $(n+4)$-th derivative of $r(x)$ should be $O(e^{\epsilon x})$ is thus not a very serious additional restriction.

$F(s)$ and $G(s)$ are throughout defined by equations (9). It is evident that, under the conditions imposed, they exist for $\Re s > 0$.

1.4. We prove the following theorems:

THEOREM I. (i) *There is a constant σ_0 such that $F(s)$ has no zeros for $\Re s \geqslant \sigma_0$.*

(ii) *Suppose that $\mu \geqslant \lambda$. Then, for $x > 0$, the integral*

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs}\frac{G(s)}{F(s)}\,ds, \tag{12}$$

where $\sigma > 0$, $\sigma > \sigma_0$, and where the integral is taken along a straight line, converges if $\mu > \lambda$, and exists as a Cesàro integral if $\mu = \lambda$. In either case, if we denote its value by $p(x)$,

$$g(x) = df(x) + \int_0^x p(x-t)f(t)\,dt, \tag{13}$$

where d is a constant which is equal to 0 in the case $\mu > \lambda$.

THEOREM II. *In order that summability (f) should imply summability (g), it is necessary and sufficient that*

(i) $$\mu \geqslant \lambda,$$

(ii) $$\int_0^x |p(x-t)|\,f_1(t)\,dt = O[g_1(x)] \quad as \quad x \to \infty,$$

and that

(iii) *for any fixed $\delta > 0$,*

$$\int_0^{2\delta} p(x-t)\,K_\delta(t)\,dt = o[g_1(x)] \quad as \quad x \to \infty,$$

where $K_\delta(t)$ is defined by

$$K_\delta(t) = t \qquad (0 \leqslant t \leqslant \delta),$$

$$K_\delta(t) = 2\delta - t \qquad (\delta \leqslant t \leqslant 2\delta),$$

$$K_\delta(t) = 0 \qquad \text{for all other values of } t.$$

If $f_1(t) \to \infty$ as $t \to \infty$, or if $f_1(t)$ is bounded and $g_1(t)$ is also bounded, then (iii) *is implied by* (ii), *and may therefore be omitted.*

THEOREM III. (*a*) *Suppose that*

(i) $$\mu \geqslant \lambda,$$

(ii) $$\int_0^\infty |p(x)|\,dx$$

converges, and

(iii) $$f_1(x) = O[g_1(x)] \quad as \quad x \to \infty.$$

Then summability (f) *implies summability* (g).

(*b*) *If summability* (f) *implies summability* (g). *then*

$$f_1(x) = O[g_1(x)] \quad as \quad x \to \infty.$$

If, also, $$g_1(x) = O[f_1(x)] \quad as \quad x \to \infty,$$

then $$\int_0^\infty |p(x)|\,dx \quad converges.$$

An immediate corollary of Theorem III is

THEOREM IV. *In order that summability* (f) *and summobility* (g) *should be equivalent, it is necessary and sufficient that* $\mu = \lambda$, *and that both*

$$\int_0^\infty |p(x)|\,dx \quad and \quad \int_0^\infty |q(x)|\,dx$$

should converge, where $q(x)$ *is defined in a similar manner to* $p(x)$, *but with* $f(x)$ *and* $g(x)$ *interchanged.*

2.1. In order to prove Theorem I, we require two lemmas.

LEMMA 1. *If* $\lambda > 0$, *there is a function* $h(x)$ *such that*

(i) $h(x)$ *is, in* $(0, \infty)$, *the fourth indefinite integral of a function which is* $O(e^{\epsilon x})$ *as* $x \to \infty$, *for any positive* ϵ.

(ii) $h(x)$ *tends to a non-zero constant as* $x \to 0$, *and*

(iii) $$f(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} h(t)\,dt.$$

We first consider the case $0 < \lambda < 1$. It follows from the conditions imposed on $f(x)$ that it is an indefinite integral of a function $f'(x)$. Thus

$$f(x) = \int_0^x f'(t)\,dt = \frac{1}{\Gamma(\lambda)\,\Gamma(1-\lambda)} \int_0^x f'(t)\,dt \int_t^x (x-u)^{\lambda-1}(u-t)^{-\lambda}\,du$$

$$= \frac{1}{\Gamma(\lambda)\,\Gamma(1-\lambda)} \int_0^x (x-u)^{\lambda-1}\,du \int_0^u f'(t)\,(u-t)^{-\lambda}\,dt.$$

Hence, if we put

$$h(x) = \frac{1}{\Gamma(1-\lambda)} \int_0^x f'(t)\,(x-t)^{-\lambda}\,dt,$$

then (iii) is satisfied. Moreover,

$$h(x) \sim k \int_0^x t^{\lambda-1}(x-t)^{-\lambda}\,dt = k \quad \text{as} \quad x \to 0,$$

so that (ii) is satisfied. In order to prove that (i) is satisfied, we write $h(x) = h_1(x) + h_2(x)$, where

$$h_1(x) = \frac{1}{\Gamma(1-\lambda)} \int_0^x \left\{ \frac{d}{dt} \left[kt^{\lambda} + ct^{\lambda+1} + ct^{\lambda+2} + ct^{\lambda+3} + ct^{\lambda+4} \right] \right\} (x-t)^{-\lambda}\,dt,$$

and

$$h_2(x) = \frac{1}{\Gamma(1-\lambda)} \int_0^x \left\{ \frac{d}{dt}\, r(t) \right\} (x-t)^{-\lambda}\,dt.$$

It is evident that $h_1(x)$ satisfies (i), so we need consider only $h_2(x)$. Now $r(x)$ is the fifth indefinite integral of a function $s(x)$, say. Thus

$$h_2(x) = k \int_0^x (x-t)^{-\lambda}\,dt \int_0^t (t-u)^3 s(u)\,du$$

$$= k \int_0^x (x-t)^3\,dt \int_0^t (t-u)^{-\lambda} s(u)\,du.$$

Thus $h_2(x)$ is a fourth indefinite integral of

$$k \int_0^x (x-u)^{-\lambda} s(u)\,du,$$

and the result is now evident.

Next suppose that $\lambda = 1$. In this case the result is trivial; we have only to take

$$h(x) = \frac{df(x)}{dx}.$$

If $\lambda > 1$, suppose that $\lambda = m + \theta$, where m is an integer and $0 < \theta \leqslant 1$. Then $f(x)$ is an m-th indefinite integral of a function $f^{(m)}(x)$ which satisfies conditions similar to those imposed on $f(x)$, but with θ instead of λ. It follows from the cases already dealt with that there is a function $h(x)$ satisfying conditions (i) and (ii), and such that

$$f^{(m)}(x) = \frac{1}{\Gamma(\theta)} \int_0^x (x-t)^{\theta-1} h(t)\,dt.$$

Now
$$f(x) = k\int_0^x (x-t)^{m-1} f^{(m)}(t)\,dt$$
$$= k\int_0^x (x-t)^{\lambda-1} h(t)\,dt,$$

and the result follows.

LEMMA 2. *For large* $|s|$,
$$F(s) = ks^{-\lambda-1} + cs^{-\lambda-2} + cs^{-\lambda-3} + O(|s|^{-\lambda-4})$$
uniformly in $\Re s \geqslant c > 0$.

Suppose first that $\lambda = 0$. Then, by integrating by parts four times, we have
$$F(s) = \int_0^\infty e^{-xs} f(x)\,dx = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{f'''(0)}{s^4} + \frac{1}{s^4}\int_0^\infty e^{-xs} f^{\text{iv}}(x)\,dx,$$
and the result follows at once. Next suppose that $\lambda > 0$. Then, by Lemma 1,
$$F(s) = \frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-xs}\,dx \int_0^x (x-t)^{\lambda-1} h(t)\,dt$$
$$= \frac{1}{\Gamma(\lambda)}\int_0^\infty h(t)\,dt \int_t^\infty e^{-xs}(x-t)^{\lambda-1}\,dx$$
$$= s^{-\lambda}\int_0^\infty e^{-ts} h(t)\,dt,$$
and the result follows from the case $\lambda = 0$. Finally, if $\lambda < 0$, we have
$$F(s) = s\int_0^\infty e^{-xs} f_1(x)\,dx.$$
Now $f_1(x)$ satisfies conditions similar to those imposed upon $f(x)$, but with $\lambda+1$ in place of λ, and the result follows at once from the case $\lambda > 0$.

2.2. We are now in a position to prove Theorem I. The first part follows at once from Lemma 2. In order to prove (ii), we suppose first that $\mu > \lambda$. Since, for $\sigma > 0$, $e^{-x\sigma}\, g(x)$ is absolutely integrable in $(0, \infty)$, and since $g(x)$ is continuous and of bounded variation in any finite interval which does not include the origin, it follows from Fourier's Integral Theorem that, for $\sigma > 0$, $x > 0$,
$$e^{-x\sigma} g(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{itx}\,dt \int_0^\infty e^{-ity-y\sigma} g(y)\,dy.$$

Writing $\sigma+it=s$, we deduce that

$$g(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{xs}G(s)\,ds. \tag{14}$$

Now it follows from Lemma 2 that

$$\frac{G(s)}{F(s)}=ks^{\lambda-\mu}+O(|s|^{\lambda-\mu-1}). \tag{15}$$

The integral (12) is thus convergent for all real x, except possibly $x=0$. If x is negative, the integral

$$\frac{1}{2\pi i}\int e^{xs}\frac{G(s)}{F(s)}\,ds,$$

taken round that part of the circle $|z|=R$ which lies to the right of the contour in (12), tends to 0 as $R\to\infty$. It follows at once that, when $x<0$, $p(x)=0$. Hence

$$\begin{aligned}\int_0^x p(x-t)f(t)\,dt&=\int_0^\infty p(x-t)f(t)\,dt\\&=\frac{1}{2\pi i}\int_0^\infty f(t)\,dt\int_{\sigma-i\infty}^{\sigma+i\infty}e^{(x-t)s}\frac{G(s)}{F(s)}\,ds\\&=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{xs}\frac{G(s)}{F(s)}\,ds\int_0^\infty e^{-ts}f(t)\,dt\end{aligned}$$

(the inversion in the order of integration being easily justified)

$$\begin{aligned}&=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{xs}G(s)\,ds\\&=g(x),\end{aligned}$$

by (14), which proves the theorem. We must now consider the case $\mu=\lambda$. We then have

$$\frac{g(x)}{f(x)}\to d \quad \text{as} \quad x\to 0,$$

where d is a constant. It is evident that $g(x)-df(x)\sim cx^{\lambda+1}$ as $x\to 0$, and hence, by the case $\mu>\lambda$, we see that, if we put

$$\bar{p}(x)=\int_{\sigma-i\infty}^{\sigma+i\infty}e^{xs}\frac{G(s)-dF(s)}{F(s)}\,ds,$$

then $$g(x)-df(x)=\int_0^x \bar{p}(x-t)f(t)\,dt.$$

But $$\int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs}\,ds=0,$$

if the integral is interpreted as a Cesàro integral. It follows that $\bar{p}(x)=p(x)$, and the theorem is now proved.

3.1. In what follows, we write

$$A^{(f)}(x)=\int_0^x A(t)f(x-t)\,dt, \tag{16}$$

$$A^{(g)}(x)=\int_0^x A(t)g(x-t)\,dt, \tag{17}$$

and we use a similar notation when $A(x)$ is replaced by $A_r(x)$ or by $B(x)$. In order to prove Theorem II, we require some more lemmas.

LEMMA 3. *Suppose that* $\mu\geqslant\lambda$. *Then*

$$A^{(g)}(x)=dA^{(f)}(x)+\int_0^x A^{(f)}(t)\,p(x-t)\,dt,$$

$p(t)$ and d being defined as in Theorem I.

We have, by Theorem I,

$$\begin{aligned}A^{(g)}(x)&=\int_0^x A(u)\left\{df(x-u)+\int_u^x p(x-t)f(t-u)\,dt\right\}du\\&=dA^{(f)}(x)+\int_0^x p(x-t)\,dt\int_0^t A(u)f(t-u)\,du\\&=dA^{(f)}(x)+\int_0^x A^{(f)}(t)\,p(x-t)\,dt,\end{aligned}$$

and the lemma is now proved.

LEMMA 4. *Suppose that* $D\geqslant 0$. *Let* $\phi(x)$ *be any function such that* $\phi(x)=0$ *for* $0\leqslant x\leqslant D$, *and such that* $\phi(x+D)$ *satisfies conditions similar to those imposed on* $f(x)$, *but with* λ *replaced by* ν, *where* $\nu>\lambda+1$. *Then we can choose* $A(x)$ *so that* $A^{(f)}(x)=\phi(x)$.

If $D=0$, the result follows from Theorem I; we have only to observe that (with the notation of Theorem I), if $\mu>\lambda+1$, then, by Lemma 2, the integral (12) converges uniformly for x in any finite interval, and hence that $p(x)$ is bounded for x in any finite interval. If $D>0$, it follows

from the case $D=0$ that we can choose $B(x)$ so that $B^{(f)}(x)=\phi(x+D)$. If we now choose

$$A(x)=0 \qquad (x\leqslant D)$$
$$A(x)=B(x-D) \quad (x>D)$$

it is evident that the required condition is satisfied.

LEMMA 5. *Suppose that, given any positive numbers K, X, we can determine $A(x)$ so that*

(i) *$A^{(f)}(x)$ satisfies the conditions of Lemma 4, with ν an integer,*

(ii) *$|A^{(f)}(x)|\leqslant 1$ for all x,*

(iii) *$A^{(f)}(x)=0$ for $x\geqslant X$, and*

(iv) *$|A^{(g)}(X)|\geqslant K$.*

Then summability (f) does not imply summability (g).

If the above conditions are satisfied, then, by (iii), $A(x)$ is summable (f) to 0. If $A^{(g)}(x)$ is not $o[g_1(x)]$ as $x\to\infty$, then $A(x)$ is not summable (g) to 0, and the result follows. We may therefore suppose that whenever the above conditions are satisfied,

$$A^{(g)}(x)=o[g_1(x)] \quad \text{as} \quad x\to\infty. \tag{18}$$

We next note that, if we put

$$A(x)=0 \qquad (x\leqslant Y)$$
$$A(x)=B(x-Y) \qquad (x>Y),$$

then

$$A^{(f)}(x)=0 \qquad (x\leqslant Y)$$
$$A^{(f)}(x)=B^{(f)}(x-Y) \quad (x>Y),$$

and that a similar result holds for $A^{(g)}(x)$. It follows at once that, given positive numbers X, Y, K, where $X>Y$, we can determine $A(x)$ so that the above conditions are satisfied, and so that, in addition,

$$A(x)=0 \quad \text{for} \quad x\leqslant Y,$$

and hence also

$$A^{(f)}(x)=A^{(g)}(x)=0 \quad \text{for } x\leqslant Y.$$

We now define numbers X_r, Y_r, and functions $A_r(x)$, by induction as follows. We choose $Y_1>0$ arbitrarily. Y_r having been chosen, we take

X_r as any number greater than Y_r, and then choose $A_r(x)$ so that $A_r^{(f)}(x)$ satisfies condition (i) of this lemma, and so that

$$A_r^{(f)}(x) = 0 \quad \text{for} \quad x \geqslant X_r \tag{19}$$

$$A_r(x) = 0 \quad \text{for} \quad x \leqslant Y_r \tag{20}$$

$$|A_r^{(f)}(x)| \leqslant \frac{1}{r} f_1(Y_r) \quad \text{for all } x \tag{21}$$

and

$$|A_r^{(g)}(X_r)| \geqslant r g_1(X_r). \tag{22}$$

Finally, X_s, Y_s and $A_s(x)$ having been chosen for all $s < r$, we choose Y_r so that $Y_r > X_{r-1}$, and so that

$$\sum_{s=1}^{r-1} |A_s^{(g)}(x)| \leqslant g_1(x) \quad \text{for} \quad x \geqslant Y_r; \tag{23}$$

it follows from (18) that this is possible. Now $A_r^{(f)}(x)$ differs from 0 only in the interval (Y_r, X_r), and the intervals (Y_r, X_r) are non-overlapping. It is thus evident that

$$\sum_{r=1}^{\infty} A_r^{(f)}(x)$$

satisfies the conditions of Lemma 4. We can therefore choose $A(x)$ so that

$$A^{(f)}(x) = \sum_{r=1}^{\infty} A_r^{(f)}(x),$$

and it follows from Lemma 3 that we then have

$$A^{(g)}(x) = \sum_{r=1}^{\infty} A_r^{(g)}(x).$$

When $A(x)$ is so chosen, it follows from (19), (20) and (21), and the fact that $f_1(x)$ is an increasing function of x, that $A(x)$ is summable (f) to 0 as $x \to \infty$. On the other hand, since

$$A_s^{(g)}(x) = 0 \quad \text{for} \quad x \leqslant Y_s,$$

we have

$$|A^{(g)}(X_r)| = \left| \sum_{s=1}^{r} A_s^{(g)}(X_r) \right| \geqslant (r-1) g_1(X_r),$$

by (22) and (23). Thus $A(x)$ is not summable (g), and the lemma is proved.

COROLLARY. *The result still holds when condition* (iv) *is replaced by*

$$|A^{(g)}(x)| \geqslant K, \tag{iv a}$$

where x is a given number satisfying $0 < x < X$.

The modifications of the proof which are necessary to establish this result are evident.

LEMMA 6. *Suppose that conditions* (i) *and* (ii) *of Theorem* II *are satisfied. Then*

$$f_1(x) = O[g_1(x)] \quad as \quad x \to \infty.$$

If $p(x)$ is identically zero, it follows from (13) that

$$g(x) = df(x),$$

and the result is trivial. We therefore suppose that $p(x)$ is not identically zero. We can then find X such that

$$\int_0^X |p(x)|\,dx > 0. \tag{24}$$

We next note that, since $f(x) = o[f_1(x)]$, we can find x_0 such that, for $x > x_0$,

$$f(x) \leqslant \frac{1}{2X} f_1(x).$$

Since $f_1(x)$ is an increasing function, it follows that, for $x \geqslant t \geqslant x - X \geqslant x_0$,

$$f_1(t) = f_1(x) - \int_t^x f(u)\,du \geqslant \tfrac{1}{2} f_1(x).$$

Hence, for $x \geqslant x_0 + X$,

$$f_1(x) \int_{x-X}^{x} |p(x-t)|\,dt \leqslant 2 \int_{x-X}^{x} f_1(t)\,|p(x-t)|\,dt = O[g_1(x)] \quad \text{as} \quad x \to \infty.$$

The result now follows with the aid of equation (24).

LEMMA 7. *Suppose that conditions* (i) *and* (ii) *of Theorem* II *are satisfied. Let δ be any fixed positive number. Then, if $f_1(x) \to \infty$ as $x \to \infty$,*

$$\int_{x-\delta}^{x} |p(t)|\,dt = o[g_1(x)] \quad as \quad x \to \infty,$$

and, in any case,

$$\int_{x-\delta}^{x} |p(t)|\,dt = O[g_1(x)] \quad as \quad x \to \infty.$$

If $X > 0$, we have

$$\int_{x-\delta}^{x} |p(t)|\,dt \leqslant \frac{1}{f_1(X)} \int_{x-\delta}^{x} |p(t)| f_1(X+x-t)\,dt$$

$$\leqslant \frac{K g_1(X+x)}{f_1(X)},$$

by condition (ii), where K is a constant independent of x and X. Since

$$g(x) = o[g_1(x)] \quad \text{as} \quad x \to \infty,$$

it follows that, for fixed X,

$$g_1(x+X) \sim g_1(x) \quad \text{as} \quad x \to \infty,$$

and the result is now evident.

LEMMA 8. *If $\sigma > 0$, $\nu > 0$, $x > 0$, and the integral is taken along a straight line, then*

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{xs}}{s^{\nu}}\,ds = \frac{x^{\nu-1}}{\Gamma(\nu)}.$$

This result may easily be verified.

LEMMA 9. *Suppose that $\mu \geqslant \lambda$. Then $p(t)$ is continuous for $t > 0$.*

By Lemma 2,

$$p(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs} \frac{G(s)}{F(s)}\,ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs} \{cs^{\lambda-\mu} + cs^{\lambda-\mu-1} + R(s)\}\,ds,$$

where $$R(s) = O(|s|^{\lambda-\mu-2}).$$

The integral $$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs} R(s)\,ds$$

is uniformly convergent for x in any finite interval. It therefore represents a continuous function, and the result follows with the aid of Lemma 8.

LEMMA 10. *Suppose that $\mu \geqslant \lambda$, and that summability (f) implies summability (g). Suppose also that, for any fixed X,*

$$\int_0^X |p(x-t)| f_1(t)\,dt = O[g_1(x)] \quad \textit{as} \quad x \to \infty. \tag{25}$$

Then $$\int_0^x |p(x-t)| f_1(t)\,dt = O[g_1(x)] \quad \textit{as} \quad x \to \infty.$$

We suppose that the conclusion is false, and show that this leads to a contradiction. We define numbers X_r, K_r and functions $A_r(x)$ by induction as follows. We choose $X_1 > 0$ arbitrarily. X_r having been chosen, we choose K_r so that

$$\int_0^{X_r} |p(x-t)| f_1(t)\,dt \leqslant K_r g_1(x) \quad \text{for} \quad x \geqslant X_r; \tag{26}$$

it follows from (25) that this is possible. X_r and K_r having been chosen, we choose $X_{r+1} > X_r$ so that

$$\int_0^{X_{r+1}} |p(X_{r+1}-t)| f_1(t)\,dt \geqslant (r^2+1) K_r g_1(X_{r+1}); \tag{27}$$

it follows from (26) and (27) that

$$\int_{X_r}^{X_{r+1}} |p(X_{r+1}-t)| f_1(t)\,dt \geqslant r^2 K_r g_1(X_{r+1}). \tag{28}$$

Finally, X_r, X_{r+1} and K_r having been chosen, it follows from Lemmas 4 and 9, and equation (28), that we can choose $A_r(x)$ so that $A_r^{(f)}(x)$ satisfies the conditions of Lemma 4, with ν an integer, and so that

$$A_r^{(f)}(x) = 0 \quad \text{for} \quad x \leqslant X_r \quad \text{and for} \quad x \geqslant X_{r+1}, \tag{29}$$

$$|A_r^{(f)}(x)| \leqslant \frac{1}{r} f_1(x) \quad \text{everywhere}, \tag{30}$$

and

$$\int_{X_r}^{X_{r+1}} p(X_{r+1}-t) A_r^{(f)}(t)\,dt \geqslant \tfrac{1}{2} r K_r g_1(X_{r+1}). \tag{31}$$

Using Lemma 3, we now deduce from (26), (29) and (30) that, for $x \geqslant X_{r+1}$,

$$|A_r^{(g)}(x)| = \left| \int_0^{X_{r+1}} p(x-t) A_r^{(f)}(t)\,dt \right| \leqslant \frac{1}{r} K_{r+1} g_1(x); \tag{32}$$

we also deduce from (31) that

$$A_r^{(g)}(X_{r+1}) \geqslant \tfrac{1}{2} r K_r g_1(X_{r+1}). \tag{33}$$

As in the proof of Lemma 5, we can now choose $A(x)$ so that

$$A^{(f)}(x) = \sum_{r=1}^{\infty} A_r^{(f)}(x).$$

It then follows from (29) and (30) that $A(x)$ is summable (f) to 0 as $x \to \infty$. But, by (32) and (33),

$$A^{(g)}(X_{r+1}) = \sum_{s=1}^{r} A_s^{(g)}(X_{r+1})$$

$$\geqslant g_1(X_{r+1}) \left[\tfrac{1}{2} r K_r - \sum_{s=1}^{r-1} \frac{1}{s} K_{s+1} \right].$$

Since it is evident from (26) and (27) that $\{K_r\}$ is an increasing sequence, it follows that $A(x)$ is not summable (g). Our contradiction has now been obtained.

3.2. We are now in a position to prove Theorem II. We first show that the conditions are sufficient. Suppose, then, that the conditions are satisfied. If $A(x)$ is summable (f) to s, we have

$$A^{(f)}(x) = s f_1(x) + r(x), \tag{34}$$

where
$$r(x) = o[f_1(x)] \quad \text{as} \quad x \to \infty. \tag{35}$$

Now it follows from Theorem I that

$$g_1(x) = d f_1(x) + \int_0^x p(x-t) f_1(t)\, dt. \tag{36}$$

Using Lemma 3, we deduce from (34) and (36) that

$$A^{(g)}(x) = s g_1(x) + d r(x) + \int_0^x p(x-t) r(t)\, dt. \tag{37}$$

We next note that, by Lemma 6, it follows from (35) that

$$r(x) = o\,[g_1(x)] \quad \text{as} \quad x \to \infty,$$

so that it remains only to consider the integral on the right of (37). Now let ϵ be any given positive number. Then we can find X such that

$$|r(x)| \leqslant \epsilon f_1(x) \quad \text{for} \quad x \geqslant X. \tag{38}$$

Hence, if $x > X$,

$$\left| \int_0^x p(x-t) r(t)\, dt \right| \leqslant \epsilon I_1 + I_2,$$

where
$$I_1 = \int_X^x |p(x-t)| f_1(t)\, dt = O[g_1(x)] \quad \text{as} \quad x \to \infty.$$

by condition (ii), and where

$$I_2 = \left| \int_0^X p(x-t)\, r(t)\, dt \right|.$$

If $f_1(x) \to \infty$ as $x \to \infty$, it follows from Lemma 7 that

$$I_2 = O\left[\int_0^X |p(x-t)|\, dt \right] = o[g_1(x)] \quad \text{as} \quad x \to \infty,$$

and the result follows. We next consider the case when $f_1(x)$ is bounded. Suppose, then, that $f_1(x) \to M$ as $x \to \infty$. It follows from (38) that $|r(X)| \leqslant M\epsilon$. Since $r(x)$ is continuous, and since $r(0) = 0$, it follows that we can determine an integer N and a continuous function $k(x)$, defined in $(0, X)$, which is linear in each of the intervals $\big(n\delta, (n+1)\delta\big)$ (where $N\delta = X$, $n = 0, 1, \ldots, N-1$), and which is such that $k(0) = k(X) = 0$, and

$$|k(x) - r(x)| \leqslant 2M\epsilon. \tag{39}$$

It follows from (39) that

$$I_2 \leqslant 2M\epsilon\, I_{21} + I_{22},$$

where $$I_{21} = \int_0^X |p(x-t)|\, dt = O[g_1(x)] \quad \text{as} \quad x \to \infty,$$

by Lemma 7, and where

$$I_{22} = \left| \int_0^X p(x-t)\, k(t)\, dt \right|.$$

Now it follows from condition (iii) that, for any fixed h,

$$\int_h^{h+2\delta} p(x-t)\, K_\delta(t-h)\, dt$$

$$= \int_0^{2\delta} p(x-h-t)\, K_\delta(t)\, dt = o[g_1(x-h)] = o[g_1(x)] \quad \text{as} \quad x \to \infty.$$

Since it is a consequence of the conditions satisfied by $k(x)$ that it is the sum of terms of the form $cK_\delta(x-n\delta)$, it follows at once that

$$I_{22} = o[g_1(x)] \quad \text{as} \quad x \to \infty,$$

and the proof that the conditions are sufficient is now completed.

3.3. We next show that condition (i) is necessary; that is to say, that if $\mu < \lambda$, then summability (f) does not imply summability (g). Consider first the case in which $\lambda - 1 < \mu < \lambda$. Since

$$\int_0^\infty e^{-xs} g_1(x)\,dx = \frac{G(s)}{s},$$

it follows from Theorem I that, if we write

$$p_1(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs}\frac{G(s)}{sF(s)}\,ds,$$

then

$$g_1(x) = \int_0^x p_1(x-t) f(t)\,dt. \tag{40}$$

Now it follows from Lemma 2 that

$$\frac{G(s)}{F(s)} = ks^{\lambda-\mu} + cs^{\lambda-1-\mu} + R(s),$$

where

$$R(s) = O(|s|^{\lambda-2-\mu}).$$

We note that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs}\frac{R(s)}{s}\,ds$$

is an indefinite integral of

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{xs} R(s)\,ds,$$

since the latter integral is absolutely convergent. Using Lemma 8, we deduce that $p_1(x)$ is, for $x > 0$, the indefinite integral of a function $p(x)$, where

$$p(x) \sim kx^{-\lambda+\mu-1} \quad \text{as} \quad x \to 0. \tag{41}$$

Thus, if $X > 0$,

$$p_1(x) = B + \int_X^x p(t)\,dt, \tag{42}$$

where, for fixed X, B is a constant. Now

$$\int_0^x A^{(g)}(t)\,dt = \int_0^x A(u)\, g_1(x-u)\,du$$

$$= \int_0^x A^{(f)}(t)\, p_1(x-t)\,dt,$$

by (40), using an argument similar to the proof of Lemma 3. We deduce from (42) that, if $x \geqslant X$,

$$\int_0^x A^{(g)}(t)\,dt = B\int_0^x A^{(f)}(t)\,dt + \int_0^{x-X} A^{(f)}(t)\,dt \int_X^{x-t} p(u)\,du - \int_{x-X}^{x} A^{(f)}(t)\,dt \int_{x-t}^{X} p(u)\,du$$

$$= B\int_0^x A^{(f)}(t)\,dt + \int_X^x p(u)\,du \int_0^{x-u} A^{(f)}(t)\,dt - \int_0^X p(u)\,du \int_{x-u}^{x} A^{(f)}(t)\,dt.$$

If $A^{(f)}(x)$ satisfies the conditions of Lemma 4, we deduce by differentiation* that, for $x \geqslant X$,

$$A^{(g)}(x) = BA^{(f)}(x) + \int_X^x A^{(f)}(x-u)\,p(u)\,du - \int_0^X p(u)[A^{(f)}(x) - A^{(f)}(x-u)]\,du,$$

and, in particular,

$$A^{(g)}(X) = BA^{(f)}(X) - \int_0^X p(u)[A^{(f)}(X) - A^{(f)}(X-u)]\,du. \tag{43}$$

By Lemma 4, we can now choose $A(x)$ so that (43) is satisfied, and so that

$$A^{(f)}(x) = 0 \quad \text{for } x \geqslant X,$$

$$0 \leqslant A^{(f)}(x) \leqslant 1 \quad \text{for all } x,$$

$$A^{(f)}(x) = 1 \quad \text{for } \tfrac{1}{2}X \leqslant x \leqslant X - \eta.$$

It follows at once from (41) and (43) that, X being given, we can make $A^{(g)}(X)$ arbitrarily large by making η sufficiently small. The result follows at once with the aid of Lemma 5.

We next suppose that $\mu = \lambda - 1$. Defining $p_1(x)$ as before, we have

$$g_1(x) = kf(x) + \int_0^x p_1(x-t)\,f(t)\,dt.$$

Now
$$\frac{G(s)}{F(s)} = ks + c + cs^{-1} + R(s),$$

where
$$R(s) = O(|s|^{-2}).$$

---

* The result holds everywhere, since both sides are continuous. Of course, much less stringent conditions than those of Lemma 4 would suffice to ensure the truth of this equation.

It follows as before from Lemma 8 that, for $x \geqslant 0$, $p_1(x)$ is an indefinite integral of a function $p(x)$. Thus

$$p_1(x) = c + \int_0^x p(t)\,dt.$$

Again, we see as before that

$$\int_0^x A^{(g)}(t)\,dt = kA^{(f)}(x) + \int_0^x A^{(f)}(t)\,p_1(x-t)\,dt.$$

Hence, if $A^{(f)}(x)$ is differentiable,

$$A^{(g)}(x) = k\frac{dA^{(f)}(x)}{dx} + cA^{(f)}(x) + \int_0^x A^{(f)}(t)\,p(x-t)\,dt.$$

Now, X being given, we see that, if

$$|A^{(f)}(x)| \leqslant 1 \text{ for all } x, \tag{44}$$

then $$\left| cA^{(f)}(x) + \int_0^x A^{(f)}(t)\,p(x-t)\,dt \right| \leqslant c + \int_0^X |p(t)|\,dt = c,$$

for $x \leqslant X$. Now it follows from Lemma 4 that we can choose $A(x)$ so that (44) is satisfied, so that

$$A^{(f)}(x) = 0 \quad \text{for} \quad x \geqslant X,$$

and so that $$\left| \frac{dA^{(f)}(x)}{dx} \right|$$

is arbitrarily large for a given x satisfying $0 < x < X$. Thus the conditions of Lemma 5, Corollary, are satisfied, and the result follows.

Lastly suppose that $\mu < \lambda - 1$. It is evident that summability (g) implies summability (g_1), and hence, by induction, summability (g_n), where

$$g_n(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} g(t)\,dt.$$

Hence, if summability (f) implies summability (g), it must also imply summability (g_n). But, if we choose n so that

$$\lambda - 1 \leqslant \mu + n < \lambda,$$

this is impossible, by the cases already dealt with. The result follows.

3.4. The condition (i) having been proved necessary, we may now suppose it to be satisfied. We shall next show that the condition* that, for $\delta > 0$,

$$\int_{x-\delta}^{x} |p(t)|\,dt = O[g_1(x)] \quad \text{as} \quad x \to \infty, \tag{45}$$

is necessary. We suppose that summability (f) implies summability (g), but that (45) is not satisfied, and deduce that this leads to a contradiction. Let $\{\eta_r\}$ be any given sequence decreasing to 0. We define numbers X_r, Y_r and functions $A_r(x)$ by induction as follows. We choose $Y_1 > 0$ arbitrarily. Y_r having been chosen, we choose $X_r > Y_r$ so that

$$\eta_r f_1(Y_r) \int_{X_r - Y_r}^{X_r - Y_r + \delta} |p(t)|\,dt \geqslant g_1(X_r); \tag{46}$$

this is possible, since (45) is false, and since, for fixed Y_r,

$$g_1(x - Y_r + \delta) \sim g_1(x) \quad \text{as} \quad x \to \infty.$$

Next it follows from Lemmas 4 and 9, and equation (46), that, X_r and Y_r having been chosen, we can choose $A_r(x)$ so that $A_r^{(f)}(x)$ satisfies the conditions of Lemma 4, with ν an integer, and so that

$$A_r^{(f)}(x) = 0 \quad \text{for } x \leqslant Y_r - \delta, \text{ and for } x \geqslant Y_r, \tag{47}$$

$$|A_r^{(f)}(x)| \leqslant \eta_r f_1(Y_r) \quad \text{for all } x, \tag{48}$$

and

$$A_r^{(g)}(X_r) = \int_0^{X_r} A_r^{(f)}(x-t)\,p(t)\,dt \geqslant \tfrac{1}{2} g_1(X_r). \tag{49}$$

Finally, suppose X_s, Y_s and $A_s(x)$ chosen for all $s < r$. Since

$$\sum_{s=1}^{r-1} A_s^{(f)}(x) = 0 \quad \text{for all sufficiently large } x,$$

and since summability (f) implies summability (g), it follows that

$$\sum_{s=1}^{r-1} A_s^{(g)}(x) = o[g_1(x)] \quad \text{as} \quad x \to \infty.$$

Hence we can choose Y_r so that $Y_r > X_{r-1} + \delta$, and so that, for $x \geqslant Y_r$,

$$\left| \sum_{s=1}^{r-1} A_s^{(g)}(x) \right| \leqslant \tfrac{1}{4} g_1(x). \tag{50}$$

* It is evident that, if (45) holds for some positive δ, then it holds for any given positive δ.

As in the proof of Lemma 5, we can now choose $A(x)$ so that

$$A^{(f)}(x) = \sum_{r=1}^{\infty} A_r^{(f)}(x).$$

It then follows from (47) and (48) that $A(x)$ is summable (f) to 0. But, by (49) and (50),

$$A^{(g)}(X_r) = \sum_{s=1}^{r} A_s^{(g)}(X_r) \geqslant \tfrac{1}{4} g_1(X_r).$$

Thus $A(x)$ is not summable (g) to 0, and our contradiction has been obtained.

Now it follows at once from (45) that, for fixed X,

$$\int_0^X |p(x-t)| f_1(t)\, dt = O[g_1(x)],$$

and hence, by Lemma 10, that condition (ii) is necessary.

It remains to show that (iii) is necessary. Suppose, then, that summability (f) implies summability (g); we have proved that conditions (i) and (ii) are satisfied. Let ϵ be any positive number. It follows from Lemma 4 that we can choose $A(x)$ so that

$$A^{(f)}(x) = 0 \quad \text{for } x \geqslant 2\delta, \tag{51}$$

and

$$|A^{(f)}(x) - K_\delta(x)| \leqslant \epsilon.$$

Thus

$$\left| \int_0^{2\delta} p(x-t)\, K_\delta(t)\, dt \right| \leqslant \epsilon I_1 + I_2,$$

where

$$I_1 = \int_0^{2\delta} |p(x-t)|\, dt = O[g_1(x)] \quad \text{as} \quad x \to \infty,$$

by Lemma 7, and where

$$I_2 = \left| \int_0^{2\delta} p(x-t)\, A^{(f)}(t)\, dt \right|.$$

It follows from Lemma 3 and equation (51) that, for $x > 2\delta$,

$$I_2 = |A^{(g)}(x)|,$$

and hence that

$$I_2 = o\,[g_1(x)] \quad \text{as} \quad x \to \infty.$$

The result follows.

Finally, the last part of the theorem is an immediate corollary of Lemma 7.

3.5. We now come to the proof of Theorem III. If the conditions of Theorem III (a) are satisfied, then

$$\int_0^x |p(x-t)|\,f_1(t)\,dt \leqslant f_1(x)\int_0^\infty |p(t)|\,dt$$
$$= O[g_1(x)] \quad \text{as} \quad x\to\infty.$$

Again, it follows from the convergence of

$$\int_0^\infty |p(x)|\,dx \tag{52}$$

that $$\int_{x-2\delta}^x |p(t)|\,dt = o(1) = o[g_1(x)] \quad \text{as} \quad x\to\infty,$$

and hence that condition (iii) of Theorem II is satisfied. It now follows from Theorem II that summability (f) implies summability (g).

Next suppose that summability (f) implies summability (g). It then follows at once from Theorem II and Lemma 6 that

$$f_1(x) = O[g_1(x)] \quad \text{as} \quad x\to\infty.$$

Again, by Theorem II, we have, for $x \geqslant c > 0$,

$$\int_0^x |p(x-t)|\,f_1(t)\,dt \leqslant Kg_1(x), \tag{53}$$

where K is a constant. Now let X be any given positive number. Since

$$f(x) = o[f_1(x)] \quad \text{as} \quad x\to\infty,$$

it follows that, for x sufficiently large,

$$f_1(t) \geqslant \tfrac{1}{2}f_1(x) \quad \text{for} \quad t \geqslant x-X. \tag{54}$$

Suppose x chosen so that (54) is satisfied. Then

$$\int_0^X |p(t)|\,dt \leqslant \frac{2}{f_1(x)}\int_{x-X}^x |p(x-t)|\,f_1(t)\,dt \leqslant \frac{2Kg_1(x)}{f_1(x)}, \tag{55}$$

by (53). If $$g_1(x) = O[f_1(x)] \quad \text{as} \quad x\to\infty,$$

the expression on the right of (55) is less than a constant independent of X. It follows that the integral (52) converges, and the proof of Theorem III is now completed.

In order to prove Theorem IV, we note that, since

$$g_1(x) = df_1(x) + \int_0^x p(x-t)\,f_1(t)\,dt,$$

it follows that, if the integral (52) converges, then

$$|g_1(x)| \leqslant f_1(x)\left\{|d|+\int_0^\infty |p(t)|\,dt\right\}$$

$$= O[f_1(x)] \quad \text{as} \quad x\to\infty.$$

It now follows at once from Theorem III (a) that the conditions are sufficient. The result that they are necessary is an immediate consequence of Theorem III (b).

3.6. We have deduced Theorem III (and hence Theorem IV) from Theorem II. An examination of the proof of Theorem III shows, however, that in order to deduce Theorem III we do not need the full result of Theorem II; it is enough to use the weaker result:

THEOREM II′. *If conditions* (i) *and* (ii) *of Theorem II are satisfied, and if, for fixed* $\delta > 0$,

$$\int_{x-\delta}^{x} |p(t)|\,dt = o[g_1(x)] \quad \textit{as} \quad x\to\infty, \tag{56}$$

then summability (f) *implies summability* (g). *If summability* (f) *implies summability* (g), *then conditions* (i) *and* (ii) *of Theorem II are satisfied.*

Thus, if our sole object were to prove Theorems III and IV, it would be enough to prove Theorem II′ instead of Theorem II, and a certain simplification of the proof would therefore be possible; the replacement of condition (iii) of Theorem II by (56) simplifies the investigation of I_2 in § 3.2, and, also, we should no longer have to prove that condition (iii) is necessary. I have, however, stated and proved Theorem II (instead of Theorem II′) in order to obtain a solution of the problem of determining necessary and sufficient conditions under which summability (f) implies summability (g).

It may be remarked that, once Theorem II has been obtained, the proof of Theorems III and IV is exactly analogous to Riesz's deduction of Theorems B and C from Theorem A. In proving Theorem A, Riesz appeals to the general theorems of Toeplitz*, whereas here we have no such general theorems to which we can appeal. In order to compare the proof of Theorem II with that of Theorem A, we should imagine the proof

* O. Toeplitz, "Über allgemeine lineare Mittelbildungen", *Prace Mat.-Fiz.*, 22 (1911), 113–119. See also J. Schur, "Über lineare Transformationen in der Theorie der unendlichen Reihen", *Journal für Math.*, 151 (1921), 79–111.

of Toeplitz's general theorems set out for the special case for which they are required in Theorem A. If we do this we observe that, although there are some difficulties arising in the proof of Theorem II which have no counterpart in that of Theorem A, yet some parts of the proof are similar; in particular, the proof of Lemma 10 was suggested to me by a consideration of Toeplitz's work. The differences arise for the following reasons. Toeplitz considers the transformation between the sequences $\{x_n\}$ and $\{y_n\}$ given by

$$y_n = \sum_{m=1}^{\infty} a_{mn} x_m. \tag{57}$$

By taking the special case in which all the x's but one are zero, he proves at once that the condition

$$a_{mn} \to 0 \quad \text{as} \quad n \to \infty \tag{58}$$

is necessary for the transformation to be regular. In our case, equation (57) is replaced by the result of Lemma 3. We cannot now prove in an analogous way that the condition

$$p(x) = o[g_1(x)] \quad \text{as} \quad x \to \infty$$

is necessary for summability (f) to imply summability (g). We prove instead that condition (iii) of Theorem II is necessary. But, whereas Toeplitz uses the necessity of (58) in proving the necessity of his other conditions, we have to use the necessity of condition (ii) in order to prove that of condition (iii).

4.1. We now consider some examples of the application of the theorems just obtained. Throughout we shall write

$$P(s) = \frac{G(s)}{F(s)} \tag{59}$$

so that, if $\mu \geqslant \lambda$,

$$p(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{xs} P(s)\, ds. \tag{60}$$

$F(s)$, $G(s)$ and $P(s)$ are defined, in the first place, only for $\Re s > 0$. But they may be capable of extension by analytic continuation, and we then use the same symbols to denote the extended functions.

We first consider the case in which

$$f(x) = \frac{d}{dx} \frac{\Gamma(\kappa + x)}{\Gamma(x)}$$

and

$$g(x) = x^{\kappa - 1},$$

where $\kappa > 0$. With the notation of C.L.F., summability (f) is then the same as summability (C, κ), while summability (g) is summability (R, κ). It was proved in C.L.F. that, if κ is an integer and if $F(s)$ has zeros for $\Re s > 0$, then summability (C, κ) does not imply summability (R, κ). We now prove the more general result that, if $\kappa > 1$, a necessary and sufficient condition that summability (C, κ) should imply summability (R, κ) is that $F(s)$ should have no zeros for* $\Re s \geqslant 0$. On the other hand, if $\kappa < 1$, summability (C, κ) does not imply summability (R, κ) in any case.

We first require to investigate the properties of $F(s)$. Since, for large $|z|$,

$$f(z) = O(|z|^{\kappa-1}) \tag{61}$$

uniformly in $|\arg z| \leqslant c < \pi$, it is evident that, if $\Re s > 0$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-xs} f(x)\, dx \\ &= \int_0^{\infty \exp(-i \arg s)} e^{-zs} f(z)\, dz \\ &= \frac{1}{s} \int_0^\infty e^{-x} f\left(\frac{x}{s}\right) dx. \end{aligned} \tag{62}$$

It follows that $F(s)$, when extended by analytic continuation, is a function regular in the entire plane with the exception of the negative real axis.

In what follows, equations giving approximations for functions of s hold uniformly in $|\arg s| \leqslant c < \pi$; we state this here to avoid constant repetition. Using (61), and the result that, for small z,

$$f(z) = k + O(|z|), \tag{63}$$

we deduce from (62) that, for large $|s|$,

$$\begin{aligned} F(s) &= O\left\{|s|^{-\kappa} \int_{|s|}^\infty e^{-x} x^{\kappa-1}\, dx\right\} + \frac{k}{s} \int_0^{|s|} e^{-x}\, dx + O\left\{\frac{1}{|s|^2} \int_0^{|s|} e^{-x} x\, dx\right\} \\ &= \frac{k}{s} + O\left(\frac{1}{|s|^2}\right). \end{aligned} \tag{64}$$

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* We shall see that $F(s)$ can be defined by analytic continuation at all points of $\Re s = 0$ except $s = 0$.

Again, since

$$\frac{d^n F(s)}{ds^n} = (-1)^n \int_0^\infty e^{-xs} x^n f(x)\, dx$$

$$= \frac{(-1)^n}{s^{n+1}} \int_0^\infty e^{-x} x^n f\left(\frac{x}{s}\right) dx, \tag{65}$$

we can deduce from (61) and (63), by a similar argument, that, for large $|s|$,

$$F'(s) = \frac{k}{s^2} + O\left(\frac{1}{|s|^3}\right), \tag{66}$$

$$F''(s) = \frac{k}{s^3} + O\left(\frac{1}{|s|^4}\right). \tag{67}$$

In order to investigate the behaviour of $F(s)$ for small s, we use the results that, for $|z|$ large,

$$f_1(z) = \frac{\Gamma(\kappa+z)}{\Gamma(z)} = kz^\kappa + cz^{\kappa-1} + O(|z|^{\kappa-2})$$

uniformly in $|\arg z| \leqslant c < \pi$, and that $f_1(z)$ is bounded in the neighbourhood of the origin. It follows from (62) that

$$F(s) = \int_0^\infty e^{-x} f_1\left(\frac{x}{s}\right) dx$$

$$= ks^{-k} \int_{|s|}^\infty e^{-x} x^\kappa\, dx + cs^{-\kappa+1} \int_{|s|}^\infty e^{-x} x^{\kappa-1}\, dx$$

$$+ O\left\{|s|^{-\kappa+2} \int_{|s|}^\infty e^{-x} x^{\kappa-2}\, dx\right\} + O\left\{\int_0^{|s|} e^{-x}\, dx\right\}$$

$$= ks^{-\kappa} I_1 + cs^{-\kappa+1} I_2 + O(|s|^{-\kappa+2} I_3) + O(I_4), \quad \text{say.}$$

Now, if $\kappa > 1$, $I_3 = O(1)$, and, if $\kappa < 1$, $I_3 = O(|s|^{\kappa-1})$. If $\kappa = 1$, the term in I_3 may be omitted, as may easily be verified. Moreover, in any case

$$I_1 = \Gamma(\kappa+1) + O(|s|^{\kappa+1}),$$

and I_2 may be similarly dealt with. It follows that, for small s,

$$F(s) = as^{-\kappa} + bs^{-\kappa+1} + O(|s|) + O(|s|^{-\kappa+2}). \tag{68}$$

where a and b are constants, and where $a \neq 0$. Again, since, by (65),

$$\frac{d^n F(s)}{ds^n} = \frac{(-1)^n}{s^n} \int_0^\infty e^{-x} (x^n - nx^{n-1}) f_1\left(\frac{x}{s}\right) dx,$$

we can deduce by a similar argument that

$$F'(s) = -\kappa a s^{-\kappa-1} + (1-\kappa) b s^{-\kappa} + O(1) + O(|s|^{-\kappa+1}); \tag{69}$$

$$F''(s) = \kappa(1+\kappa) a s^{-\kappa-2} - \kappa(1-\kappa) b s^{-\kappa-1} + O(|s|^{-\kappa}). \tag{70}$$

We next note that

$$G(s) = \Gamma(\kappa) s^{-\kappa}. \tag{71}$$

It now follows from (64), (66), (67) and (71) that, for large $|s|$,

$$P(s) = k s^{1-\kappa} + O(|s|^{-\kappa}); \tag{72}$$

$$P'(s) = O(|s|^{-\kappa}); \tag{73}$$

$$P''(s) = O(|s|^{-\kappa-1}). \tag{74}$$

It also follows from (68), (69), (70) and (71) that, for small s,

$$P(s) = O(1); \tag{75}$$

$$P'(s) = O(1); \tag{76}$$

$$P''(s) = O(1) + O(|s|^{\kappa-1}). \tag{77}$$

We have now obtained those properties of $P(s)$ which we require, and are thus in a position to apply Theorem III. It is evident that the constants λ and μ are now given by

$$\lambda = 0; \quad \mu = \kappa - 1.$$

If $\kappa < 1$, then $\mu < \lambda$, and hence, by Theorem II, summability (C, κ) does not imply summability (R, κ). The case $\kappa = 1$ is trivial, since we then have $f(x) = g(x)$. It remains to consider the case $\kappa > 1$. We now have $\mu > \lambda$. Moreover, it is evident that

$$f_1(x) \sim g_1(x) \quad \text{as} \quad x \to \infty.$$

It therefore follows from Theorem III that a necessary and sufficient condition for summability (C, κ) to imply summability (R, κ) is that the integral

$$\int_0^\infty |p(x)|\, dx \tag{78}$$

should converge.

If $F(s)$ has no zeros for $\Re s \geqslant 0$, then it follows from (60), (72) and (75) that

$$p(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{xs} P(s)\, ds,$$

where the path of integration is a straight line. Integrating by parts twice, we deduce that

$$p(x) = \frac{1}{2\pi i x^2} \int_{-i\infty}^{i\infty} e^{xs} P''(s)\, ds.$$

It follows with the aid of (74) and (77) that, for large x,

$$p(x) = O\left(\frac{1}{x^2}\right), \tag{79}$$

and hence that the integral (78) converges. Hence summability (C, κ) implies summability (R, κ).

Next suppose that $F(s)$ has zeros for $\Re s \geqslant 0$. It follows from (64) and (68) that there are only a finite number of such zeros. Suppose, then, that these zeros occur at the points $s = s_r$, $r = 1, 2, \ldots, n$. Since $G(s)$ is nowhere zero, $P(s)$ has a pole at each of the points $s = s_r$. We now have

$$p(x) = \bar{p}(x) + S(x),$$

where

$$\bar{p}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{xs} P(s)\, ds,$$

the contour of integration being bent, if necessary, so as to pass to the left of the points $s = s_r$, but to the right of any other poles of the integrand, and where $S(x)$ denotes the sum of the residues of $e^{xs} P(s)$ at the points $s = s_r$. Now, if $P(s)$ has a pole of order m_r at $s = s_r$, the residue of $e^{xs} P(s)$ is

$$(kx^{m_r-1} + cx^{m_r-2} + \ldots + c)\, e^{xs_r}.$$

Thus

$$S(x) = \sum_{r=1}^{n} (kx^{m_r-1} + cx^{m_r-2} + \ldots + c)\, e^{xs_r}.$$

We can show as before that, for large x,

$$\bar{p}(x) = O\left(\frac{1}{x^2}\right),$$

and it follows that

$$p(x) = \sum_{r=1}^{n} (kx^{m_r-1} + cx^{m_r-2} + \ldots + c)\, e^{xs_r} + O\left(\frac{1}{x^2}\right).$$

Hence the integral (78) is divergent, so that summability (C, κ) does not imply summability (R, κ).

We have thus reduced the problem of determining whether or not summability (C, κ) implies summability (R, κ) to the problem of investi-

gating the zeros of $F(s)$. We next examine whether, with the same values of $f(x)$ and $g(x)$, summability (g) implies summability (f). That is to say, we examine whether summability (R, κ) implies summability (C, κ). If $\kappa > 1$, summability (R, κ) in no circumstances implies summability (C, κ), since we then have $\mu > \lambda$. On the other hand, if $\kappa < 1$, we have $\mu < \lambda$. Since $G(s)$ is nowhere zero, an argument similar to that used in proving (79) shows that, for large x,

$$q(x) = O\left(\frac{1}{x^2}\right),$$

and hence that the integral

$$\int_0^\infty |q(x)|\, dx$$

is convergent. But

$$f_1(x) \sim g_1(x) \quad \text{as} \quad x \to \infty,$$

and it follows from Theorem III that summability (R, κ) implies summability (C, κ).

4.2. We now consider the case in which

$$f(x) = \frac{d}{dx} \frac{\Gamma(\kappa+x)}{\Gamma(x)}$$

(as before), and in which

$$g(x) = x^{\kappa+\delta-1},$$

where $\delta > 0$. That is to say, we investigate whether summability (C, κ) implies summability $(R, \kappa+\delta)$. The values of λ and μ are now given by

$$\lambda = 0; \quad \mu = \kappa+\delta-1.$$

If $\kappa+\delta < 1$, then $\mu < \lambda$, and thus summability (C, κ) does not imply summability $(R, \kappa+\delta)$. We now consider the case $\kappa+\delta \geqslant 1$. Since it is now no longer true that

$$g_1(x) = O\,[f_1(x)] \quad \text{as} \quad x \to \infty,$$

Theorem III does not give a necessary and sufficient condition for summability (C, κ) to imply summability $(R, \kappa+\delta)$. We therefore apply Theorem II.

The value of $F(s)$ is the same as before, but $G(s)$ is now given by

$$G(s) = \Gamma(\kappa+\delta)\, s^{-\kappa-\delta}.$$

Thus, for large $|s|$,

$$P(s) = ds^{1-\kappa-\delta} + O(|s|^{-\kappa-\delta});$$

$$P'(s) = d(1-\kappa-\delta)\, s^{-\kappa-\delta} + O(|s|^{-\kappa-\delta-1}), \tag{80}$$

where d is a non-zero constant, and, for small s,

$$P(s) = O(|s|^{-\delta});$$

$$P'(s) = O(|s|^{-1-\delta}). \tag{81}$$

If $F(s)$ has no zeros for $\Re s \geqslant 0$, we have, as before, if $\kappa+\delta > 1$,

$$p(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{xs} P(s)\, ds, \tag{82}$$

the contour being bent so as to pass to the right of the origin. If $\kappa+\delta = 1$, we see, as in the proof of the case $\mu = \lambda$ of Theorem I, that (82) is replaced by

$$p(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{xs}\Big(P(s)-d\Big)\, ds.$$

In any case we obtain, on integration by parts,

$$p(x) = -\frac{1}{2\pi i x}\int_{-i\infty}^{i\infty} e^{xs} P'(s)\, ds.$$

We take as the contour of integration the part of the circle $|s| = 1/x$ lying to the right of the imaginary axis, and the imaginary axis with the exception of the part lying within this circle. It follows from (80) and (81) that, for large x,

$$p(x) = O(x^{\delta-1}).$$

Thus
$$\int_0^x |p(x-t)|\, f_1(t)\, dt = O\left\{\int_0^x (x-t)^{\delta-1} t^{\kappa}\, dt\right\}$$
$$= O(x^{\kappa+\delta}) = O[g_1(x)],$$

so that* summability (C, κ) implies summability $(R, \kappa+\delta)$.

If $F(s)$ has zeros for $\Re s \geqslant 0$, we suppose, as in § 4.1, that they occur

* Since summability (R, κ) implies summability $(R, \kappa+\delta)$, this follows from the results of § 4.1 when $\kappa \geqslant 1$ (but not when $\kappa < 1$).

at the points $s = s_r$. We obtain, as in § 4 . 1,

$$p(x) = \sum_{r=1}^{n} (kx^{m_r-1} + cx^{m_r-2} + \ldots + c)\, e^{xs_r} + O(x^{\delta-1}).$$

If any zeros occur for $\Re s > 0$, it is evident that

$$\int_0^x |p(x-t)| f_1(t)\, dt \tag{83}$$

is not $O[g_1(x)]$ If no zeros occur for $\Re s > 0$, that is to say, if all the points s_r lie on the imaginary axis, let the value of the greatest m_r be m. It is then evident that the expression (83) is

$$kx^{\kappa+m} + O(x^{\kappa+m-1}) + O(x^{\kappa+\delta}),$$

and this is $O[g_1(x)]$, provided that $\delta \geqslant m$.

Combining the results just obtained with those of § 4 . 1, we can now state the result that, *if* $\kappa+\delta \geqslant 1$, $\delta \geqslant 0$, *a necessary and sufficient condition that summability* (C, κ) *should imply summability* $(R, \kappa+\delta)$ *is that* $F(s)$ *should possess no zeros for* $\Re s > 0$, *and no zeros of order greater* than* $[\delta]$ *for* $\Re s = 0$.

Lastly, we consider the case in which

$$f(x) = \frac{d}{dx} \frac{\Gamma(\kappa+x)}{\Gamma(x)},$$

$$g(x) = \frac{d}{dx} \frac{\Gamma(\kappa+\delta+x)}{\Gamma(x)},$$

where $\delta > 0$. We now have $\lambda = \mu = 0$. We can deal with this case in a way similar to the previous one, but the result cannot be stated in quite the same form, since the zeros of $F(s)$ do not now necessarily give rise to poles of $P(s)$, as they may coincide with zeros of $G(s)$. We thus have the result that *a necessary and sufficient condition that summability* (C, κ) *should imply summability* $(C, \kappa+\delta)$ *is that* $P(s)$ *should possess no poles for* $\Re s > 0$, *and no poles of order greater than* $[\delta]$ *for* $\Re s = 0$.

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* If $\delta < 1$, this must be interpreted as meaning no zeros at all.

THE MEAN VALUE OF THE ZETA-FUNCTION ON THE CRITICAL LINE

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1. *Introduction.* Two proofs have been given of the result

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt \sim \frac{1}{2\pi^2} T \log^4 T \quad \text{as} \quad T \to \infty,$$

one by Ingham*, who used the approximate functional equation for $\zeta^2(s)$, and one by Titchmarsh†, who used Parseval's theorem to prove the equivalent result

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-t} dt \sim \frac{1}{2\pi^2 \delta} \log^4 \frac{1}{\delta} \quad \text{as} \quad \delta \to 0.$$

In this paper we use a different method to prove a better asymptotic formula for the latter integral. This is

THEOREM I. *There are constants A, B, C, D, E, such that,*

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t} dt$$
$$= \frac{1}{\delta}\left(A \log^4 \frac{1}{\delta} + B \log^3 \frac{1}{\delta} + C \log^2 \frac{1}{\delta} + D \log \frac{1}{\delta} + E\right) + O\left(\left(\frac{1}{\delta}\right)^{\frac{13}{14}+\epsilon}\right),$$

as $\delta \to 0$, for any fixed $\epsilon > 0$. We have, in fact,

$$A = \frac{1}{2\pi^2}, \quad B = -\frac{1}{\pi^2}\left(2 \log 2\pi - 6\gamma + \frac{24\zeta'(2)}{\pi^2}\right).$$

We need two special results in proving this, a formula in the analytic theory of numbers, (3.1), and an asymptotic formula, (5.3), due to Estermann, for sums of the form $\sum_m d(m)\, d(m+k)$.

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* A. E. Ingham, *Proc. London Math. Soc.* (2), 27 (1927), 296–300.

† E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 27 (1927), 137–150.

We use ϵ throughout to denote an arbitrarily small fixed positive quantity, not necessarily the same at each occurrence; a dash on a summation sign indicates that an infinite term is to be omitted.

2. *A preliminary approximation.* We commence by showing that it is sufficient to consider a slightly different integral. We have

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt - \frac{e^{-\frac{1}{2}i\delta}}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\zeta^2(s)\,\zeta^2(1-s)}{2\cos\frac{1}{2}s\pi}\, e^{-is(\frac{1}{2}\pi-\delta)}\,ds$$

$$= \int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt - \int_{-\infty}^{\infty} \frac{|\zeta(\tfrac{1}{2}+it)|^4 e^{-\frac{1}{4}i\pi+\frac{1}{2}\pi t} e^{-\delta t}}{e^{-\frac{1}{4}i\pi+\frac{1}{2}\pi t}+e^{\frac{1}{4}i\pi-\frac{1}{2}\pi t}}\,dt$$

$$= \int_0^\infty \frac{|\zeta(\tfrac{1}{2}+it)|^4 e^{-\frac{1}{4}i\pi-\frac{1}{2}\pi t} e^{\delta t}}{e^{-\frac{1}{4}i\pi-\frac{1}{2}\pi t}+e^{\frac{1}{4}i\pi+\frac{1}{2}\pi t}}\,dt + \int_0^\infty \frac{|\zeta(\tfrac{1}{2}+it)|^4 e^{\frac{1}{4}i\pi-\frac{1}{2}\pi t} e^{-\delta t}}{e^{\frac{1}{4}i\pi-\frac{1}{2}\pi t}+e^{-\frac{1}{4}i\pi+\frac{1}{2}\pi t}}\,dt; \quad (2.1)$$

both these integrals are uniformly convergent for $|\delta| < \theta < \pi$, and so they represent functions analytic for $|\delta| < \pi$. Now

$$\frac{e^{-\frac{1}{2}i\delta}}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\zeta^2(s)\,\zeta^2(1-s)}{2\cos\frac{1}{2}s\pi}\, e^{-is(\frac{1}{2}\pi-\delta)}\,ds = 2\,\frac{e^{-\frac{1}{2}i\delta}}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma^2(s)\cos\tfrac{1}{2}s\pi\, e^{-is(\frac{1}{2}\pi-\delta)}\,\zeta^4(s)\,ds,$$

and if we move the contour to the right this becomes

$$R(\delta) + \frac{e^{-\frac{1}{2}i\delta}}{i}\int_{2-i\infty}^{2+i\infty} \Gamma^2(s)(e^{i\delta s}+e^{-is(\pi-\delta)}) \sum_{n=1}^{\infty} d_4(n)\,n^{-s}\,ds$$

$$= R(\delta) + 4\pi e^{-\frac{1}{2}i\delta} \sum_{n=1}^{\infty} d_4(n)\{K_0(4\pi\sqrt{n}\,e^{-\frac{1}{2}\delta}) + K_0(4\pi i\sqrt{n}\,e^{-\frac{1}{2}i\delta})\}, \quad (2.2)$$

where $R(\delta)$, the contribution of the triple pole at $s=1$, is a polynomial is $e^{\frac{1}{2}i\delta}$ and δ, whose exact value does not concern us since it is an integral function.

Now
$$K_0(z) = \sqrt{(\tfrac{1}{2}\pi)}\,z^{-\frac{1}{2}}e^{-z}\big(1-8z^{-1}+O(z^{-2})\big), \quad (2.3)$$

uniformly for $|\arg z| < \theta < \frac{3}{2}\pi$, $|z| \geqslant 1$, and hence the series

$$\sum_{n=1}^{\infty} d_4(n)\,K_0\big(4\pi\sqrt{n}\exp(-\tfrac{1}{2}i\delta)\big)$$

is uniformly convergent for $|\delta| < \theta < \pi$; this series therefore represents a function analytic for $|\delta| < \pi$.

We have therefore, by (2.1) and (2.2),

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt = 4\pi e^{-\frac{1}{2}i\delta} \sum_{n=1}^{\infty} d_4(n)\,K_0(4\pi i\sqrt{n}\,e^{-\frac{1}{2}i\delta}) + \phi(\delta), \quad (2.4)$$

where $\phi(\delta)$ is analytic in the region $|\delta| < \pi$.

We proceed to replace the Bessel functions by exponentials. We have, by (2.3),

$$e^{-\frac{1}{4}i\delta}\sum_{n=1}^{\infty} d_4(n)\,K_0(4\pi i\sqrt{n}\,e^{-\frac{1}{2}i\delta})$$

$$=\frac{1}{2\sqrt{2}}\,e^{-\frac{1}{4}i(\delta+\pi)}\sum_{n=1}^{\infty} d_4(n)\,n^{-\frac{1}{4}}\exp(-4\pi i\sqrt{n}\,e^{-\frac{1}{2}i\delta})\left(1+O(n^{-\frac{1}{2}})\right). \quad (2.5)$$

The contribution of the term $O(n^{-\frac{1}{2}})$ is

$$O\Big(\sum_{n=1}^{\infty} d_4(n)\,n^{-\frac{3}{4}}\exp(-4\pi\sqrt{n}\sin\tfrac{1}{2}\delta)\Big),$$

and, since $\sum_{n<x} d_4(n) = O(x\log^3 x)$, this is, by partial summation,

$$O\Big(\sum_{n=1}^{\infty} n^{-\frac{3}{4}}\log^3 n\exp(-4\pi\sqrt{n}\sin\tfrac{1}{2}\delta)\Big) = O(\delta^{-\frac{1}{2}}\log^3\delta).$$

Next we may replace $\exp(-4\pi i\sqrt{n}\,e^{-\frac{1}{2}i\delta})$ in (2.5) by

$$\exp\left(-4\pi i\sqrt{n}(1-\tfrac{1}{2}i\delta)\right)$$

with error

$$O\Big(\delta^2\sum_{n=1}^{\infty} d_4(n)\,n^{\frac{1}{4}}e^{-2\pi\delta\sqrt{n}}\Big) = O\Big(\delta^2\sum_{n=1}^{\infty} n^{\frac{1}{4}}\log^3 n\,e^{-2\pi\delta\sqrt{n}}\Big) = O(\delta^{-\frac{1}{2}}\log^3\delta).$$

Lastly the factor $e^{-\frac{1}{4}i(\delta+\pi)}$ in (2.5) may be replaced by $e^{-\frac{1}{4}i\pi}$ with error

$$O\Big(\delta\sum_{n=1}^{\infty} d_4(n)\,n^{-\frac{1}{4}}e^{-2\pi\delta\sqrt{n}}\Big) = O(\delta^{-\frac{1}{2}}\log^3\delta).$$

Combining these results with (2.4) and (2.5) we get

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt = \pi\sqrt{2}\sum_{n=1}^{\infty} d_4(n)\,n^{-\frac{1}{4}}e^{-4\pi i\sqrt{n}-\frac{1}{4}i\pi-2\pi\delta\sqrt{n}}+O(\delta^{-\frac{1}{2}}\log^3\delta). \quad (2.6)$$

Now let us write

$$D(x) = \sum_{n<x^2} d_4(n)\,n^{-\frac{1}{4}}e^{-4\pi i\sqrt{n}-\frac{1}{4}i\pi},$$

$$D_1(x) = \sum_{n<x^2} d_4(n)\,n^{-\frac{1}{4}}e^{-4\pi i\sqrt{n}-\frac{1}{4}i\pi}(x-\sqrt{n}). \quad (2.7)$$

Then

$$\sum_{n=1}^{\infty} d_4(n)\,n^{-\frac{1}{4}}e^{-4\pi i\sqrt{n}-\frac{1}{4}i\pi-2\pi\delta\sqrt{n}} = 2\pi\delta\int_1^{\infty} D(x)\,e^{-2\pi x\delta}\,dx$$

$$= 4\pi^2\delta^2\int_1^{\infty} D_1(x)e^{-2\pi x\delta}\,dx,$$

by integration by parts. Hence, finally,

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt = 4\pi^3\delta^2\sqrt{2}\int_1^\infty D_1(x)\,e^{-2\pi x\delta}\,dx + O(\delta^{-\frac{1}{2}}\log^3\delta), \quad (2.8)$$

where $D_1(x)$ is defined by (2.7).

3. *The fundamental formula.* We have now arrived at the kernel of the problem, the determination of an asymptotic formula for $D_1(x)$. We deduce this from a subsidiary theorem which runs as follows:

THEOREM II. *If l is an integer and x^2 is not an integer, $x^2 > \frac{1}{2}$, then*

$$\sum_{n<x^2} d(n)(ln)^{-\frac{1}{4}} e^{-4\pi i\sqrt{(ln)}-\frac{1}{4}i\pi}(x-\sqrt{n})$$

$$= \frac{x^2}{\sqrt{(2l)}}\,d(l) - \frac{1}{8\pi^2\sqrt{2}}\sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}}\,\frac{e^{4\pi ix(\sqrt{n}-\sqrt{l})}}{(\sqrt{n}-\sqrt{l})^2} + \frac{xl\sqrt{2}}{\pi^2}\sum_{n=1}^{\infty}{}' \frac{d(n)\{i\pi+\log(l/n)\}}{n^2-l^2}$$

$$+ \frac{\sqrt{l}}{\pi^2\sqrt{2}}\sum_{n=1}^{\infty}{}' \frac{d(n)(l^2+n^2)}{(n^2-l^2)^2} - \frac{x}{\sqrt{2}}(\gamma+\tfrac{1}{2}\log l+\tfrac{1}{2}i\pi)$$

$$+ O(x^{\frac{3}{2}}l^{-\frac{3}{4}}\log x) + O(xl^{-\frac{1}{2}}\log x) + O(xl^{-1}\log^2 x). \quad (3.1)$$

The proof is somewhat lengthy, and proceeds by integration over $(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ of the two sides of the following identity:

LEMMA 1. *For $|\arg s| < \frac{1}{2}\pi$,*

$$\frac{2s^2}{\pi^2}\sum_{n=1}^{\infty} d(n)\,\frac{\log(s^2/n)}{s^4-n^2} = \log s+\gamma+\frac{\log(2\pi s)}{\pi^2 s^2}+4\sum_{n=1}^{\infty} d(n)\,K_0(4\pi s\sqrt{n}). \quad (3.2)$$

We have, for $|\arg z| < \pi$,

$$\frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{z^{-u}\,du}{\sin^2\frac{1}{2}u\pi} = \frac{4}{\pi^2}\,\frac{\log(1/z)}{1-z^2},$$

and hence, for $|\arg s| < \frac{1}{2}\pi$,

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{d(n)}{s^2}\,\frac{\log(s^2/n)}{1-n^2/s^4} = \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\frac{\zeta^2(u)\,s^{2u-2}}{\sin^2\frac{1}{2}u\pi}\,du;$$

when we move the contour to the left, and use the functional equation for the zeta-function, this becomes

$$2\log s+2\gamma+\frac{2\log(2\pi s)}{\pi^2 s^2}+\frac{4}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty}\zeta^2(u)\,\Gamma^2(u)(4\pi^2 s^2)^{-u}\,du$$

$$= 2\log s+2\gamma+\frac{2\log(2\pi s)}{\pi^2 s^2}+8\sum_{n=1}^{\infty} d(n)\,K_0(4\pi s\sqrt{n}),$$

which proves Lemma 1.

We now multiply the two sides of this identity by $e^{4\pi x(s-i\sqrt{l})}/(s-i\sqrt{l})^2$, and integrate term by term over $(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. We have first of all to justify this process. In the case of the right-hand side of (3.2) it is sufficient to prove that, as $T\to\infty$,

$$\sum_{n=1}^{\infty} d(n) \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\infty} K_0(4\pi s\sqrt{n})\, \frac{e^{4\pi x(s-i\sqrt{l})}}{(s-i\sqrt{l})^2}\, ds \to 0.$$

But by (2.3) this expression is, for $T > 2\sqrt{l}$,

$$O\Big(\sum_{n=1}^{\infty} d(n) n^{-\frac{1}{4}} e^{2\pi(x-\sqrt{n})} T^{-\frac{3}{2}}\Big) \to 0 \quad (T\to\infty).$$

To justify the process for the left-hand side of (3.2) it is sufficient to prove that, as $T\to\infty$,

$$\sum_{n=1}^{\infty} d(n) \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\infty} \frac{s^2 \log(s^2/n)}{s^4-n^2}\, \frac{e^{4\pi x(s-i\sqrt{l})}}{(s-i\sqrt{l})^2}\, ds \to 0.$$

Let us write this sum as

$$\sum_{n\leqslant \frac{1}{2}T^2} + \sum_{\frac{1}{2}T^2<n\leqslant 2T^2} + \sum_{n>2T^2} = \Sigma_1+\Sigma_2+\Sigma_3,$$

and let us take $T > 2\sqrt{l}$, so that $s^2/(s-i\sqrt{l})^2 = O(1)$.

Then, in Σ_3, we may write

$$\begin{aligned}\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\infty} &= \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\sqrt{(\frac{1}{2}n)}} + \int_{\frac{1}{2}+i\sqrt{(\frac{1}{2}n)}}^{\frac{1}{2}+2i\sqrt{n}} + \int_{\frac{1}{2}+2i\sqrt{n}}^{\frac{1}{2}+i\infty} \\ &= O\Big(\sqrt{n}\,\frac{\log n}{n^2}\Big) + O\Big(n^{-\frac{3}{2}}\int_{-\sqrt{(\frac{1}{2}n)}}^{\sqrt{n}} \frac{dt}{1+|t|}\Big) + O\Big(\int_{\sqrt{n}}^{\infty} t^{-4}\log t\, dt\Big) \\ &= O(n^{-\frac{3}{2}}\log n),\end{aligned}$$

and similarly in Σ_2. In Σ_1 we have

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\infty} = O\Big(\int_T^{\infty} t^{-4}\log t\, dt\Big) = O(T^{-3}\log T).$$

Hence, for $T > 2\sqrt{l}$,

$$\begin{aligned}\sum_{n=1}^{\infty} d(n) &\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\infty} \frac{s^2\log(s^2/n)}{s^4-n^2}\, \frac{e^{4\pi x(s-i\sqrt{l})}}{(s-i\sqrt{l})^2}\, ds \\ &= O\Big(T^{-3}\log T \sum_{n\leqslant \frac{1}{2}T^2} d(n)\Big) + O\Big(\sum_{n\geqslant \frac{1}{2}T^2} d(n) n^{-\frac{3}{2}}\log n\Big) \\ &\to 0 \text{ as } T\to\infty.\end{aligned}$$

Now let us consider the contribution to the integral of the right-hand side of (3.2). This is

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\left(\log s+\gamma+\frac{\log(2\pi s)}{\pi^2 s^2}\right)\frac{e^{4\pi x(s-i\sqrt{l})}}{(s-i\sqrt{l})^2}\,ds$$

$$+4\sum_{n=1}^{\infty}d(n)\,\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}K_0(4\pi s\sqrt{n})\,\frac{e^{4\pi x(s-i\sqrt{l})}}{(s-i\sqrt{l})^2}\,ds.$$

We deal first with the term $\log s+\gamma$; we bend the contour back on to the negative real axis, described twice. The pole at $s=i\sqrt{l}$ gives us

$$4\pi x(\tfrac{1}{2}i\pi+\tfrac{1}{2}\log l+\gamma)-i/\sqrt{l}.$$

The negative real axis gives

$$O\left(\int_0^{\infty}\frac{e^{-4\pi xs}}{(s+\sqrt{l})^2}\,ds\right)=O(l^{-\frac{1}{2}}).$$

For the term $\log(2\pi s)/\pi^2 s^2$ we bend back the contour on to the strip $(-\infty, -x^{-1})$, described twice, together with a circle of radius x^{-1} and centre the origin. This circle gives

$$O\left(\frac{1}{x}\,\frac{\log x}{(1/x)^2}\,\frac{1}{l}\right)=O(xl^{-1}\log x).$$

The rest of the contour gives

$$O\left(\int_{x^{-1}}^{\infty}\frac{e^{-4\pi xs}\,|\log(2\pi s)|}{s^2(s+\sqrt{l})^2}\,ds\right)=O(xl^{-1}\log x).$$

The pole at $s=i\sqrt{l}$ gives $O(xl^{-1}\log l)$.

We come now to the contribution of the Bessel functions. For $n>x^2$ we have, by (2.3),

$$e^{4\pi xs}K_0(4\pi s\sqrt{n})=O\left(s^{-\frac{1}{2}}n^{-\frac{1}{4}}\exp\left(-(\sqrt{n}-x)\Re(s)\right)\right),$$

and hence, by moving the contour to the right, we get

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}K_0(4\pi s\sqrt{n})\,\frac{e^{4\pi x(s-i\sqrt{l})}}{(s-i\sqrt{l})^2}\,ds=0\quad(n>x^2).$$

For $n<x^2$ we bend the contour back on to the negative real axis, described twice. The strip $(0, -n^{-\frac{1}{2}})$ gives

$$O\left(l^{-1}\int_0^{n^{-\frac{1}{2}}}|K_0(4\pi s\sqrt{n}e^{-\pi i})-K_0(4\pi s\sqrt{n}\,e^{\pi i})|\,ds\right)=O(l^{-1}n^{-\frac{1}{2}}).$$

This contributes altogether

$$O\Big(l^{-1}\sum_{n<x^2} d(n)\,n^{-\frac{1}{2}}\Big) = O(xl^{-1}\log x).$$

The strip $(-n^{-\frac{1}{2}}, -\infty)$ gives, by (2.3),

$$O\Big(\int_{n-\frac{1}{2}}^{\infty} \frac{e^{-4\pi s(x-\sqrt{n})}}{(s+\sqrt{l})^2}\,ds\Big) = O\Big(\min\Big(\frac{1}{\sqrt{l}},\ \frac{1}{l(x-\sqrt{n})}\Big)\Big).$$

This gives altogether

$$O\Big(\sum_{n<x^2} d(n)\min\Big(\frac{1}{\sqrt{l}},\ \frac{1}{l(x-\sqrt{n})}\Big)\Big).$$

The range $n < \frac{1}{2}x^2$ in this sum gives

$$O\Big(x^{-1}l^{-1}\sum_{n<\frac{1}{2}x^2} d(n)\Big) = O(xl^{-1}\log^2 x).$$

The range $\frac{1}{2}x^2 < n \leqslant x^2 - x$ gives

$$O\Big(x\,l^{-1}\sum_{\frac{1}{2}x^2<n\leqslant x^2-x} \frac{d(n)}{x^2-n}\Big) = O(xl^{-1}\log^2 x).$$

by partial summation. The range $x^2 - x < n < x^2$ gives

$$O\Big(l^{-\frac{1}{2}}\sum_{x^2-x<n<x^2} d(n)\Big) = O(xl^{-\frac{1}{2}}\log x).$$

We are left to consider the contribution of the pole at $s = i\sqrt{l}$. This is

$$4\pi x\,K_0\big(4\pi i\sqrt{(ln)}\big) + 4\pi\sqrt{n}\,K_0'\big(4\pi i\sqrt{(ln)}\big)$$
$$= \sqrt{(\tfrac{1}{2}\pi)}\,\frac{e^{-4\pi i\sqrt{(ln)}}}{\big(4\pi i\sqrt{(ln)}\big)^{\frac{1}{2}}}\,(4\pi x - 4\pi\sqrt{n}) + O(xl^{-\frac{1}{4}}n^{-\frac{1}{4}}).$$

The error term $O(xl^{-\frac{1}{4}}n^{-\frac{1}{4}})$ gives altogether

$$O\Big(xl^{-\frac{1}{4}}\sum_{n<x^2} d(n)\,n^{-\frac{1}{4}}\Big) = O(x^{\frac{3}{2}}l^{-\frac{1}{4}}\log x).$$

Hence the contribution of the right-hand side of (3.2) to the integral is

$$4\pi\sqrt{2}\sum_{n<x^2} d(n)(ln)^{-\frac{1}{4}}\,e^{-4\pi i\sqrt{(ln)}-\frac{1}{4}i\pi}(x-\sqrt{n}) + 4\pi x(\gamma + \tfrac{1}{2}\log l + \tfrac{1}{2}i\pi)$$
$$+ O(xl^{-1}\log^2 x) + O(xl^{-\frac{1}{2}}\log x) + O(x^{\frac{3}{2}}l^{-\frac{1}{4}}\log x). \quad (3.3)$$

Now we consider the contribution of the left-hand side of (3.2). We have to evaluate

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{e^{4\pi x(s-i\sqrt{l})}}{(s-i\sqrt{l})^2}\log\left(\frac{s^2}{n}\right)\left(\frac{1}{s^2-n}+\frac{1}{s^2+n}\right)ds.$$

We bend the contour back on to the negative real axis described twice, semicircular indentations of radius $\frac{1}{2}$ and centre $-\sqrt{n}$, being made to each side of this axis. On each of these indentations the integrand is

$$O(e^{-4\pi x(\sqrt{n}-\frac{1}{2})}l^{-1}n^{-\frac{1}{2}}),$$

and hence their total contribution is

$$O\left(l^{-1}\sum_{n=1}^{\infty}d(n)n^{-\frac{1}{2}}e^{-4\pi x(\sqrt{n}-\frac{1}{2})}\right)=O(l^{-1}).$$

The remainder of the negative real axis gives

$$O\left(\int_1^{\infty}e^{-4\pi xs}\left(\left|\log\frac{s^2}{n}\right|+1\right)s^2\,ds l^{-1}n^{-\frac{3}{2}}\right)=O(l^{-1}n^{-\frac{3}{2}}\log n),$$

which contributes altogether

$$O\left(l^{-1}\sum_{n=1}^{\infty}d(n)n^{-\frac{3}{2}}\log n\right)=O(l^{-1}).$$

We are left with the contributions of the poles at $s=i\sqrt{l}$, $\pm i\sqrt{n}$. The pole at $s=i\sqrt{n}$ gives

$$O\left(n^{-\frac{1}{2}}(\sqrt{n}+\sqrt{l})^{-2}\right),$$

and so, altogether,

$$O\left(\sum_{n=1}^{\infty}d(n)n^{-\frac{1}{2}}(\sqrt{n}+\sqrt{l})^{-2}\right)=O(l^{-\frac{1}{2}}\log l).$$

When $n=l$ the pole at $s=i\sqrt{l}$ gives

$$\frac{4\pi^3x^2}{\sqrt{l}}+O\left(\frac{x}{l}\right)$$

When $n\neq l$ the pole at $s=i\sqrt{l}$ gives

$$4\pi x\left(\pi i+\log\frac{l}{n}\right)\frac{2l}{n^2-l^2}+\frac{4i\sqrt{l}}{l^2-n^2}-\left(\pi i+\log\frac{l}{n}\right)\frac{4i\sqrt{l}(l^2+n^2)}{(l^2-n^2)^2}.$$

The second term gives altogether

$$O\Big(\sqrt{l}\sum_{n=1}^{\infty}{}'\frac{d(n)}{|n^2-l^2|}\Big)=O\Big(\sqrt{l}\sum_{n\leqslant\frac{1}{2}l}\frac{d(n)}{l^2}+l^{-\frac{1}{2}}\sum_{\frac{1}{2}l<n\leqslant 2l}{}'\frac{n^\epsilon}{|n-l|}+\sqrt{l}\sum_{n>2l}\frac{d(n)}{n^2}\Big)$$
$$=O(l^{\epsilon-\frac{1}{2}}).$$

That part of the third term in $\log(l/n)$ gives altogether

$$O\Big(\sqrt{l}\sum_{n=1}^{\infty}{}'d(n)\Big|\log\frac{l}{n}\Big|\frac{l^2+n^2}{(l^2-n^2)^2}\Big)=O(l^{\epsilon-\frac{1}{2}}),$$

by the same argument.

When $n\neq l$ the pole at $s=i\sqrt{n}$ gives

$$\frac{e^{4\pi ix(\sqrt{n}-\sqrt{l})}}{(i\sqrt{n}-i\sqrt{l})^2}\cdot\pi i\cdot\frac{1}{2i\sqrt{n}},$$

and so, altogether,

$$-\frac{\pi}{2}\sum_{n=1}^{\infty}{}'\frac{d(n)}{\sqrt{n}}\frac{e^{4\pi ix(\sqrt{n}-\sqrt{l})}}{(\sqrt{n}-\sqrt{l})^2}.$$

Hence the contribution of the left-hand side of (3.2) to the integral is

$$\frac{4\pi x^2}{\sqrt{l}}d(l)-\frac{1}{2\pi}\sum_{n=1}^{\infty}{}'\frac{d(n)}{\sqrt{n}}\frac{e^{4\pi ix(\sqrt{n}-\sqrt{l})}}{(\sqrt{n}-\sqrt{l})^2}+\frac{8xl}{\pi}\sum_{n=1}^{\infty}{}'\frac{d(n)\{\pi i+\log(l/n)\}}{n^2-l^2}$$
$$+\frac{4\sqrt{l}}{\pi}\sum_{n=1}^{\infty}{}'\frac{d(n)(l^2+n^2)}{(l^2-n^2)^2}+O(xl^{\epsilon-1})+O(l^{\epsilon-\frac{1}{2}}). \quad (3.4)$$

Comparison of this with (3.3) completes the proof of Theorem II.

4. *Transformation of* $D_1(x)$. We have, by (2.7),

$$D_1(x)=\sum_{n<x^2}d_4(n)n^{-\frac{1}{2}}e^{-4\pi i\sqrt{n}-\frac{1}{4}i\pi}(x-\sqrt{n})$$
$$=2\sum_{m<x}d(m)\sum_{n<x^2/m}d(n)(mn)^{-\frac{1}{2}}e^{-4\pi i\sqrt{(mn)}-\frac{1}{4}i\pi}\Big(x-\sqrt{(mn)}\Big)$$
$$-\sum_{m<x}d(m)\sum_{n<x}d(n)(mn)^{-\frac{1}{2}}e^{-4\pi i\sqrt{(mn)}-\frac{1}{4}i\pi}\Big(x-\sqrt{(mn)}\Big)$$
$$=2\sum_{m<x}d(m)\sqrt{m}\sum_{n<x^2/m}d(n)(mn)^{-\frac{1}{2}}e^{-4\pi i\sqrt{(mn)}-\frac{1}{4}i\pi}\Big(\frac{x}{\sqrt{m}}-\sqrt{n}\Big)$$
$$-\sum_{m<x}d(m)\sqrt{m}\sum_{n<x}d(n)(mn)^{-\frac{1}{2}}e^{-4\pi i\sqrt{(mn)}-\frac{1}{4}i\pi}(\sqrt{x}-\sqrt{n})$$
$$-\sum_{m<x}d(m)\sqrt{x}\sum_{n<x}d(n)(mn)^{-\frac{1}{2}}e^{-4\pi i\sqrt{(mn)}-\frac{1}{4}i\pi}(\sqrt{x}-\sqrt{m})$$
$$=2S_1-S_2-S_3, \quad \text{say.} \quad (4.1)$$

We now substitute the result of Theorem II in each of these sums. We get

$$S_1 = \frac{x^2}{\sqrt{2}} \sum_{m<x} \frac{d^2(m)}{m} - \frac{1}{8\pi^2\sqrt{2}} \sum_{m<x} d(m)\sqrt{m} \sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i x(\sqrt{n}-\sqrt{m})/\sqrt{m}}}{(\sqrt{n}-\sqrt{m})^2}$$
$$+ \frac{x\sqrt{2}}{\pi^2} \sum_{m<x} d(m)\, m \sum_{n=1}^{\infty}{}' \frac{d(n)(i\pi+\log m/n)}{n^2-m^2}$$
$$+ \frac{1}{\pi^2\sqrt{2}} \sum_{m<x} d(m)\, m \sum_{n=1}^{\infty}{}' \frac{d(n)(n^2+m^2)}{(m^2-n^2)^2}$$
$$- \frac{x}{\sqrt{2}} \sum_{m<x} d(m)(\gamma + \tfrac{1}{2}i\pi + \tfrac{1}{2}\log m) + O(x^{\frac{3}{2}}\log^3 x)$$
$$= \frac{x^2}{\sqrt{2}} S_{11} - \frac{1}{8\pi^2\sqrt{2}} S_{12} + \frac{x\sqrt{2}}{\pi^2} S_{13} + \frac{1}{\pi^2\sqrt{2}} S_{14} - \frac{x}{\sqrt{2}} S_{15} + O(x^{\frac{3}{2}}\log^3 x). \tag{4.2}$$

$$S_2 = \frac{x}{\sqrt{2}} \sum_{m<x} d^2(m) - \frac{1}{8\pi^2\sqrt{2}} \sum_{m<x} d(m)\sqrt{m} \sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i\sqrt{x}(\sqrt{n}-\sqrt{m})}}{(\sqrt{n}-\sqrt{m})^2}$$
$$+ \frac{\sqrt{(2x)}}{\pi^2} \sum_{m<x} d(m)\, m^{\frac{1}{2}} \sum_{n=1}^{\infty}{}' \frac{d(n)(i\pi+\log m/n)}{n^2-m^2}$$
$$+ \frac{1}{\pi^2\sqrt{2}} \sum_{m<x} d(m)\, m \sum_{n=1}^{\infty}{}' \frac{d(n)(m^2+n^2)}{(m^2-n^2)^2}$$
$$- \sqrt{\left(\frac{x}{2}\right)} \sum_{m<x} d(m)\sqrt{m}(\gamma + \tfrac{1}{2}i\pi + \tfrac{1}{2}\log m) + O(x^{\frac{3}{2}}\log^2 x)$$
$$= \frac{x}{\sqrt{2}} S_{21} - \frac{1}{8\pi^2\sqrt{2}} S_{22} + \frac{\sqrt{(2x)}}{\pi^2} S_{23} + \frac{1}{\pi^2\sqrt{2}} S_{24}$$
$$- \sqrt{\left(\frac{x}{2}\right)} S_{25} + O(x^{\frac{3}{2}}\log^2 x), \tag{4.3}$$

$$S_3 = \sqrt{x} \sum_{n<x} d(n) \sum_{m<x} d(m)(mn)^{-\frac{1}{2}} e^{-4\pi i\sqrt{(mn)}-\frac{1}{2}i\pi}(\sqrt{x}-\sqrt{m})$$
$$= \frac{x^{\frac{3}{2}}}{\sqrt{2}} \sum_{m<x} \frac{d^2(m)}{\sqrt{m}} - \frac{\sqrt{x}}{8\pi^2\sqrt{2}} \sum_{m<x} d(m) \sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i\sqrt{x}(\sqrt{n}-\sqrt{m})}}{(\sqrt{n}-\sqrt{m})^2}$$
$$+ \frac{x\sqrt{2}}{\pi^2} \sum_{m<x} d(m) \sum_{n=1}^{\infty}{}' \frac{d(n)(i\pi+\log m/n)}{n^2-m^2}$$
$$+ \frac{\sqrt{x}}{\pi^2\sqrt{2}} \sum_{m<x} d(m)\sqrt{m} \sum_{n=1}^{\infty}{}' \frac{d(n)(m^2+n^2)}{(m^2-n^2)^2}$$
$$- \frac{x}{\sqrt{2}} \sum_{m<x} d(m)(\gamma + \tfrac{1}{2}i\pi + \tfrac{1}{2}\log m) + O(x^{\frac{3}{2}}\log^2 x)$$
$$= \frac{x^{\frac{3}{2}}}{\sqrt{2}} S_{31} - \frac{\sqrt{x}}{8\pi^2\sqrt{2}} S_{32} + \frac{x\sqrt{2}}{\pi^2} S_{33} + \frac{\sqrt{x}}{\pi^2\sqrt{2}} S_{34} - \frac{x}{\sqrt{2}} S_{35}$$
$$+ O(x^{\frac{3}{2}}\log^2 x). \tag{4.4}$$

5. *Estermann's formula and the sums* S_{12}, S_{22}, S_{32}. In what follows we use $c_1, c_2, c_3, \ldots$ to denote various absolute constants, not necessarily real, with whose actual values we shall not be concerned. We deal now with with the sums S_{12}, S_{22}, S_{32} defined by (4.2), (4.3), (4.4) respectively.

LEMMA 2.

$$S_{12} = x^2(c_1 \log^2 x + c_2 \log x + c_3) + O(x^{2-\frac{1}{14}+\epsilon}).$$

We have
$$S_{12} = \sum_{m<x} d(m)\sqrt{m} \sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i x(\sqrt{(n/m)}-1)}}{(\sqrt{n}-\sqrt{m})^2}$$
$$= \sum_{m<\sqrt{x}} + \sum_{\sqrt{x}<m<x} = S'_{12} + S''_{12}.$$

Now

$$\sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i x(\sqrt{(n/m)}-1)}}{(\sqrt{n}-\sqrt{m})^2}$$
$$= \sum_{n\leqslant \frac{1}{2}m} + \sum_{\frac{1}{2}m<n\leqslant 2m} + \sum_{n>2m}$$
$$= O\Big(m^{-1} \sum_{n\leqslant \frac{1}{2}m} d(n) n^{-\frac{1}{2}}\Big) + O\Big(m^{\frac{1}{2}} \sum_{n\leqslant m} m^{\epsilon} n^{-2}\Big) + O\Big(\sum_{n>2m} d(n) n^{-\frac{3}{2}}\Big)$$
$$= O(m^{\frac{1}{2}+\epsilon}).$$

Hence
$$S'_{12} = O\Big(\sum_{m<\sqrt{x}} d(m) m^{1+\epsilon}\Big) = O(x^{1+\epsilon}).$$

Again

$$S''_{12} = \sum_{\sqrt{x}<m<x} d(m)\sqrt{m} \sum_{m-x^{\beta}\leqslant n\leqslant m+x^{\beta}} \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i x(\sqrt{(n/m)}-1)}}{(\sqrt{n}-\sqrt{m})^2}$$
$$+ \sum_{\sqrt{x}<m<x} d(m)\sqrt{m} \sum_{\substack{n>m+x^{\beta} \\ n<m-x^{\beta}}} \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i x(\sqrt{(n/m)}-1)}}{(\sqrt{n}-\sqrt{m})^2}$$
$$= S^{(3)}_{12} + S^{(4)}_{12},$$

where β is a constant to be chosen later such that $0 < \beta < \frac{1}{6}$.

Now

$$\sum_{n<m-x^{\beta}} \frac{d(n)}{\sqrt{n}(\sqrt{n}-\sqrt{m})^2} = O\Big(\sum_{n\leqslant \frac{1}{2}m} \frac{d(n)}{m\sqrt{n}}\Big) + O\Big(\sum_{\frac{1}{2}m<n<m-x^{\beta}} \frac{m^{\frac{1}{2}+\epsilon}}{(m-n)^2}\Big)$$
$$= O(m^{-\frac{1}{2}} \log m) + O(m^{\frac{1}{2}+\epsilon} x^{-\beta}),$$

and similarly for the sum in which $n > m+x^{\beta}$. Hence

$$S^{(4)}_{12} O\Big(\sum_{m<x} d(m) m^{1+\epsilon} x^{-\beta}\Big) = O(x^{2-\beta+\epsilon}).$$

Collecting these results we get, after rearrangement,

$$S_{12} = \sum_{1 \leqslant \pm k \leqslant x^\beta} \sum_{\sqrt{x} < m < x} d(m)\, d(m+k) \sqrt{\left(\frac{m}{m+k}\right)} \frac{e^{4\pi i x\left(\sqrt{((m+k)/m)}-1\right)}}{\left(\sqrt{(m+k)}-\sqrt{m}\right)^2} + O(x^{2-\beta+\epsilon}). \quad (5.1)$$

But

$$\sqrt{\left(\frac{m}{m+k}\right)} \frac{e^{4\pi i x\left(\sqrt{((m+k)/m)}-1\right)}}{\left(\sqrt{(m+k)}-\sqrt{m}\right)^2} = \frac{4m}{k^2}\, e^{2\pi i x k/m} + O\left(\frac{x}{m}\right),$$

and the error term here gives in (5.1) altogether

$$O\left(x \sum_{1 \leqslant \pm k < x^\beta} \sum_{m < x} x^\epsilon m^{-1}\right) = O(x^{1+\beta+\epsilon} \log x).$$

Hence, finally,

$$S_{12} = 4 \sum_{1 \leqslant \pm k \leqslant x^\beta} k^{-2} \sum_{\sqrt{x} < m < x} d(m)\, d(m+k)\, m e^{2\pi i x k/m} + O(x^{2-\beta+\epsilon}). \quad (5.2)$$

We now apply the following result, essentially due to Estermann*:

For $1 \leqslant \pm k \leqslant \sqrt{n} - \frac{1}{2}$, *we have*

$$\sum_{\max(1-k,\, 1) \leqslant m \leqslant n} d(m)\, d(m+k) = n(a_{1k} \log^2 n + a_{2k} \log n + a_{3k}) + O(n^{\frac{1}{2}} k^{\frac{1}{2}} \log^3 n), \quad (5.3)$$

where

$$a_{1k} = \frac{6}{\pi^2}\, \sigma_{-1}(|k|),$$

$$a_{2k} = \left\{\frac{12}{\pi^2}\,(2\gamma - 1) + 4a'\right\} \sigma_{-1}(|k|) - \frac{24}{\pi^2}\, \sigma'_{-1}(|k|),$$

$$a_{3k} = \left\{\frac{6}{\pi^2}\,(2\gamma-1)^2 + \frac{6}{\pi^2} + 4a'(2\gamma-1) + 4a''\right\} \sigma_{-1}(|k|) - \left\{\frac{24}{\pi^2}\,(2\gamma-1) + 8a'\right\} \sigma'_{-1}(|k|) + \frac{24}{\pi^2}\, \sigma''_{-1}(|k|),$$

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* T. Estermann, *Journal für Math.*, 164 (1931), 173–182. The result is given there for fixed positive k. To obtain a result holding uniformly in k all we need to do, apart from minor modifications in the *Elementare Abschätzungen* is to take account of k when applying the final lemma, Hilfssatz 7. The extension to negative values of k is trivial. Dr. Estermann points out that this result can be improved by means of the improvement of Kloosterman's inequality effected by Salié, *Math. Zeitschrift*, 36 (1933), 263–278, and by Davenport, *Journal für Math.*, 169 (1933), 158–176. This gives an error term $O(n^{\frac{5}{8}+\epsilon} k^\epsilon)$ in (5.3), and in the end an error term $O(\delta^{-\frac{8}{5}-\epsilon})$ in the result of the present paper.

and $$a' = -\sum_{\nu=1}^{\infty} \mu(\nu)\,\nu^{-2}\log\nu, \quad a'' = \sum_{\nu=1}^{\infty} \mu(\nu)\,\nu^{-2}\log^2\nu.$$

We have, therefore,

$$a_{rk} = O(\log^3 k) \quad (r = 1, 2, 3).$$

Write $$a_{1k} = a_{1k}, \quad a_{2k} = 2a_{1k} + a_{2k}, \quad a_{3k} = a_{2k} + a_{3k},$$

so that, for $r = 1, 2, 3$,

$$a_{rk} = O(\log^3 k). \tag{5.4}$$

Then (5.3) gives

$$\sum_{\sqrt{x}<m<x} d(m)\,d(m+k)\,m e^{2\pi i x k/m}$$

$$= \int_{\sqrt{x}}^{x} y e^{2\pi i x k/y}(a_{1k}\log^2 y + a_{2k}\log y + a_{3k})\,dy$$

$$+ O\left(\int_{\sqrt{x}}^{x} \frac{xk}{y}\, y^{\frac{11}{12}} k^{\frac{1}{6}} \log^3 y\,dy\right) + O(k^{\frac{1}{6}} x^{\frac{23}{12}} \log^3 x)$$

$$= x^2 \int_{x^{-\frac{1}{2}}}^{1} y e^{2\pi i k/y}\Big(a_{1k}\log^2 x + (2a_{1k}\log y + a_{2k})\log x$$

$$+ a_{1k}\log^2 y + a_{2k}\log y + a_{3k}\Big)\,dy + O(k^{\frac{7}{6}} x^{\frac{23}{12}} \log^3 x)$$

$$= x^2 \log^2 x \int_0^1 a_{1k}\, y\, e^{2\pi i k/y}\,dy + x^2 \log x \int_0^1 y\, e^{2\pi i k/y}(2a_{1k}\log y + a_{2k})\,dy$$

$$+ x^2 \int_0^1 y\, e^{2\pi i k/y}(a_{1k}\log^2 y + a_{2k}\log y + a_{3k})\,dy$$

$$+ O(x^{\frac{1}{2}} k^{-1}\log^2 x \log^3 k) + O(k^{\frac{7}{6}} x^{\frac{23}{12}} \log^3 x)$$

$$= A_{1k} x^2 \log^2 x + A_{2k} x^2 \log x + A_{3k} x^2 + O(k^{\frac{7}{6}} x^{\frac{23}{12}} \log^3 x),$$

where $$A_{rk} = O(k^{-1}\log^3 k). \tag{5.5}$$

Substituting this in (5.2) we have

$$S_{12} = 4x^2\left\{\log^2 x \sum_{1 \leqslant \pm k \leqslant x^\beta} A_{1k} k^{-2} + \log x \sum_{1 \leqslant \pm k \leqslant x^\beta} A_{2k} k^{-2} + \sum_{1 \leqslant \pm k \leqslant x^\beta} A_{3k} k^{-2}\right\}$$

$$+ O\left(x^{\frac{23}{12}} \log^3 x \sum_{1 \leqslant \pm k \leqslant x^\beta} k^{-\frac{5}{6}}\right) + O(x^{2-\beta+\epsilon}).$$

But, by (5.5),

$$\sum_{\pm k>x^\beta} \frac{A_{rk}}{k^2} = O(x^{-2\beta}\log^3 x).$$

Hence, putting $$c_r = 4\sum_{\pm k=1}^{\infty} \frac{A_{rk}}{k^2} \quad (r=1,\ 2,\ 3),$$

we have

$$S_{12} = x^2(c_1\log^2 x + c_2\log x + c_3) + O(x^{\frac{23}{12}+\frac{1}{6}\beta}\log^3 x) + O(x^{2-\beta+\epsilon}).$$

Putting $\beta = \frac{1}{14}$ we get Lemma 2.

LEMMA 3.

$$S_{22} = x^2(c_4\log^2 x + c_5\log x + c_6) + O(x^{2-\frac{1}{14}+\epsilon}).$$

We have

$$S_{22} = \sum_{m<x} d(m)\sqrt{m}\sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i\sqrt{x}(\sqrt{n}-\sqrt{m})}}{(\sqrt{n}-\sqrt{m})^2},$$

and hence, by the reasoning of Lemma 2,

$$S_{22} = \sum_{1\leqslant \pm k\leqslant x^\beta} \sum_{\sqrt{x}<m<x} d(m)\,d(m+k)\sqrt{\left(\frac{m}{m+k}\right)} \frac{e^{4\pi i\sqrt{x}(\sqrt{(m+k)}-\sqrt{m})}}{\left(\sqrt{(m+k)}-\sqrt{m}\right)^2} + O(x^{2-\beta+\epsilon}). \quad (5.6)$$

But

$$\sqrt{\left(\frac{m}{m+k}\right)} \frac{e^{4\pi i\sqrt{x}(\sqrt{(m+k)}-\sqrt{m})}}{\left(\sqrt{(m+k)}-\sqrt{m}\right)^2} = \frac{4m}{k^2} e^{2\pi ik\sqrt{(x/m)}} + O\left(\sqrt{\left(\frac{x}{m}\right)}\right),$$

and the error term here gives in (5.6) altogether

$$O\left(x^{\frac{1}{2}}\sum_{k\leqslant x^\beta}\sum_{m<x} x^\epsilon m^{-\frac{1}{2}}\right) = O(x^{1+\beta+\epsilon}).$$

Hence, finally,

$$S_{22} = 4\sum_{1\leqslant \pm k\leqslant x^\beta} k^{-2} \sum_{\sqrt{x}<m<x} d(m)\,d(m+k)\,m\,e^{2\pi ik\sqrt{(x/m)}} + O(x^{2-\beta+\epsilon}), \quad (5.7)$$

β being some constant between 0 and $\frac{1}{6}$.

We now apply Estermann's result, (5.3). We get

$$\sum_{\sqrt{x}<m<x} d(m)\,d(m+k)\,m\,e^{2\pi ik\sqrt{(x/m)}}$$

$$= \int_{\sqrt{x}}^{x} y\,e^{2\pi ik\sqrt{(x/y)}}\,(a_{1k}\log^2 y + a_{2k}\log y + a_{3k})\,dy$$

$$+ O\Big(\int_{\sqrt{x}}^{x} k\sqrt{(x/y)}\,y^{\frac{11}{12}}k^{\frac{1}{6}}\log^3 y\,dy\Big) + O(k^{\frac{1}{6}}x^{\frac{23}{12}}\log^3 x)$$

$$= x^2\int_{x^{-\frac{1}{2}}}^{1} y\,e^{2\pi ik/\sqrt{y}}\Big(a_{1k}\log^2 x + (2a_{1k}\log y + a_{2k})\log x$$

$$+ a_{1k}\log^2 y + a_{2k}\log y + a_{3k}\Big)\,dy + O(k^{\frac{1}{6}}x^{\frac{23}{12}}\log^3 x)$$

$$= x^2\log^2 x\int_0^1 a_{1k}\,y\,e^{2\pi ik/\sqrt{y}}\,dy + x^2\log x\int_0^1 e^{2\pi ik/\sqrt{y}}(2a_{1k}\log y + a_{2k})\,dy$$

$$+ x^2\int_0^1 y\,e^{2\pi ik/\sqrt{y}}(a_{1k}\log^2 y + a_{2k}\log y + a_{3k})\,dy + O(x^{\frac{3}{4}}k^{-1}\log^2 x\log^3 k)$$

$$+ O(k^{\frac{7}{6}}x^{\frac{23}{12}}\log^3 x)$$

$$= B_{1k}x^2\log^2 x + B_{2k}x^2\log x + B_{3k}x^2 + O(k^{\frac{7}{6}}x^{\frac{23}{12}}\log^3 x),$$

where
$$B_{rk} = O(k^{-1}\log^2 k). \tag{5.8}$$

Substituting this in (5.7) we get

$$S_{22} = 4x^2\Big\{\log^2 x\sum_{1\leqslant \pm k\leqslant x^\beta} B_{1k}k^{-2} + \log x\sum_{1\leqslant \pm k\leqslant x^\beta} B_{2k}k^{-2} + \sum_{1\leqslant \pm k\leqslant x^\beta} B_{3k}k^{-2}\Big\}$$

$$+ O\Big(x^{\frac{23}{12}}\log x^3\sum_{1\leqslant \pm k\leqslant x^\beta} k^{-\frac{5}{6}}\Big) + O(x^{2-\beta+\epsilon}).$$

But, by (5.8),

$$\sum_{\pm k>x^\beta}\frac{B_{rk}}{k^2} = O(x^{-2\beta}\log^3 x) \quad (r = 1, 2, 3).$$

Hence, putting

$$c_4 = 4\sum_{\pm k=1}^{\infty} B_{1k}k^{-2}, \quad c_5 = 4\sum_{\pm k=1}^{\infty} B_{2k}k^{-2}, \quad c_6 = 4\sum_{\pm k=1}^{\infty} B_{3k}k^{-2},$$

we have

$$S_{22} = x^2(c_4\log^2 x + c_5\log x + c_6) + O(x^{\frac{23}{12}+\frac{1}{6}\beta}\log^3 x) + O(x^{2-\beta+\epsilon}).$$

Putting $\beta = \frac{1}{14}$ we get Lemma 3.

LEMMA 4.

$$S_{32} = x^{\frac{3}{2}}(c_7 \log^2 x + c_8 \log x + c_9) + O(x^{\frac{3}{2}-\frac{1}{14}+\epsilon}).$$

We have

$$S_{32} = \sum_{m<x} d(m) \sum_{n=1}^{\infty}{}' \frac{d(n)}{\sqrt{n}} \frac{e^{4\pi i \sqrt{x}(\sqrt{n}-\sqrt{m})}}{(\sqrt{n}-\sqrt{m})^2},$$

and hence, by the reasoning of lemma 2,

$$S_{32} = \sum_{1 \leqslant \pm k \leqslant x^\beta} \sum_{\sqrt{x}<m<x} d(m)\, d(m+k) \frac{1}{\sqrt{(m+k)}} \frac{e^{4\pi i \sqrt{x}(\sqrt{(m+k)}-\sqrt{m})}}{\left(\sqrt{(m+k)}-\sqrt{m}\right)^2} + O(x^{\frac{3}{2}-\beta+\epsilon}). \quad (5.9)$$

But

$$\frac{1}{\sqrt{(m+k)}} \frac{e^{4\pi i \sqrt{x}(\sqrt{(m+k)}-\sqrt{m})}}{\left(\sqrt{(m+k)}-\sqrt{m}\right)^2} = \frac{4\sqrt{m}}{k^2} e^{2\pi i k \sqrt{(x/m)}} + O\left(\frac{\sqrt{x}}{m}\right),$$

and the error term here gives in (5.9) altogether

$$O\left(x^{\frac{1}{2}} \sum_{k \leqslant x^\beta} \sum_{m<x} x^\epsilon m^{-1}\right) = O(x^{\frac{1}{2}+\beta+\epsilon} \log x).$$

Hence, finally,

$$S_{32} = 4 \sum_{1 \leqslant \pm k \leqslant x^\beta} k^{-2} \sum_{\sqrt{x}<m<x} d(m)\, d(m+k) \sqrt{m}\, e^{2\pi i k \sqrt{(x/m)}} + O(x^{\frac{3}{2}-\beta+\epsilon}). \quad (5.10)$$

We now once more apply Estermann's result (5.3). We get

$$\sum_{\sqrt{x}<m<x} d(m)\, d(m+k) \sqrt{m}\, e^{2\pi i k \sqrt{(x/m)}}$$

$$= \int_{\sqrt{x}}^{x} \sqrt{y}\, e^{2\pi i k(x/y)}(a_{1k} \log^2 y + a_{2k} \log y + a_{3k})\, dy + O\left(\int_{\sqrt{x}}^{x} k \sqrt{(x/y)}\, y^{\frac{1}{12}} k^{\frac{1}{6}} \log^3 y\, dy\right) + O(x^{\frac{17}{24}} k^{\frac{1}{6}} \log^3 x)$$

$$= x^{\frac{3}{2}} \int_{x^{-\frac{1}{2}}}^{1} \sqrt{y}\, e^{2\pi i k/\sqrt{y}} \left(a_{1k} \log^2 x + (2a_{1k} \log y + a_{2k}) \log x + a_{1k} \log^2 y + a_{2k} \log y + a_{3k}\right) dy + O(k^{\frac{7}{6}} x^{\frac{17}{24}} \log^3 x)$$

$$= x^{\frac{3}{2}} \log^2 x \int_0^1 \sqrt{y}\, a_{1k} e^{2\pi i k/\sqrt{y}}\, dy + x^{\frac{3}{2}} \log x \int_0^1 \sqrt{y}\, e^{2\pi i k/\sqrt{y}} (2a_{1k} \log y + a_{2k})\, dy + x^{\frac{3}{2}} \int_0^1 \sqrt{y}\, e^{2\pi i k/\sqrt{y}} (a_{1k} \log^2 y + a_{2k} \log y + a_{3k})\, dy + O(x^{\frac{1}{2}} k^{-1} \log^2 x \log^3 k) + O(k^{\frac{7}{6}} x^{\frac{17}{24}} \log^3 x)$$

$$= C_{1k} x^{\frac{3}{2}} \log^2 x + C_{2k} x^{\frac{3}{2}} \log x + C_{3k} x^{\frac{3}{2}} + O(k^{\frac{7}{6}} x^{\frac{17}{24}} \log^3 x),$$

where $$C_{rk} = O(k^{-1}\log^3 k). \tag{5.11}$$

Substituting this in (5.10) we get

$$S_{32} = 4x^{\frac{3}{2}}\left\{\log^2 x \sum_{1\leqslant \pm k\leqslant x^\beta} C_{1k}k^{-2} + \log x \sum_{1\leqslant \pm k\leqslant x^\beta} C_{2k}k^{-2} + \sum_{1\leqslant \pm k\leqslant x^\beta} C_{3k}\right\}$$
$$+O\left(x^{\frac{17}{12}}\log^3 x \sum_{1\leqslant \pm k\leqslant x^\beta} k^{-\frac{5}{6}}\right) + O(x^{\frac{3}{2}-\beta+\epsilon}).$$

But, by (5.11),

$$\sum_{\pm k>x^\beta} \frac{C_{rk}}{k^{\frac{3}{2}}} = O(x^{-2\beta}\log^3 x) \quad (r = 1, 2, 3).$$

Hence, putting

$$c_7 = 4\sum_{\pm k=1}^{\infty} C_{1k}k^{-2}, \quad c_8 = 4\sum_{\pm k=1}^{\infty} C_{2k}k^{-2}, \quad c_9 = 4\sum_{\pm k=1}^{\infty} C_{3k}k^{-2},$$

we have

$$S_{32} = x^{\frac{3}{2}}(c_7\log^2 x + c_8\log x + c_9) + O(x^{\frac{17}{12}+\frac{1}{6}\beta}\log^3 x) + O(x^{\frac{3}{2}-\beta+\epsilon}).$$

Taking $\beta = \frac{1}{14}$ we get Lemma 4.

6. *The elementary sums* S_{13}, S_{23}, S_{33}. We shall now use the elementary formula

$$\sum_{n<x} d(n) = x\log x + x(2\gamma-1) + O(x^{\frac{1}{2}}), \tag{6.1}$$

to deal with the sums S_{13}, S_{23}, S_{33}, defined in (4.2), (4.3), (4.4) respectively. We have

LEMMA 5.

$$S_{13} = x(c_{10}\log^2 x + c_{11}\log x + c_{12}) + O(x^{\frac{1}{2}+\epsilon}).$$

We have

$$S_{13} = \sum_{m<x} d(m)\, m \sum_{n=1}^{\infty}{}' \frac{d(n)\left(i\pi+\log(m/n)\right)}{n^2-m^2}$$
$$= \sum_{m<x} d(m)\, m \sum_{n<x}{}' + \sum_{m<x} d(m)\, m \sum_{n>x}$$
$$= S'_{13} + S''_{13}. \tag{6.2}$$

Now $$S'_{13} = -\sum_{m<x} d(m) \sum_{n<m} d(n)\left(\frac{i\pi}{m+n} + \frac{\log(m/n)}{m-n}\right).$$

But, by (6.1),

$$\sum_{n<m}\frac{d(n)}{m+n}=\int_0^m\frac{\log y+2\gamma}{m+y}\,dy+O\Big(\int_0^m\frac{\sqrt{y}\,dy}{(m+y)^2}\Big)+O(m^{-\frac{1}{2}})$$
$$=\int_0^1(\log m+\log y+2\gamma)\,\frac{dy}{1+y}+O(m^{-\frac{1}{2}})$$
$$=\log m\log 2+c_{13}+O(m^{-\frac{1}{2}}).$$

Also, by (6.1),

$$\sum_{n<m}d(n)\,\frac{\log(m/n)}{m-n}=\int_0^m\frac{(\log y+2\gamma)\log(m/y)}{m-y}\,dy+O(m^{-\frac{1}{2}})$$
$$+O\Big(\int_0^m\sqrt{y}\,\Big|\frac{\log(m/y)-(m-y)/y}{(m-y)^2}\Big|\,dy\Big)$$
$$=\int_0^m\frac{(\log y+2\gamma)\log(m/y)}{m-y}\,dy+O(m^{-\frac{1}{2}}).$$
$$=\int_0^1\frac{(\log m+\log y+2\gamma)\log(1/y)}{1-y}\,dy+O(m^{-\frac{1}{2}})$$
$$=c_{14}\log m+c_{15}+O(m^{-\frac{1}{2}}).$$

Substituting these expressions in S'_{13} we get

$$S'_{13}=-\sum_{m<x}d(m)\{\log m(i\pi\log 2+c_{14})+i\pi c_{13}+c_{15}+O(m^{-\frac{1}{2}})\}$$
$$=c_{16}x\log^2x+c_{17}x\log x+c_{18}x+O(x^{\frac{1}{2}}\log x).\qquad(6.3)$$

We are left to consider

$$S''_{13}=\sum_{m<x}d(m)\,m\sum_{n>x}d(n)\,\frac{\big(i\pi+\log(m/n)\big)}{n^2-m^2}.$$

Now, writing x' for the integral part of x, we have

$$\sum_{n>x}\frac{d(n)\big(i\pi+\log(m/n)\big)}{n^2-m^2}$$
$$=\int_{x'+1}^\infty\frac{(\log y+2\gamma)\big(i\pi+\log(m/y)\big)}{m^2-y^2}\,dy$$
$$+O\Big(\int_{x'+1}^\infty\sqrt{y}\,\Big|\frac{2y\big(i\pi+\log(m/y)\big)-(m^2-y^2)/y}{(m^2-y^2)^2}\Big|\,dy\Big)+O\Big(\frac{\sqrt{x}}{(x'+1)^2-m^2}\Big),$$
$$=\frac{1}{m}\int_{(x'+1)/m}^\infty\frac{(\log m+\log y+2\gamma)(i\pi-\log y)}{1-y^2}\,dy+O\Big(\frac{x^{-\frac{1}{2}}}{x'+1-m}\Big)$$
$$+O\Big(m^{-\frac{3}{2}}\int_{(x'+1)/m}^\infty\sqrt{y}\,\Big|\frac{2y(i\pi-\log y)-(1-y^2)/y}{(1-y^2)^2}\Big|\,dy\Big)$$
$$=\frac{1}{m}\int_{(x'+1)/m}^\infty\frac{(\log m+\log y+2\gamma)(i\pi-\log y)}{1-y^2}\,dy+O\Big(\frac{m^{-\frac{1}{2}}}{x'+1-m}\Big).\qquad(6.4)$$

Hence

$$S''_{13} = \sum_{m<x} d(m) \int_{(x'+1)/m}^{\infty} \frac{(\log m + \log y + 2\gamma)(i\pi - \log y)}{y^2-1}\, dy + O(x^{\frac{1}{2}+\epsilon}). \quad (6.5)$$

But

$$\sum_{m<x} d(m) \int_{(x'+1)/m}^{\infty} \frac{(\log y + 2\gamma)(i\pi - \log y)}{y^2-1}\, dy$$

$$= \int_{1+1/x'}^{\infty} \frac{(\log y + 2\gamma)(i\pi - \log y)}{y^2-1} \sum_{(x'+1)/y < m \leqslant x'} d(m)\, dy$$

$$= \int_{1+1/x'}^{\infty} \frac{(\log y + 2\gamma)(i\pi - \log y)}{y^2-1} \left\{ x \log x \left(1 - \frac{1}{y}\right) - \frac{x \log y}{y} \right.$$

$$\left. + (2\gamma - 1) x \left(1 - \frac{1}{y}\right) + O(x^{\frac{1}{2}}) \right\} dy.$$

But
$$\int_{1+1/x'}^{\infty} \left(1 - \frac{1}{y}\right) \frac{dy}{y^2-1} = \int_1^{\infty} \frac{dy}{y(1+y)} + O\left(\frac{1}{x}\right),$$

and similarly for

$$\int_{1+1/x'}^{\infty} \frac{\log y}{\log^2 y} \left(1 - \frac{1}{y}\right) \frac{dy}{y^2-1}.$$

Again

$$\int_{1+1/x'}^{\infty} \log y \, \frac{dy}{y(y^2-1)} = \int_1^{\infty} \log y \, \frac{dy}{y(y^2-1)} + O\left(\frac{1}{x}\right),$$

and similarly for the integrals in $\log^2 y$, $\log^3 y$.

Lastly

$$\int_{1+1/x'}^{\infty} \frac{dy}{y^2-1} = O(\log x), \quad \int_{1+1/x'}^{\infty} \frac{\log y}{\log^2 y} \frac{dy}{y^2-1} = O(1).$$

Hence

$$\sum_{m<x} d(m) \int_{(x'+1)/m}^{\infty} \frac{(\log y + 2\gamma)(i\pi - \log y)}{y^2-1}\, dy = x(c_{19} \log x + c_{20}) + O(x^{\frac{1}{2}} \log x).$$

Similarly

$$\sum_{m<x} d(m) \int_{(x'+1)/m}^{\infty} \frac{\log m (i\pi - \log y)}{y^2-1}\, dy$$

$$= x(c_{21} \log^2 x + c_{22} \log x + c_{23}) + O(x^{\frac{1}{2}} \log^2 x).$$

Inserting these results in (6.5) we get

$$S''_{13} = x\Big(c_{21}\log^2 x+(c_{22}+c_{19})\log x+(c_{23}+c_{20})\Big)+O(x^{\frac{1}{2}+\epsilon}). \qquad (6.6)$$

Combining (6.6) with (6.3) and (6.2) we get lemma 5.

LEMMA 6.

$$S_{23} = x^{\frac{3}{2}}(c_{24}\log^2 x+c_{25}\log x+c_{26})+O(x^{1+\epsilon}).$$

We have

$$\begin{aligned} S_{23} &= \sum_{m<x} d(m)\, m^{\frac{3}{2}} \sum_{n=1}^{\infty}{}' \frac{d(n)\Big(i\pi+\log(m/n)\Big)}{n^2-m^2} \\ &= \sum_{m<x} d(m)\, m^{\frac{3}{2}} \sum_{n<x}{}' + \sum_{m<x} d(m)\, m^{\frac{3}{2}} \sum_{n>x} \\ &= S'_{23}+S''_{23}. \end{aligned} \qquad (6.7)$$

Then

$$S'_{23} = -\sum_{m<x} d(m) \sum_{n<m} d(n)\left(i\pi\,\frac{m^{\frac{3}{2}}-n^{\frac{3}{2}}}{m^2-n^2}+\frac{m^{\frac{3}{2}}+n^{\frac{3}{2}}}{m^2-n^2}\log\frac{m}{n}\right).$$

But

$$\begin{aligned} \sum_{n<m} d(n)\,\frac{m^{\frac{3}{2}}-n^{\frac{3}{2}}}{m^2-n^2} &= \int_0^m \frac{(\log y+2\gamma)(m^{\frac{3}{2}}-y^{\frac{3}{2}})}{m^2-y^2}\,dy+O(1) \\ &\qquad +O\left(\int_0^m \sqrt{y}\left|\frac{d}{dy}\left(\frac{m^{\frac{3}{2}}-y^{\frac{3}{2}}}{m^2-y^2}\right)\right|dy\right) \\ &= \sqrt{m}\int_0^1 \frac{(\log m+\log y+2\gamma)(1-y^{\frac{3}{2}})}{1-y^2}\,dy+O(1) \\ &= c_{27}\sqrt{m}\log m+c_{28}\sqrt{m}+O(1). \end{aligned}$$

Also

$$\begin{aligned} &\sum_{n<m} d(n)\,(m^{\frac{3}{2}}+n^{\frac{3}{2}})\,\frac{\log(m/n)}{m^2-n^2} \\ &\quad = \int_0^m (\log y+2\gamma)(m^{\frac{3}{2}}+y^{\frac{3}{2}})\,\frac{\log(m/y)}{m^2-y^2}\,dy \\ &\qquad\qquad +O(1)+O\left(\int_0^m \sqrt{y}\left|\frac{d}{dy}\left(\frac{(m^{\frac{3}{2}}+y^{\frac{3}{2}})\log(m/y)}{m^2-y^2}\right)\right|dy\right) \\ &\quad = \sqrt{m}\int_0^1 (\log m+\log y+2\gamma)(1+y^{\frac{3}{2}})\,\frac{\log(1/y)}{1-y^2}\,dy+O(1) \\ &\quad = c_{29}\sqrt{m}\log m+c_{30}\sqrt{m}+O(1). \end{aligned}$$

Hence

$$S'_{23} = -\sum_{m<x} d(m)\{(c_{27}i\pi+c_{29})\sqrt{m}\log m+(c_{28}i\pi+c_{30})\sqrt{m}+O(1)\}$$

$$= x^{\frac{3}{2}}(c_{31}\log^2 x+c_{32}\log x+c_{33})+O(x\log x). \tag{6.8}$$

It remains to consider

$$S''_{23} = \sum_{m<x} d(m)\, m^{\frac{3}{2}} \sum_{n>x} d(n)\left(\frac{i\pi+\log(m/n)}{n^2-m^2}\right).$$

By (6.4) we have

$$S''_{23} = \sum_{m<x} d(m)\sqrt{m}\int_{(x'+1)/m}^{\infty} \frac{(\log m+\log y+2\gamma)(i\pi-\log y)}{y^2-1}\,dy+O(x^{1+\epsilon}), \tag{6.9}$$

where x' denotes the integral part of x. Now

$$\sum_{m<x} d(m)\sqrt{m}\int_{(x'+1)/m}^{\infty} \frac{(\log y+2\gamma)(i\pi-\log y)}{y^2-1}\,dy$$

$$= \int_{1+1/x'}^{\infty} \frac{(\log y+2\gamma)(i\pi-\log y)}{y^2-1} \sum_{(x'+1)/y<m\leqslant x'} d(m)\sqrt{m}\,dy$$

$$= \int_{1+1/x'}^{\infty} \frac{(\log y+2\gamma)(i\pi-\log y)}{y^2-1} \{\tfrac{2}{3}x^{\frac{3}{2}}\log x\,.\,(1-y^{-\frac{3}{2}})-\tfrac{2}{3}x^{\frac{3}{2}}y^{-\frac{3}{2}}\log y$$

$$+\tfrac{4}{9}x^{\frac{3}{2}}(3\gamma-1)(1-y^{-\frac{3}{2}})+O(x)\}\,dy.$$

But

$$\int_{1+1/x'}^{\infty} \frac{1-y^{-\frac{3}{2}}}{y^2-1}\,dy = \int_1^{\infty} \frac{1-y^{-\frac{3}{2}}}{y^2-1}\,dy+O\left(\frac{1}{x}\right),$$

and similarly for

$$\int_{1+1/x'}^{\infty} \frac{\log y}{\log^2 y}\,\frac{1-y^{-\frac{3}{2}}}{y^2-1}\,dy.$$

Again

$$\int_{1+1/x'}^{\infty} y^{-\frac{3}{2}}\log y\,\frac{dy}{y^2-1} = \int_1^{\infty} y^{-\frac{3}{2}}\log y\,\frac{dy}{y^2-1}+O\left(\frac{1}{x}\right),$$

and similarly for the integrals in $\log^2 y$, $\log^3 y$.

Lastly

$$\int_{1+1/x'}^{\infty} \frac{dy}{y^2-1} = O(\log x), \qquad \int_{1+1/x'}^{\infty} \frac{\log y}{\log^2 y}\,\frac{dy}{y^2-1} = O(1).$$

Hence

$$\sum_{m<x} d(m)\sqrt{m}\int_{(x'+1)/m}^{\infty} \frac{(\log y+2\gamma)(i\pi-\log y)}{y^2-1}\,dy$$

$$= x^{\frac{3}{2}}(c_{34}\log x+c_{35})+O(x\log x).$$

Likewise

$$\sum_{m<x} d(m)\sqrt{m}\int_{(x'+1)/m}^{\infty}\frac{\log m(i\pi-\log y)}{y^2-1}\,dy$$

$$=x^{\frac{3}{2}}(c_{36}\log^2 x+c_{37}\log x+c_{38})+O(x\log^2 x).$$

Inserting these results in (6.9) we get

$$S''_{23}=x^{\frac{3}{2}}(c_{39}\log^2 x+c_{40}\log x+c_{41})+O(x^{1+\epsilon}). \tag{6.10}$$

Combining (6.10) with (6.8) and (6.7) we get Lemma 6.

LEMMA 7.

$$S_{33}=x(c_{10}\log^2 x+c_{11}\log x+c_{12})+O(x^{\frac{1}{2}+\epsilon}).$$

Proof. Lemma 5 and $S_{13}\equiv S_{33}$.

7. *Estermann's formula and the sums* S_{14}, S_{24}, S_{34}.

LEMMA 8.

$$S_{14}=x^2(c_{42}\log^2 x+c_{43}\log x+c_{44})+O(x^{\frac{23}{13}}\log^3 x).$$

We have, by (4.2),

$$S_{14}=\sum_{m<x} d(m)\,m\sum_{n=1}^{\infty}{}'\frac{d(n)(m^2+n^2)}{(m^2-n^2)^2}$$

$$=\sum_{m<\sqrt{x}} d(m)\,m\sum_{n=1}^{\infty}{}'+\sum_{\sqrt{x}<m<x} d(m)\,m\sum_{n=1}^{\infty}{}'$$

$$=S'_{14}+S''_{14}.$$

But

$$\sum_{n=1}^{\infty}{}'\frac{d(n)(m^2+n^2)}{(m^2-n^2)^2}=\sum_{n\leqslant\frac{1}{2}m}+\sum_{\frac{1}{2}m<n\leqslant 2m}{}'+\sum_{n>2m}$$

$$=O\Big(m^{-2}\sum_{n\leqslant\frac{1}{2}m} d(n)\Big)+O\Big(m^{\epsilon}\sum_{\frac{1}{2}m<n\leqslant 2m}{}'(n-m)^{-2}\Big)$$

$$+O\Big(\sum_{n>2m} d(n)\,n^{-2}\Big)$$

$$=O(m^{\epsilon}).$$

Hence $$S'_{14}=O\Big(\sum_{m<\sqrt{x}} d(m)\,m^{1+\epsilon}\Big)=O(x^{1+\epsilon}).$$

Again

$$S_{14}'' = \sum_{\sqrt{x}<m<x} d(m)\, m \sum_{m-x^\beta \leqslant n \leqslant m+x^\beta} + \sum_{\sqrt{x}<m<x} d(m)\, m \sum_{\substack{n>m+x^\beta \\ n<m-x^\beta}}$$

$$= S_{14}^{(3)} + S_{14}^{(4)},$$

where β is a constant to be chosen later such that $0 < \beta < \frac{1}{6}$. But

$$\sum_{n<m-x^\beta} \frac{d(n)(m^2+n^2)}{(m^2-n^2)^2} = O\Big(m^\epsilon \sum_{n<m-x^\beta} (m-n)^{-2}\Big) = O(m^\epsilon x^{-\beta}).$$

Hence $$S_{14}^{(4)} = O\Big(x^{-\beta} \sum_{m<x} d(m)\, m^{1+\epsilon}\Big) = O(x^{2-\beta+\epsilon}).$$

Collecting these results we get, after rearrangement,

$$S_{14} = \sum_{1 \leqslant \pm k \leqslant x^\beta} \sum_{\sqrt{x}<m<x} d(m)\, d(m+k)\, m \frac{m^2+(m+k)^2}{\big((m+k)^2-m^2\big)} + O(x^{2-\beta+\epsilon}). \quad (7.1)$$

But $$\frac{m^2+(m+k)^2}{\big((m+k)^2-m^2\big)^2} = \frac{1}{2k^2} + O\Big(\frac{1}{km}\Big).$$

The error term here gives in (7.1) altogether

$$O\Big(\sum_{1 \leqslant \pm k \leqslant x^\beta} k^{-1} \sum_{m<x} d(m)\, d(m+k)\Big) = O(x^{1+\epsilon}).$$

Hence, finally,

$$S_{14} = \tfrac{1}{2} \sum_{1 \leqslant \pm k \leqslant x^\beta} k^{-2} \sum_{\sqrt{x}<m<x} m d(m)\, d(m+k) + O(x^{2-\beta+\epsilon}). \quad (7.2)$$

Now we have, by (5.3),

$$\sum_{\sqrt{x}<m<x} m d(m)\, d(m+k) = x^2(b_{1k} \log^2 x + b_{2k} \log x + b_{3k}) + O(x^{\frac{23}{12}} k^{\frac{1}{6}} \log^3 x),$$

the b_{1k}, b_{2k}, b_{3k} being certain fixed linear combinations of the a_{1k}, a_{2k}, a_{3k} of (5.3) so that, for $r = 1, 2, 3$,

$$b_{rk} = O(\log^3 k). \quad (7.3)$$

Substituting this in (7.2) we get

$$S_{14} = \tfrac{1}{2} x^2 \Big\{ \log^2 x \sum_{1 \leqslant \pm k \leqslant x^\beta} b_{1k} k^{-2} + \log x \sum_{1 \leqslant \pm k \leqslant x^\beta} b_{2k} k^{-2} + \sum_{1 \leqslant \pm k \leqslant x^\beta} b_{3k} k^{-2} \Big\}$$

$$+ O(x^{2-\beta+\epsilon}) + O(x^{\frac{23}{12}} \log^3 x).$$

But, by (7.3),

$$\sum_{\pm k > x^\beta} \frac{b_{rk}}{k^2} = O(x^{-\beta} \log^3 x) \quad (r = 1, 2, 3).$$

Hence, putting

$$c_{42} = \tfrac{1}{2} \sum_{\pm k=1}^{\infty} b_{1k} k^{-2}, \quad c_{43} = \tfrac{1}{2} \sum_{\pm k=1}^{\infty} b_{2k} k^{-2}, \quad c_{44} = \tfrac{1}{2} \sum_{\pm k=1}^{\infty} b_{3k} k^{-2},$$

we have

$$S_{14} = x^2(c_{42} \log^2 x + c_{43} \log x + c_{44}) + O(x^{2-\beta+\epsilon}) + O(x^{\frac{23}{12}} \log^3 x).$$

Taking $\beta > \frac{1}{12}$ we get Lemma 8.

LEMMA 9.

$$S_{24} = x^2(c_{42} \log^2 x + c_{43} \log x + c_{44}) + O(x^{\frac{23}{12}} \log^2 x).$$

Proof. $S_{14} \equiv S_{24}$ and Lemma 8.

LEMMA 10.

$$S_{34} = x^{\frac{3}{2}}(c_{45} \log^2 x + c_{46} \log x + c_{47}) + O(x^{\frac{17}{12}} \log^3 x).$$

We have, by (4.4),

$$S_{34} = \sum_{m<x} d(m) \sqrt{m} \sum_{n=1}^{\infty}{}' \frac{d(n)(m^2+n^2)}{(m^2-n^2)^2},$$

and hence, as in Lemma 8,

$$S_{34} = \sum_{1 \leqslant \pm k \leqslant x^\beta} \sum_{\sqrt{x}<m<x} d(m)\, d(m+k) \sqrt{m}\, \frac{(m+k)^2+m^2}{\big((m+k)^2-m^2\big)^2} + O(x^{\frac{3}{2}-\beta+\epsilon})$$

$$= \tfrac{1}{2} \sum_{1 \leqslant \pm k \leqslant x^\beta} k^{-2} \sum_{\sqrt{x}<m<x} d(m)\, d(m+k) \sqrt{m} + O(x^{\frac{3}{2}-\beta+\epsilon}). \qquad (7.4)$$

But, by (5.3),

$$\sum_{\sqrt{x}<m<x} d(m)\, d(m+k) \sqrt{m} = x^{\frac{3}{2}}(d_{1k} \log^2 x + d_{2k} \log x + d_{3k}) + O(x^{\frac{17}{12}} k^{\frac{1}{6}} \log^3 x),$$

the d_{1k}, d_{2k}, d_{3k}, being certain fixed linear combinations of the a_{1k}, a_{2k}, a_{3k} of (5.3), so that, for $r = 1, 2, 3$,

$$d_{rk} = O(\log^3 k). \qquad (7.5)$$

Substituting this in (7.4) we get

$$S_{34} = x^{\frac{3}{2}} \Big\{ \log^2 x \sum_{1 \leqslant \pm k \leqslant x^\beta} d_{1k} k^{-2} + \log x \sum_{1 \leqslant \pm k \leqslant x^\beta} d_{2k} k^{-2} + \sum_{1 \leqslant \pm k \leqslant x^\beta} d_{3k} k^{-2} \Big\}$$

$$+ O(x^{\frac{3}{2}-\beta+\epsilon}) + O(x^{\frac{17}{12}} \log^3 x).$$

The reasoning now proceeds as in Lemma 8.

8. *The trivial sums* S_{15}, S_{25}, S_{35}. We have

$$S_{15} = \sum_{m<x} d(m)(\gamma+\tfrac{1}{2}i\pi+\tfrac{1}{2}\log m)$$
$$= x(c_{48}\log^2 x+c_{49}\log x+c_{50})+O(x^{\frac{1}{2}}\log x). \quad (8.1)$$

Also

$$S_{25} = \sum_{m<x} d(m)\sqrt{m}(\gamma+\tfrac{1}{2}i\pi+\tfrac{1}{2}\log m)$$
$$= x^{\frac{3}{2}}(c_{51}\log^2 x+c_{74}\log x+c_{75})+O(x\log x). \quad (8.2)$$

Lastly

$$S_{35} \equiv S_{15}$$
$$= x(c_{48}\log^2 x+c_{49}\log x+c_{50})+O(x^{\frac{1}{2}}\log x). \quad (8.3)$$

9. *A Ramanujan formula and the sums* S_{11}, S_{21}, S_{31}. It is known* that

$$\sum_{m<x} d^2(m) = S_{21} = \frac{1}{\pi^2}x\log^3 x+\left(\frac{12\gamma-3}{\pi^2}-\frac{36\zeta'(2)}{\pi^4}\right)x\log^2 x$$
$$+c_{51}x\log x+c_{52}x+O(x^{\frac{1}{2}+\epsilon}). \quad (9.1)$$

Hence

$$S_{11} = \sum_{m<x} d^2(m)m^{-1}$$
$$= \frac{1}{4\pi^2}\log^4 x+\left(\frac{4\gamma}{\pi^2}-\frac{12\zeta'(2)}{\pi^4}\right)\log^3 x+c_{53}\log^2 x+c_{54}\log x$$
$$+c_{53}x+O(x^{-\frac{1}{2}+\epsilon}). \quad (9.2)$$

Finally,

$$S_{31} = \sum_{m<x} d^2(m)m^{-\frac{1}{2}}$$
$$= \frac{2}{\pi^2}\sqrt{x}\log^3 x+c_{56}\sqrt{x}\log^2 x+c_{57}\sqrt{x}\log x+c_{58}\sqrt{x}+O(x^{\epsilon}). \quad (9.3)$$

10. *Calculation of* $D_1(x)$ *and proof of Theorem I.* We have, thus, by (4.2), lemmas 2, 5 and 8, (8.1) and (9.2),

$$S_1 = x^2\left\{\frac{1}{4\pi^2\sqrt{2}}\log^4 x+\frac{1}{\sqrt{2}}\left(\frac{4\gamma}{\pi^2}-\frac{12\zeta'(2)}{\pi^4}\right)\log^3 x+c_{59}\log^2 x\right.$$
$$\left.+c_{60}\log x+c_{61}\right\}+O(x^{2-\frac{1}{14}+\epsilon}). \quad (10.1)$$

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* S. Ramanujan, *Messenger of Math*, 45 (1916), 81–84; B. M. Wilson, *Proc. London Math. Soc.* (2), 21 (1922), 235–255.

Again, from (4.3), Lemmas 3, 6, and 9 (8.2) and (9.1), we have

$$S_2 = x^2\left\{\frac{1}{\pi^2\sqrt{2}}\log^3 x + c_{62}\log^2 x + c_{63}\log x + c_{64}\right\} + O(x^{2-\frac{1}{14}+\epsilon}). \quad (10.2)$$

Lastly, from (4.4), Lemmas 4, 7, and 10, (8.3) and (9.3), we have

$$S_3 = x^2\left\{\frac{\sqrt{2}}{\pi^2}\log^3 x + c_{65}\log^2 x + c_{66}\log x + c_{67}\right\} + O(x^{2-\frac{1}{14}+\epsilon}). \quad (10.3)$$

Hence, from (4.1), (10.1), (10.2), and (10.3), we have

$$D_1(x) = x^2\left\{\frac{1}{2\pi^2\sqrt{2}}\log^4 x + \frac{1}{\pi^2\sqrt{2}}\left(8\gamma - 3 - \frac{24\zeta'(2)}{\pi^2}\right)\log^3 x + c_{68}\log^2 x + c_{69}\log x + c_{70}\right\} + O(x^{2-\frac{1}{14}+\epsilon}). \quad (10.4)$$

Inserting (10.4) in (2.8) we have

$$\begin{aligned}
&\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt \\
&= 4\pi^3\delta^2\sqrt{2}\int_1^\infty x^2\left\{\frac{1}{2\pi^2\sqrt{2}}\log^4 x + \frac{1}{\pi^2\sqrt{2}}\left(8\gamma - 3 - \frac{24\zeta'(2)}{\pi^2}\right)\log^3 x \right. \\
&\qquad \left. + c_{68}\log^2 x + c_{69}\log x + c_{70} + O(x^{2-\frac{1}{14}+\epsilon})\right\} e^{-2\pi\delta x}\,dx + O(\delta^{-\frac{1}{2}}\log^3\delta) \\
&= \frac{1}{\delta\sqrt{2}}\int_0^\infty x^2\left\{\frac{1}{2\pi^2\sqrt{2}}\Big(\log x - \log(2\pi\delta)\Big)^4 \right. \\
&\quad + \frac{1}{\pi^2\sqrt{2}}\left(8\gamma - 3 - \frac{24\zeta'(2)}{\pi^2}\right)\Big((\log x - \log(2\pi\delta)\Big)^3 \\
&\quad \left. + c_{68}\Big(\log x - \log(2\pi\delta)\Big)^2 + c_{69}\Big(\log x - \log(2\pi\delta)\Big) + c_{70}\right\} e^{-x}\,dx + O\left(\left(\frac{1}{\delta}\right)^{\frac{13}{14}+\epsilon}\right) \\
&= \frac{1}{\delta}\log^4\frac{1}{\delta}\left(\frac{1}{4\pi^2}\int_0^\infty x^2 e^{-x}\,dx\right) - \frac{1}{\delta}\log^3\frac{1}{\delta}\left(\frac{\log 2\pi}{\pi^2}\int_0^\infty x^2 e^{-x}\,dx\right) \\
&\quad + \frac{1}{\delta}\log^3\frac{1}{\delta}\left(\frac{1}{\pi^2}\int_0^\infty x^2 e^{-x}\log x\,dx\right) \\
&\quad + \frac{1}{\delta}\log^3\frac{1}{\delta}\left(8\gamma - 3 - \frac{24\zeta'(2)}{\pi^2}\right)\left(\frac{1}{2\pi^2}\int_0^\infty x^2 e^{-x}\,dx\right) \\
&\quad + c_{71}\frac{1}{\delta}\log^2\frac{1}{\delta} + c_{72}\frac{1}{\delta}\log\frac{1}{\delta} + c_{73}\frac{1}{\delta} + O\left(\left(\frac{1}{\delta}\right)^{\frac{13}{14}+\epsilon}\right). \quad (10.5)
\end{aligned}$$

But
$$\int_0^\infty x^2 e^{-x}\,dx = \Gamma(3) = 2,$$

$$\int_0^\infty x^2 e^{-x} \log x\,dx = \Gamma'(3) = 3 - 2\gamma.$$

Inserting these values in (10.5) we complete the proof of Theorem I.

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ON FINITELY MEAN VALENT FUNCTIONS

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1. We shall say that a function $f(z)$, regular in the unit circle, has valency p if $f(z)$ takes no value w more than p times. More generally, let $W(R)$ denote the area (regions covered multiply being counted multiply) of that portion of the transform of the unit circle by $w=f(z)$ which lies in the circle $|w|\leqslant R$; then, if

$$W(R)\leqslant p\pi R^2 \tag{1.1}$$

for all $R>0$, where p is a positive number (not necessarily integral), we shall say* that $f(z)$ has mean valency p. Let $\mathbf{A}(\rho)$ be the area of the transform of $|z|\leqslant\rho$ by $f(z)$, $M(\rho, f)$ the maximum modulus of $f(z)$ in the circle $|z|=\rho$. Then

$$\mathbf{A}(\rho)\leqslant W[M(\rho, f)]\leqslant p\pi M^2(\rho, f).$$

Functions $f(z)$ satisfying the condition, less restrictive than (1.1),

$$\mathbf{A}(\rho)\leqslant p\pi M^2(\rho, f), \tag{1.2}$$

we shall describe as having weak mean valency p.

We use the notation

$$M_\lambda(\rho, f)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\rho e^{i\theta})|^\lambda d\theta\right)^{1/\lambda} \quad (\lambda>0),$$

where $\rho<1$. By A we mean a positive absolute constant; by $K(\alpha, \beta, \ldots)$ a positive number depending only upon the parameters shown explicitly. A's and K's are not necessarily the same in different contexts. A

* This definition was suggested to me by Prof. J. E. Littlewood, to whom I am also indebted for suggesting this problem, and for much guidance in the preparation of this paper.

function $f(z)$ such that, for $\alpha > 0$,

$$M(\rho, f) = O\left((1-\rho)^{-\alpha}\right)$$

is said to be of order α. We say that $f(z)$ is of exact order α, where $\alpha > 0$, if the ratio $M(\rho, f)/(1-\rho)^{-\alpha}$ lies between two positive constants.

2. I prove the following theorems:

THEOREM 1. *If* $f(z)$ has mean valency $p \geqslant 1$, $f(0) = 0$, and $\lambda > 0$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^{\lambda} d\theta \leqslant p\Lambda \int_0^{\rho} \frac{M^{\lambda}(r, f)}{r}\, dr, \tag{2.1}$$

where

$$\Lambda = \begin{cases} \lambda & (0 < \lambda \leqslant 2) \\ \frac{1}{2}\lambda^2 & (\lambda > 2). \end{cases}$$

If in addition $\alpha > 0$ and $f(z)$ satisfies†

$$M(\rho, f) \leqslant C(1-\rho)^{-\alpha},$$

then

$$M_{\lambda}(\rho, f) \leqslant CK(p, \lambda)(1-\rho)^{-\alpha+1/\lambda}, \tag{2.2}$$

provided that $\lambda > 1/\alpha$.

The constant on the right-hand side of (2.1) is best possible when p is integral and $0 < \lambda \leqslant 2$.

THEOREM 2. *If $f(z)$ satisfies the conditions of Theorem 1, $\alpha \geqslant 0$, and*

$$M(\rho, f) \leqslant C(1-\rho)^{-\alpha},$$

then‡

$$M_{\lambda}(\rho, f') \leqslant CK(p, \lambda, \alpha)(1-\rho)^{-\alpha-1+1/\lambda}, \tag{2.3}$$

provided that§

$$\lambda > \frac{1}{\alpha+\frac{1}{2}}. \tag{2.4}$$

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* See the Note at the end of the paper.

† In this and the following two theorems C represents the same positive number.

‡ (2.3) is equivalent to (2.2) for $\lambda > 1/\alpha$, but (2.3) is stronger than (2.2) for

$$1/(\alpha+\tfrac{1}{2}) < \lambda \leqslant 1/\alpha.$$

In fact, the truth of (2.3) for any particular $\lambda_1 \leqslant 1/\alpha$ implies (2.2) for all $\lambda > 1/\alpha$. Compare Hardy and Littlewood (**4**), Theorems 27 and 46.

§ If $\alpha = 0$, (2.3) is true (and trivial) for $\lambda \geqslant 2$.

THEOREM 3. *If*

$$f(z) = \Sigma a_n z^n$$

satisfies the conditions of Theorem 2, *then*

$$|a_n| \leqslant CK(p, \alpha) n^{\alpha-1}, \tag{2.5}$$

provided that $\alpha > \frac{1}{2}$.

COROLLARY. *If*

$$f_k(z) = a_1 z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \ldots$$

has valency $p \geqslant 1$, *then*

$$|a_n| < K(p, k) \mu_q n^{2p/k-1} \tag{2.6}$$

provided that $2p/k > \frac{1}{2}$, *where* $\mu_q = \max\{|a_1|, |a_{k+1}|, \ldots, |a_{(q-1)k+1}|\}$ *and* q $(\leqslant p)$ *is the number of zeros of* $f_k(z)$ *in the unit circle.*

Theorem 1 is a known result for schlicht functions [Littlewood (**6**)], and for p-valent functions [Biernacki (**1**)]. Biernacki's method differs from that of Littlewood, and is based on a transformation of Prawitz (**10**), rediscovered by Mandelbrojt (**9**). Theorems 2*, 3, and the corollary are known for schlicht functions. This has been proved in effect† by Littlewood and Paley (**8**), and Levin (**5**)‡. Biernacki (**1**) has given the case $k = 1$ of the corollary as a consequence of (2.1) for p-valent functions and the inequality of Cartwright (**2**). Finally, Robertson (**11**), using a method equivalent to that of Littlewood and Paley, has established the corollary for functions f_k which have no zeros other than the origin.

3. Some additional remarks (of a negative character) are perhaps desirable. (i) It is natural to suppose that the inequality (2.3) of Theorem 2 would remain valid so long as the index (of $1-\rho$) on the right-hand side is strictly negative, *i.e.*, so long as $\lambda > 1/(\alpha+1)$. That this is false is shown by Theorem 4 below. Whether or not the lower bound for λ [as given by (2.4)] is the right one remains an unsolved problem. It is

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* At any rate for $\lambda = 1$.

† The authors confine themselves to functions of the form $f_k(z)$, but their method, without modification, yields the more general results stated here.

‡ Levin's argument is substantially that of Littlewood and Paley.

quite possible that this bound is correct, as the following considerations indicate. For a bounded schlicht function

$$\mathbf{A}(\rho)=\int_0^{\rho}\int_{-\pi}^{\pi}|f'|^2 r\,dr\,d\theta<A,$$

and so

$$M_2(f')=o\left((1-\rho)^{-\frac{1}{2}}\right).$$

Therefore (*a fortiori*)

$$M_1(f')=o\left((1-\rho)^{-\frac{1}{2}}\right).$$

But (as Prof. Littlewood has suggested to me) there is nothing known that contradicts the existence of a bounded schlicht function for which

$$M_1(f')=\Omega\left((1-\rho)^{-\frac{1}{2}+\epsilon}\right),$$

for every positive ϵ. If such a function f does exist, then the lower bound for λ in Theorem 2 is best possible (since f is of order α).

(ii) It is natural to inquire how far the hypothesis of mean valency may be relaxed. Theorem 5 below shows that finite weak mean valency is certainly not enough. Theorems 1 and 2 do, however, remain true* under the weaker hypothesis (1.3) for the range $\lambda \geqslant 2$ of the index λ.

4. THEOREM 4. *Given γ satisfying $1<\gamma<\frac{1}{2}\pi$, and a k greater than a certain absolute constant, there exists a schlicht function $\Phi(z)$, of exact order $\frac{1}{2}(\log\gamma/\log k)$, satisfying*

$$M_1(\rho_n, \Phi')>A(1-\rho_n)^{-(\log\gamma/\log k)}, \tag{4.1}$$

for $\rho_n=1-1/n$, $n=1, 2, \ldots$.

THEOREM 5. *Corresponding to every α greater than a certain absolute constant† there exists a function of finite weak mean valency, and of exact order α, for which the results of Theorems 1‡, 2, and 3 are false.*

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* Theorem 2 is trivial for $\lambda \geqslant 2$. A further property of functions of finite weak mean valency is that, if $f(z)=\Sigma a_n z^n$ is of order α, then $\mathbf{F}(z)=\Sigma|a_n|z^n$ (the "exact" majorant of f) is of order α.

† Which may be taken to be $\frac{3}{2}$.

‡ Theorem 1 is, in fact, false in a drastic manner, not merely by failure of the exact constant involved.

5. We now proceed to the proof of the theorems. In the first place, Theorem 3 is an immediate consequence of Theorem 2. For

$$n|a_n| = \left|\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f'(z)}{z^n}\,dz\right| \leqslant \frac{1}{2\pi\rho^{n-1}}\int_{-\pi}^{\pi}|f'(\rho e^{i\theta})|\,d\theta$$

$$\leqslant CK(p,\alpha)\rho^{-n+1}(1-\rho)^{-\alpha}$$

by Theorem 2 (with $\lambda = 1$), provided that $\alpha > \frac{1}{2}$. Taking $\rho = 1-1/n$ in this, we get

$$n|a_n| \leqslant CK(p,\alpha)n^{\alpha},$$

which is the result of Theorem 3.

For the corollary we observe that the function

$$f_1(z) = \{f_k(z^{1/k})\}^k$$

is p-valent [compare Robertson (**11**)]. Hence, by the inequality of Cartwright (**2**),

$$M(\rho, f_1) < K(p)\mu_{q_1}(1-\rho)^{-2p},$$

where q_1 is the number of zeros of f_1, and μ_{q_1} is the maximum of the moduli of its first q_1 coefficients. It follows that

$$M(\rho, f_k) < K(p, k)\mu_{q_1}^{1/k}(1-\rho)^{-2p/k}.$$

Theorem 3 now gives the corollary with $\mu_{q_1}^{1/k}$ in place of μ_q. But, since $q_1 \leqslant q$, and the first ν coefficients of f_1 depend upon only the first ν coefficients of f_k, it is easily seen that

$$\mu_{q_1} \leqslant K(p, k)\mu_q{}^k.$$

This completes the proof of the corollary.

7. After these preliminaries we come to the proofs of Theorems 1 and 2. We base the proof of both on the following two lemmas.

LEMMA 1. *If $f(z)$ is regular in the unit circle, then, for any $s > 0$,*

$$(7.1)\qquad s^2\int_0^{\rho}\int_{-\pi}^{\pi}|f(re^{i\theta})|^{s-2}|f'(re^{i\theta})|^2 r\,dr\,d\theta = \rho\frac{d}{d\rho}\int_{-\pi}^{\pi}|f(\rho e^{i\theta})|^s\,d\theta$$

$$(7.2)\qquad = s\int_{|z|=\rho}R^s\,d\Phi,$$

where $f(\rho e^{i\theta}) = Re^{i\Phi}$.

Lemma 1 is not new, but is a combination of two well-known results. (7.1) is the Hardy-Stein identity [Hardy (3), Stein (12)]. (7.2), in the form

$$\rho \frac{d}{d\rho} \int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^s \, d\theta = s \int_{|z|=\rho} R^s \, d\Phi,$$

is a special case of the identity used by Prawitz (10).

LEMMA 2. *If $f(z)$ has mean valency p, then, for $s > 0$,*

$$s^2 \int_0^{\rho} \int_{-\pi}^{\pi} |f(re^{i\theta})|^{s-2} |f'(re^{i\theta})|^2 r \, dr \, d\theta \leqslant 2p\pi S M^s(\rho, f), \tag{7.3}$$

where

$$S = \begin{cases} s & (0 < s \leqslant 2), \\ \frac{1}{2}s^2 & (s > 2). \end{cases}$$

Let $E(\rho)$ be the transform of $|z| \leqslant \rho$ by $f(z)$ [so that $mE(\rho) = \mathbf{A}(\rho)$]. For a fixed $\mathbf{R}$ and ρ we define

$$p(\mathbf{R}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} N(\mathbf{R}e^{i\psi}) \, d\psi,$$

where $N(\mathbf{R}e^{i\psi})$ is the number of times the value $w = \mathbf{R}e^{i\psi}$ is taken by $f(z)$ in $|z| \leqslant \rho$. $p(\mathbf{R})$ (which contains the parameter ρ) is the mean value of the valency (in $|z| \leqslant \rho$) on the circumference $|w| = \mathbf{R}$. It is evidently bounded by a constant depending on ρ. The hypothesis of mean valency p implies the inequality

$$\int_0^{\mathbf{R}} p(\mathbf{R}) \, d(\pi \mathbf{R}^2) \leqslant p\pi \mathbf{R}^2$$

for every $\rho < 1$.

For brevity we write $M(\rho, f) = M$. Then we prove first

LEMMA 3. *For any $f(z)$ regular in the unit circle and $s > 0$,*

$$\tfrac{1}{2} \int_{|z|=\rho} R^s \, d\Phi = \int_0^M p(\mathbf{R}) \, d(\pi \mathbf{R}^s). \tag{7.4}$$

In fact, let $R' = R^{\frac{1}{2}s}$, $\mathbf{R}' = \mathbf{R}^{\frac{1}{2}s}$. Then (7.4) becomes

$$\tfrac{1}{2} \int R'^2 \, d\Phi = \int_0^{M^{\frac{1}{2}s}} p(\mathbf{R}'^{2/s}) \, d(\pi \mathbf{R}'^2). \tag{7.5}$$

Let $\mathfrak{P} = Pe^{i\chi}$ be a point of the (Riemann) surface $E(\rho)$, and let $E'(\rho)$ be the surface composed of the aggregate of points $\mathfrak{P}' = P^{\frac{1}{2}s}e^{i\chi}$. Since the surface $E'(\rho)$ is enclosed by the curve $w = R'e^{i\Phi}$, the left-hand side of (7.5) is the integral over this surface of a sectorial element, and is therefore equal to the area of $E'(\rho)$ (multiply covered areas being counted multiply). The integral on the right-hand side of (7.5) is [from the meaning of $p(\mathbf{R})$] plainly also equal to this same area. Hence (7.5) is true, and with it (7.4).

Returning now to Lemma 2, we have, integrating by parts,

$$(7.6)\quad \int_0^M p(\mathbf{R})\,d(\pi\mathbf{R}^s) = sM^{s-2}.\pi\int_0^M p(\mathbf{R})\,\mathbf{R}\,d\mathbf{R} + s(2-s)\,\pi\int_0^M \left(\int_0^{\mathbf{R}_1} p(\mathbf{R})\,\mathbf{R}\,d\mathbf{R}\right)\mathbf{R}_1^{s-3}\,d\mathbf{R}_1.$$

By the hypothesis of mean valency p,

$$2\pi\int_0^{\mathbf{R}_1} p(\mathbf{R})\,\mathbf{R}\,d\mathbf{R} = \int_0^{\mathbf{R}_1} p(\mathbf{R})\,d(\pi\mathbf{R}^2) \leqslant p\pi\mathbf{R}_1^2$$

for all $\mathbf{R}_1 > 0$. Hence, if $s < 2$, (7.6) gives

$$(7.8)\quad \int_0^M p(\mathbf{R})\,d(\pi\mathbf{R}^s) \leqslant sM^{s-2}.\tfrac{1}{2}p\pi M^2 + s(2-s)\int_0^M (\tfrac{1}{2}p\pi\mathbf{R}_1^2).\mathbf{R}_1^{s-3}\,d\mathbf{R}_1 = p\pi M^s.$$

If $s > 2$, we have (by 7.6), using only the hypothesis of finite weak mean valency,

$$(7.9)\quad \int_0^M p(\mathbf{R})\,d(\pi\mathbf{R}^s) \leqslant sM^{s-2}\pi\int_0^M p(\mathbf{R})\,\mathbf{R}\,d\mathbf{R} \leqslant sM^{s-2}.\tfrac{1}{2}p\pi M^2 = \tfrac{1}{2}sp\pi M^s.$$

Lemma 2 follows from (7.8), (7.9), Lemma 3, and (7.2) of Lemma 1.

8. Theorem 1 is now immediate. For, by Lemmas 1 and 2,

$$\rho\frac{d}{d\rho}\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^s\,d\theta \leqslant 2\pi pSM^s(\rho, f),$$

and the result (with s for λ) follows by integration. The function $f(z) = z^p$ shows that the constant is best possible for p integral and $0 < \lambda \leqslant 2$.

In Theorem 2 we may suppose* that $\lambda < 2$. Hence, writing q for λ, we have, by Schwarz's inequality with indices $2/q$ and $2/(2-q)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'|^q d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^{q(1-\frac{1}{2}s)} |f(z)|^{q(\frac{1}{2}s-1)} |f'(z)|^q d\theta$$

$$\leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^{q(2-s)/(2-q)} d\theta\right)^{\frac{1}{2}(2-q)} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^{s-2}|f'|^2 d\theta\right)^{\frac{1}{2}q},$$

$$M_q{}^2(r, f') \leqslant M^{2-s}_{q(2-s)/(2-q)}(r, f)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{s-2}|f'|^2 d\theta\right).$$

Multiplying both sides by $r\,dr$ and integrating from ρ to $\rho^{\frac{1}{2}}$, we get

$$\int_{\rho}^{\rho^{\frac{1}{2}}} M_q{}^2(f')\, r\, dr \leqslant \int_{\rho}^{\rho^{\frac{1}{2}}} M^{2-s}_{q(2-s)/(2-q)}(r, f)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{s-2}|f'|^2 d\theta\right) r\, dr,$$

and therefore, since the means are monotonically increasing functions of r,

$$(8.1)\quad (\rho^{\frac{1}{2}}-\rho)\, M_q{}^2(\rho, f) \leqslant M^{2-s}_{q(2-s)/(2-q)}(\rho^{\frac{1}{2}}, f)\,\frac{1}{2\pi}\int_{\rho}^{\rho^{\frac{1}{2}}}\int_{-\pi}^{\pi} |f|^{s-2}|f'|^2 d\theta\, r\, dr$$

$$< C^2 K(p, q, s)(1-\rho)^{-2\alpha-1+2/\lambda}$$

by Lemma 2 and (2.2) of Theorem 1 [the latter with $\lambda = q(2-s)/(2-q)$], provided that $q > 2/\{(2-s)\alpha+1\}$. By taking s sufficiently small, this last condition is satisfied provided that $q > 1/(\alpha+\frac{1}{2})$. (8.1) is then equivalent to the result of Theorem 2.

9. It remains to prove Theorems 4 and 5. We take Theorem 4 first, since Theorem 5 depends upon Theorem 4. We suppose throughout that k is large enough to satisfy any condition required by the argument.

To obtain the function $\Phi(z)$, we use the method employed by Littlewood (7). We cannot take the bounded (and continuous) function constructed by him, since we require a function of exact order α.

Let $$\Phi'(z) = \frac{1}{(1-z)^\sigma}\prod_{m=1}^{\infty}\phi(z^{k^m}),$$

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* In fact, using only the weak hypothesis (1.2), we have

$$\mathbf{A}(\rho) = \int_0^{\rho}\int_{-\pi}^{\pi} |f'|^2 r\, dr\, d\vartheta \leqslant p\pi M^2(\rho, f) \leqslant CK(p)(1-\rho)^{-2\alpha},$$

and so $$M_2(f') \leqslant CK(p)(1-\rho)^{-\alpha-\frac{1}{2}}.$$

This is (2.3) for $\lambda = 2$, and the result for $\lambda > 2$ follows from Theorem 23, Hardy and Littlewood (4).

where $\phi(z) = e^{\gamma z}$, $1 < \gamma < \frac{1}{2}\pi$, and $\sigma = 1-(\gamma/\log k)+\frac{1}{2}(\log\gamma/\log k)$. Then Φ' and Φ have non-negative coefficients, and a simple calculation shows that

$$A(1-\rho)^{-\sigma-(\gamma/\log k)} < M(\rho, \Phi') < A(1-\rho)^{-\sigma-(\gamma/\log k)}.$$

Hence

$$A(1-\rho)^{-\frac{1}{2}(\log\gamma/\log k)} < M(\rho, \Phi) < A(1-\rho)^{-\frac{1}{2}(\log\gamma/\log k)}, \tag{9.1}$$

so that Φ is of exact order $\frac{1}{2}(\log\gamma/\log k)$.

Next, for n of the form

$$1+k+k^2+\ldots+k^\nu,$$

the coefficients, c_n say, of the function $\prod_{m=1}^{\infty} \phi(z^{k^m})$ satisfy

$$c_n > \gamma^\nu > An^{\log\gamma/\log k}. \tag{9.2}$$

But

$$M_1(\rho, \Phi') = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi'|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|1-z|^\sigma} \prod_{m=1}^{\infty} |\phi(z^{k^m})|\,d\theta \tag{9.3}$$

$$\geqslant A\int_{-\pi}^{\pi} \prod_{m=1}^{\infty} |\phi(z^{k^m})|\,d\theta.$$

By (9.2), for $\rho_n = 1-1/n$,

$$\int_{-\pi}^{\pi} \prod_{m=1}^{\infty} |\phi(z^{k^m})|\,d\theta > An^{\log\gamma/\log k} = A(1-\rho_n)^{-\log\gamma/\log k}. \tag{9.4}$$

(9.3) and (9.4) together give (4.1).

It remains to prove that $\Phi(z)$ is schlicht; and for this the argument used by Littlewood (7) applies (with but trivial modifications*).

10. We are now able to prove Theorem 5. We observe first that, if $f(z)$ is of finite weak mean valency, and of exact order α, then all fractional derivatives $f^{(\beta)}(z)$, with $\beta > 0$, of $f(z)$ are of finite weak mean valency, and of exact order $\alpha+\beta$. In fact, let $f^{(\mu)}(z)$, $\mu > 1$, be a derivative of $f(z)$ of order μ. Then

$$M_2(f') < A(1-\rho)^{-\frac{1}{2}} M(\rho, f) < A(1-\rho)^{-\frac{1}{2}-\alpha},$$

* Since the function $\Phi(z)$ is not bounded and continuous in $|z| \leqslant 1$, it is necessary to carry through his argument on the circle $|z| = \rho$; that is, to prove that, for any $\rho < 1$, the transform of $z = \rho$ by $\Phi(z)$ is simple (has no double points). Other changes in the argument are completely trivial.

and so*

$$M_2(f^{(\mu)}) < A(1-\rho)^{-\frac{1}{2}-\alpha+1-\mu}. \tag{10.1}$$

Further†,

$$A(1-\rho)^{-\alpha-\mu} < M(f^{(\mu)}) < A(1-\rho)^{-\alpha-\mu},$$

$$A(1-\rho)^{-\alpha-\mu+1} < M(f^{(\mu-1)}) < A(1-\rho)^{-\alpha-\mu+1}.$$

Hence, by (10.1), taking $\beta = \mu - 1$,

$$\int_0^\rho \int_0^{2\pi} |f^{(\beta+1)}|^2 r\, dr\, d\theta < A(1-\rho)^{-2(\alpha+\beta)} < AM^2(f^{(\beta)}),$$

so that $f^{(\beta)}(z)$ is of finite weak mean valency, and of exact order $\alpha+\beta$.

Finally, since $\Phi(z)$ is schlicht (and therefore *a fortiori* of finite weak mean valency), and of exact order $\frac{1}{2}(\log\gamma/\log k)$, it follows that $\Phi^{(\beta)}(z)$ and $\Phi^{(\beta+1)}(z)$ are both of finite weak mean valency, and of exact orders

$$\tfrac{1}{2}\frac{\log\gamma}{\log k}+\beta, \quad \tfrac{1}{2}\frac{\log\gamma}{\log k}+\beta+1$$

respectively. Hence, if Theorem 1 were true for functions of finite weak mean valency, we should have, applying it to $\Phi^{(\beta+1)}(z)$,

$$M_\lambda(\Phi^{(\beta+1)}) < AK(\lambda)\,(1-\rho)^{-\frac{1}{2}(\log\gamma/\log k)-\beta-1+1/\lambda}, \tag{10.2}$$

for

$$\lambda > \frac{1}{\frac{1}{2}(\log\gamma/\log k)+\beta+1}. \tag{10.3}$$

But (10.2) is false, by (4.1), for $\lambda = 1$ and all $\beta \geqslant 0$‡. Similarly, if Theorem 2 were true for finite weak mean valency, we should have, applying it to $\Phi^{(\beta)}(z)$,

$$M_\lambda(\Phi^{(\beta+1)}) < AK(\lambda, \alpha)(1-\rho)^{-\frac{1}{2}(\log\gamma/\log k)-\beta-1+1/\lambda}, \tag{10.4}$$

this time for

$$\lambda > \frac{1}{\frac{1}{2}(\log\gamma/\log k)+\beta+\frac{1}{2}}. \tag{10.5}$$

This is false for $\lambda = 1$ and $\beta > \frac{1}{2} - \frac{1}{2}(\log\gamma/\log k)$.

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* See Hardy and Littlewood (**4**) for theorems concerning the order of the derivative and its means. Here we use Theorem 46.

† *Ibid.*, Theorem 39.

‡ Hardy and Littlewood (**4**), Theorem 46.

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[*Note by J. E. Littlewood.* To avoid further delay in the appearance of this paper the later proof sheets were not sent to the author in America, but have been revised by me.

In the definition of mean valency p the value $w = 0$ (more particularly) has a special position. This is of no consequence if $p \geqslant 1$; but, if $p < 1$, f cannot have a zero, and this conflicts with " normalisation " by $f(0) = 0$. The author discovered this too late, and it has fallen to me to deal with it. To reduce changes to a minimum I have made $p \geqslant 1$ throughout. It should however be observed that if, instead of supposing, in Theorems 1, 2, 3, that (1.1) holds for all R, we suppose only that it holds for $R \geqslant R_0$, then (i) the results of the Theorems remain true except that the constant factors in the upper bounds depend on the additional parameter R_0, and (ii) they become true for any positive p.]

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A DESCRIPTIVE DEFINITION OF CESÀRO-PERRON INTEGRALS

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1. *Introduction.*—It is well known that the Perron integral or equivalent special Denjoy integral satisfies the condition (C)†; that is, if $f(x)$ is defined for $a \leqslant x \leqslant b$ and integrable in the Perron sense in (α, β), whenever $a < \alpha < \beta < b$, and if

$$\lim_{\substack{\alpha \to a+0 \\ \beta \to b-0}} \int_{\alpha}^{\beta} f(t)\,dt \tag{1.1}$$

exists and is finite, then $f(x)$ is integrable in the Perron sense in (a, b) to the value (1.1).

This result has been extended by Miss Grimshaw‡ to the case when the Perron integral is replaced by the C_1-P integral and the limits are replaced by C_1-limits. The general case, when the Perron integral is replaced by the C_λ-P integral, λ being any positive number, has recently been considered by Dr. Bosanquet§, who used his result to prove that the C_λ-L integral‖ is included in the C_λ-P integral, and suggested to me that it might be possible to define an integral which would include the C_λ-L integral and correspond to the C_λ-P integral in the same way as the special Denjoy integral corresponds to the Perron integral. It is the object of this paper to define such an integral by relaxing the conditions of the descriptive definition¶ of the special Denjoy integral. Corresponding to each positive integer λ, we shall define by induction an integral which may be called the special Cesàro-Denjoy integral of order λ or C_λ-D integral,

† S. Saks, **9**, 251.
‡ M. E. Grimshaw, **6**, 17–18.
§ L. S. Bosanquet, **3**.
‖ L. S. Bosanquet, **1**, 68–70 and **2**, 274–276.
¶ *Cf.* S. Saks, **8**, 241.

and will be shown to be equivalent to the C_λ-P integral. It thus appears that the C_λ-P integral has properties corresponding to the generalized absolute continuity of the special Denjoy integral.

I should like to thank Dr. Bosanquet for the indication of the problem and for valuable criticisms and suggestions.

2. For convenience the definitions of C_λ-means, C_λ-limits, C_λ-continuity and C_λ-derivates† are given here.

Definition 1. If $F(x)$ is integrable $C_{\lambda-1}$-P in (a, b), then $C_\lambda(F, a, b)$, the λ-th Cesàro mean of $F(x)$ in (a, b), is defined by the equation

$$C_\lambda(F, a, b) = \frac{\lambda}{(b-a)^\lambda} (C_{\lambda-1}\text{-}P) \int_a^b (b-t)^{\lambda-1} F(t)\, dt.$$

The mean of order 0, $C_0(F, a, b)$, is defined to be equal to $F(b)$.

Definition 2. If $C_\lambda(F, \xi, x)$ tends to a finite limit l as x tends to ξ, we say that $F(x)$ tends to l (C_λ-sense) as x tends to ξ, and write

$$C_\lambda\text{-}\lim_{x\to\xi} F(x) = l.$$

There are corresponding definitions for C_λ-$\lim_{x\to\xi+0} F(x)$ and C_λ-$\lim_{x\to\xi-0} F(x)$.

Definition 3. If $F(\xi)$ is finite and

$$C_\lambda\text{-}\lim_{x\to\xi} F(x) = F(\xi),$$

we say that $F(x)$ is C_λ-continuous at the point ξ. There are corresponding definitions for C_λ-continuity on the right and on the left.

Definition 4. If $C_\lambda(F, \xi, \xi+h)$ exists for sufficiently small positive values of h, the upper right-hand C_λ-derivate $C_\lambda D^+F(\xi)$ is defined by the equation

$$C_\lambda D^+F(\xi) = \overline{\lim_{h\to+0}} \frac{C_\lambda(F, \xi, \xi+h) - F(\xi)}{h/(\lambda+1)}.$$

There are corresponding definitions for the other derivates $C_\lambda D_+F(\xi)$, $C_\lambda D^-F(\xi)$, $C_\lambda D_-F(\xi)$.

The upper and lower (two-sided) C_λ-derivates $C_\lambda D^*F(\xi)$, $C_\lambda D_*F(\xi)$ are defined to be $\max\{C_\lambda D^+F(\xi), C_\lambda D^-F(\xi)\}$ and $\min\{C_\lambda D_+F(\xi), C_\lambda D_-F(\xi)\}$ respectively. If they are equal, $F(x)$ is said to have a C_λ-derivative $C_\lambda DF(\xi)$ equal to their common value.

† *Cf.* J. C. Burkill, 5, 541–543.

It is clear from the definitions that C_0-limits are ordinary limits, C_0-continuity is ordinary continuity and C_0-derivates are ordinary derivates.

We next prove some lemmas† involving C_λ-means, λ being a positive integer.

LEMMA I. *If $F(x)$ is integrable $C_{\lambda-1}$-P in (a, b) and $h = (b-a)/\lambda$, then*

$$(2.1)\quad \sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r (\lambda-r)^\lambda C_\lambda\{F, a, a+(\lambda-r)h\} = \sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r (\lambda-r)^\lambda C_\lambda\{F, b, b-(\lambda-r)h\}.$$

If we write

$$F_{\lambda-1}(x) = (C_{\lambda-1}\text{-}P)\int_a^x F(t)\,dt,$$

$$F_r(x) = (C_r\text{-}P)\int_a^x F_{r+1}(t)\,dt \quad (r = 0, 1, 2, \ldots \lambda-2),$$

and integrate by parts $\lambda-1$ times, we find that

$$(2.2)\quad C_\lambda(F, x, x+h) = \frac{\lambda!}{h^\lambda}\left\{F_0(x+h) - F_0(x) - hF_1(x) - \frac{h^2}{2!}F_2(x) - \ldots - \frac{h^{\lambda-1}}{(\lambda-1)!}F_{\lambda-1}(x)\right\}.$$

Since $$\sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r (\lambda-r)^s = 0 \quad (s = 0, 1, 2, \ldots, \lambda-1),$$

it follows that

$$\sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r (\lambda-r)^\lambda C_\lambda\{F, a, a+(\lambda-r)h\} = \frac{\lambda!}{h^\lambda}\sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r F_0\{a+(\lambda-r)h\},$$

and

$$\begin{aligned}\sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r (\lambda-r)^\lambda C_\lambda\{F, b, b-(\lambda-r)h\} &= \frac{\lambda!}{(-h)^\lambda}\sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r F_0\{b-(\lambda-r)h\} \\ &= \frac{\lambda!}{(-h)^\lambda}\sum_{s=0}^{\lambda} (-1)^{\lambda-s} {}^\lambda C_{\lambda-s} F_0(b-sh) \\ &= \frac{\lambda!}{h^\lambda}\sum_{s=0}^{\lambda} (-1)^s {}^\lambda C_s F_0\{a+(\lambda-s)h\},\end{aligned}$$

so that (2.1) is satisfied.

† *Cf.* J. C. Burkill, **5**, 543–545.

LEMMA II. *If $F(x)$ is integrable $C_{\lambda-1}$-P in (a, b) and μ is any integer greater than λ, then*†

$$(2.3) \qquad \overline{\underset{a<x<b}{\text{bound}}}\,|C_\mu(F, a, x)-F(a)| \leqslant \overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(F, a, x)-F(a)|,$$

$$(2.4) \qquad \overline{\underset{a<x<b}{\text{bound}}}\,|C_\mu(F, b, x)-F(b)| \leqslant \overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(F, b, x)-F(b)|.$$

We first note that, since $\mu > \lambda$, $F(x)$ is integrable $C_{\mu-1}$-P in (a, b); the means $C_\mu(F, a, x)$ and $C_\mu(F, b, x)$ are therefore defined.

The result (2.3) is clearly true if $\overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(F, a, x)-F(a)|$ is infinite. We therefore suppose that

$$\overline{\underset{a<x<b}{\text{bound}}}\,|C_\lambda(F, a, x)-F(a)| = M,$$

where M is finite.

If we write

$$F_{\lambda-1}(x) = (C_{\lambda-1}\text{-}P)\int_a^x \{F(t)-F(a)\}\,dt,$$

$$F_r(x) = (C_r\text{-}P)\int_a^x F_{r+1}(t)\,dt \qquad (r = 0, 1, 2, \ldots, \lambda-2),$$

and integrate by parts $(\lambda-1)$ times, we find that

$$C_\lambda(F, a, x)-F(a) = \frac{\lambda!}{(x-a)^\lambda}\,F_0(x),$$

so that
$$|F_0(x)| \leqslant M(x-a)^\lambda/\lambda! \qquad (a < x < b).$$

By a similar method we find that

$$C_\mu(F, a, x)-F(a) = \frac{\mu(\mu-1)(\mu-2)\ldots(\mu-\lambda)}{(x-a)^\mu}\int_a^x (x-t)^{\mu-\lambda-1}\,F_0(t)\,dt,$$

and hence, whenever $a < x < b$,

$$|C_\mu(F, a, x)-F(a)| \leqslant \frac{M\mu(\mu-1)(\mu-2)\ldots(\mu-\lambda)}{\lambda!\,(x-a)^\mu}\int_a^x (x-t)^{\mu-\lambda-1}(t-a)^\lambda\,dt$$

$$= M.$$

The result (2.3) is therefore satisfied, and the proof of (2.4) is similar.

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† $\overline{\underset{a<x<b}{\text{bound}}}\,\phi(x)$ denotes the upper bound of $\phi(x)$ for $a < x < b$.

LEMMA III. *If $F(x)$ is integrable $C_{\lambda-1}$-P in (a, b), where $a \leqslant a < b \leqslant \beta$, if*

$$\omega = \max\left\{\overline{\text{bound}}_{a<x<b} |C_\lambda(F, a, x)-F(a)|,\ \overline{\text{bound}}_{a<x<b} |C_\lambda(F, b, x)-F(b)|\right\}$$

and k is any positive integer, then there is a constant H, depending on λ but independent of a, b, α, β and k, such that

$$|F(b)-F(a)| \leqslant H\omega, \tag{2.5}$$

$$\begin{cases} \left|\dfrac{k}{(\beta-a)^k}\displaystyle\int_a^b (\beta-t)^{k-1}\{F(t)-F(a)\}\,dt\right| \leqslant H\omega, \\ \left|\dfrac{k}{(\alpha-b)^k}\displaystyle\int_b^a (\alpha-t)^{k-1}\{F(t)-F(b)\}\,dt\right| \leqslant H\omega, \end{cases} \tag{2.6}$$

the integrals concerned being $C_{\lambda-1}$-P integrals.

We first note that it follows from (2.2) that $C_\lambda(F, x, x+h)$ is a continuous function of h for $h > 0$, so that

$$|C_\lambda(F, a, b)-F(a)| \leqslant \omega,$$

$$|C_\lambda(F, b, a)-F(b)| \leqslant \omega.$$

Let $h = (b-a)/\lambda$. Since

$$\sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r (\lambda-r)^\lambda = \lambda!,$$

it follows from Lemma I that

$$F(b)-F(a) = \frac{1}{\lambda!}\sum_{r=0}^{\lambda} (-1)^r {}^\lambda C_r (\lambda-r)^\lambda \times [C_\lambda\{F, a, a+(\lambda-r)h\}-F(a)-C_\lambda\{F, b, b-(\lambda-r)h\}+F(b)].$$

Hence, writing

$$L = 2\sum_{r=0}^{\lambda} {}^\lambda C_r (\lambda-r)^\lambda,$$

we have

$$|F(b)-F(a)| \leqslant L\omega. \tag{2.7}$$

We next consider the two integrals in (2.6) in the special cases $\beta = b$, $\alpha = a$ respectively.

Suppose that $k < \lambda$. Then

$$\frac{k}{(\lambda h)^k}(a+\lambda h-t)^{k-1} \equiv \sum_{r=1}^{\lambda} A_r \frac{\lambda}{(rh)^\lambda}(a+rh-t)^{\lambda-1},$$

where the constants $A_1, A_2, A_3, \ldots, A_\lambda$ satisfy the simultaneous equations

$$\frac{\lambda!}{s!(\lambda-1-s)!}\sum_{r=1}^{\lambda}\frac{A_r}{r^{s+1}} = \frac{k!}{s!(k-1-s)!}\frac{1}{\lambda^{s+1}} \quad (s=0, 1, 2, \ldots, k-1),$$

$$= 0, \qquad (s=k, k+1, k+2, \ldots, \lambda-1),$$

and are therefore independent of a and b, but depend in general on λ and k.

It follows that $(b-t)^{k-1}F(t)$ is integrable $C_{\lambda-1}$-P in (a, b), and

$$\left|\frac{k}{(b-a)^k}\int_a^b (b-t)^{k-1}\{F(t)-F(a)\}\,dt\right|$$

$$= \left|\sum_{r=1}^{\lambda} A_r \frac{\lambda}{(rh)^\lambda}\int_a^b (a+rh-t)^{\lambda-1}\{F(t)-F(a)\}\,dt\right|$$

$$= \left|\sum_{r=1}^{\lambda} A_r \{C_\lambda(F, a, a+rh)-F(a)\}\right.$$
$$\left. + \sum_{r=1}^{\lambda} A_r \frac{\lambda}{(rh)^\lambda}\int_{a+rh}^b (a+rh-t)^{\lambda-1}\{F(t)-F(a)\}\,dt\right|$$

$$\leqslant \sum_{r=1}^{\lambda} |A_r\{C_\lambda(F, a, a+rh)-F(a)\}|$$
$$+ \sum_{r=1}^{\lambda}\left|A_r\left(\frac{\lambda-r}{r}\right)^\lambda\{C_\lambda(F, b, a+rh)-F(b)\}\right|$$
$$+ \sum_{r=1}^{\lambda}\left|A_r\left(\frac{\lambda-r}{r}\right)^\lambda\{F(b)-F(a)\}\right|$$

$$\leqslant \omega\sum_{r=1}^{\lambda}|A_r|\left\{1+\left(\frac{\lambda-r}{r}\right)^\lambda(1+L)\right\}.$$

If $k>\lambda$, it follows from Lemma II and the continuity of $C_k(F, a, x)$ for $x=b$ that

$$\left|\frac{k}{(b-a)^k}\int_a^b (b-t)^{k-1}\{F(t)-F(a)\}\,dt\right| \leqslant \omega.$$

Similar results can be obtained for the second inequality in (2.6) with $a=a$. There is therefore a constant M, independent of a, b and k, such that

$$(2.8) \qquad \begin{cases} \left|\frac{k}{(b-a)^k}\int_a^b (b-t)^{k-1}\{F(t)-F(a)\}\,dt\right| \leqslant M\omega, \\ \left|\frac{k}{(a-b)^k}\int_a^b (a-t)^{k-1}\{F(t)-F(b)\}\,dt\right| \leqslant M\omega. \end{cases}$$

We now consider (2.6) in the general case when $a \leqslant a < b \leqslant \beta$. Then

$$\left| \frac{k}{(\beta-a)^k} \int_a^b (\beta-t)^{k-1}\{F(t)-F(a)\}\,dt \right|$$

$$= \left| \frac{k}{(\beta-a)^k} \int_a^b \sum_{s=1}^{k} {}^{k-1}C_{s-1}(\beta-b)^{k-s}(b-t)^{s-1}\{F(t)-F(a)\}\,dt \right|$$

$$\leqslant \frac{k}{(\beta-a)^k} \sum_{s=1}^{k} {}^{k-1}C_{s-1}(\beta-b)^{k-s} \left| \int_a^b (b-t)^{s-1}\{F(t)-F(a)\}\,dt \right|$$

$$\leqslant \frac{k}{(\beta-a)^k} \sum_{s=1}^{k} {}^{k-1}C_{s-1}(\beta-b)^{k-s} \frac{(b-a)^s}{s} M\omega,$$

by (2.8). Since

$$\sum_{s=1}^{k} {}^{k}C_s(\beta-b)^{k-s}(b-a)^s = (\beta-a)^k,$$

it follows that

$$\left| \frac{k}{(\beta-a)^k} \int_a^b (\beta-t)^{k-1}\{F(t)-F(a)\}\,dt \right| \leqslant M\omega.$$

The second integral in (2.6) can be considered similarly. Hence, if we take $H = \max(L, M)$, (2.5) and (2.6) are satisfied.

3. *Statement of properties of the integral.*—We take the C_0-D integral to be the C_0-P integral or special Denjoy integral. When defining the C_λ-D integral we assume that the C_r-D integral has been defined for $r = 0, 1, 2, \ldots, \lambda-1$ and has the following properties:

PROPERTY A. *For $f(x)$ to be integrable C_r-D in (a, b) it is necessary and sufficient that there should exist a function $F(x)$ such that*

(3.1) *$F(x)$ is C_r-continuous for $a \leqslant x \leqslant b$,*

(3.2) *$C_r D\,F(x) = f(x)$ almost everywhere in (a, b),*

(3.3) *the closed interval $[a, b]$ can be expressed as the sum of a denumerable number of closed sets $\{Q_n\}$ over each of which $F(x)$ is absolutely continuous and $f(x)$ absolutely integrable, while*

$$\sum_{m=1}^{\infty} \overline{\text{bound}}_{a_m^{(n)} < x < b_m^{(n)}} \left| C_r(F, a_m^{(n)}, x) - F(a_m^{(n)}) \right| < \infty,$$

$$\sum_{m=1}^{\infty} \overline{\text{bound}}_{a_m^{(n)} < x < b_m^{(n)}} \left| C_r(F, b_m^{(n)}, x) - F(b_m^{(n)}) \right| < \infty,$$

where $(a_1^{(n)}, b_1^{(n)})$, $(a_2^{(n)}, b_2^{(n)})$, $(a_3^{(n)}, b_3^{(n)})$, ..., $(a_m^{(n)}, b_m^{(n)})$, ..., *are the complementary intervals of* Q_n.

Further, if $F(x)$ *satisfies* (3.1), (3.2), (3.3), *and* $a \leqslant t \leqslant b$, *then* $f(x)$ *is integrable* C_r-D *in* (a, t) *and*

$$F(t)-F(a) = (C_r\text{-}D)\int_a^t f(x)\,dx.$$

PROPERTY B. *If* $f(x)$ *is absolutely integrable over a closed set* Q *with end points* a, b *and complementary intervals* $\{(a_n, b_n)\}$†, *while* $f(x)$ *is integrable* C_r-D *in each interval* (a_n, b_n), *and if*

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |\, C_r(F_n, a_n, x)| < \infty,$$

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |\, C_r(F_n, b_n, x)-F_n(b_n)| < \infty,$$

where $F_n(x) = (C_r\text{-}D)\int_{a_n}^{x} f(t)\,dt$ *for* $a_n \leqslant x \leqslant b_n$, *then* $f(x)$ *is integrable* C_r-D *in* (a, b) *and*

$$(C_r\text{-}D)\int_a^b f(t)\,dt = \int_Q f(t)\,dt + \sum_{n=1}^{\infty} (C_r\text{-}D)\int_{a_n}^{b_n} f(t)\,dt.$$

PROPERTY C. *The* C_r-D *integral is equivalent to the* C_r-P *integral* (*and therefore has the properties of the* C_r-P *integral*).

Since the C_0-D integral is taken to be the special Denjoy integral, it is well known that properties A, B and C are true for $r = 0$. It is therefore sufficient to assume that they are true for $r = \lambda-1$, define the C_λ-D integral and prove that they are true for $r = \lambda$, λ being any positive integer. When defining the C_λ-D integral, it would appear natural to take the integrals in the C_λ-means to be $C_{\lambda-1}$-D integrals. In view of property C, however, for $r = \lambda-1$, it is immaterial whether we take the integrals to be $C_{\lambda-1}$-D integrals or $C_{\lambda-1}$-P integrals.

We finally state as property D a subsidiary result which can be deduced from properties B and C.

PROPERTY D. *If* $F(x)$ *is absolutely continuous over a closed set* Q *with end points* a, b *and complementary intervals* $\{(a_n, b_n)\}$, *while* $F(x)$ *is integrable*

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† $\{(a_n, b_n)\}$ is used to denote the sequence of intervals

$$(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3),\ \ldots,\ (a_n, b_n),\ \ldots.$$

C_r-D in each interval (a_n, b_n), and

$$(3.4) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_{r+1}(F, a_n, x)-F(a_n)| < \infty,$$

$$(3.5) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_{r+1}(F, b_n, x)-F(b_n)| < \infty,$$

then $F(x)$ is integrable C_r-D in (a, b) and, if k is any positive integer,

$$(3.6) \quad (C_r\text{-}D)\int_a^b (b-t)^{k-1} F(t)\,dt = \int_Q (b-t)^{k-1} F(t)\,dt + \sum_{n=1}^{\infty} (C_r\text{-}D)\int_{a_n}^{b_n} (b-t)^{k-1} F(t)\,dt.$$

In view of property B, since $(b-x)^{k-1} F(x)$ is absolutely integrable over Q and integrable C_r-D in each complementary interval (a_n, b_n), it is sufficient to prove that

$$(3.7) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_r(G_n, a_n, x)| < \infty,$$

$$(3.8) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_r(G_n, b_n, x)-G_n(b_n)| < \infty,$$

where $$G_n(x) = (C_r\text{-}D)\int_{a_n}^{x} (b-t)^{k-1} F(t)\,dt.$$

We first note that, since $F(x)$ is absolutely continuous over Q, there is a constant M such that

$$(3.9) \qquad \overline{\text{bound}}_{x \in Q} |F(x)| \leqslant M.$$

If $r > 0$ and $a_n < x < b_n$, then

$$C_r(G_n, a_n, x) = \frac{r}{(x-a_n)^r} (C_{r-1}\text{-}D)\int_{a_n}^{x} (x-t)^{r-1} G_n(t)\,dt$$

$$= \frac{1}{(x-a_n)^r} (C_r\text{-}D)\int_{a_n}^{x} (x-t)^r (b-t)^{k-1} F(t)\,dt,$$

in view of property C, integrating by parts for the C_r-P integral. Also, if $r = 0$ and $a_n < x < b_n$,

$$C_r(G_n, a_n, x) = G_n(x)$$

$$= (C_0\text{-}D)\int_{a_n}^{x} (b-t)^{k-1} F(t)\,dt$$

$$= \frac{1}{(x-a_n)^r} (C_r\text{-}D)\int_{a_n}^{x} (x-t)^r (b-t)^{k-1} F(t)\,dt,$$

Hence, whether $r = 0$ or $r > 0$,

$$\overline{\underset{a_n<x<b_n}{\text{bound}}}\,|C_r(G_n, a_n, x)|$$

$$= \overline{\underset{a_n<x<b_n}{\text{bound}}}\left|\sum_{s=1}^{k} k-1C_{s-1}\frac{(b-x)^{k-s}}{(x-a_n)^r}\,(C_r\text{-}D)\int_{a_n}^{x}(x-t)^{r+s-1}F(t)\,dt\right|$$

$$\leqslant \sum_{s=1}^{k} k-1C_{s-1}(b-a)^{k-s}\frac{(b_n-a_n)^s}{r+s}\,\overline{\underset{a_n<x<b_n}{\text{bound}}}\,|C_{r+s}(F, a_n, x)|$$

$$\leqslant \frac{(b-a)^k}{r+1}\sum_{s=1}^{k} k-1C_{s-1}\left\{\overline{\underset{a_n<x<b_n}{\text{bound}}}\,|C_{r+1}(F, a_n, x)-F(a_n)|+\left(\frac{b_n-a_n}{b-a}\right)^s M\right\},$$

in view of Lemma II and (3.9). Since $\sum_{n=1}^{\infty}(b_n-a_n)^s \leqslant (b-a)^s$ for $s = 1, 2, 3, \ldots, k$, it follows from (3.4) that (3.7) is satisfied.

The result (3.8) can be proved by a similar method, and hence (3.6) is satisfied.

4. *Functions $AC^*(C_\lambda$-sense), $ACG^*(C_\lambda$-sense).* In order to relax the conditions of the descriptive definition of the Denjoy-Perron integral, the following extension of the idea of absolute continuity is introduced.

Definition 5. The function $F(x)$ is said to be AC^* (C_λ-sense) over a set E if it is integrable $C_{\lambda-1}$-D in an interval which contains E, and if to each positive number ϵ corresponds a number δ such that

$$\sum_{r=1}^{n}\overline{\underset{a_r<x<b_r}{\text{bound}}}\,|C_\lambda(F, a_r, x)-F(a_r)| < \epsilon,$$

$$\sum_{r=1}^{n}\overline{\underset{a_r<x<b_r}{\text{bound}}}\,|C_\lambda(F, b_r, x)-F(b_r)| < \epsilon,$$

for all finite sets of non-overlapping intervals

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n)$$

with end points on E and such that

$$\sum_{r=1}^{n}(b_r-a_r) \leqslant \delta.$$

It is easily seen that *if $F(x)$ is $AC^*(C_\lambda$-sense) over E, then $F(x)$ is C_λ-continuous on the right at any point of E which is not isolated on the right, and C_λ-continuous on the left at any point of E which is not isolated on the left. Also, in view of Lemma III, a function $AC^*(C_\lambda$-sense) over a set E is absolutely continuous over E.*

Definition 6. The function $F(x)$ will be said to be ACG^* (C_λ-sense) over a set E if it is C_λ-continuous at points of E, and if E is the sum of a denumerable number of sets over each of which $F(x)$ is AC^* (C_λ-sense).

We now prove a number of theorems. We assume throughout that the C_r-D integral has been defined for $r = 0, 1, 2, \ldots, \lambda-1$, and that properties A, B, C and hence D are satisfied. Unless otherwise indicated, integrals involved will be $C_{\lambda-1}$-D integrals.

THEOREM I. *If $F(x)$ is $AC^*(C_\lambda$-sense) over a set E and C_λ-continuous at points of $\bar{E}$†, then $F(x)$ is AC^* (C_λ-sense) over $\bar{E}$.*

Let ϵ be an arbitrary positive number and take a subsidiary positive number η such that $3^\lambda 2^{-\lambda}(2+6H)\eta = \frac{1}{2}\epsilon$, where H is the constant of Lemma III. Then there is a number δ such that

$$\sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} |C_\lambda(F, a_r, x) - F(a_r)| < \eta,$$

$$\sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} |C_\lambda(F, b_r, x) - F(b_r)| < \eta,$$

for all finite sets of non-overlapping intervals

$$(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3),\ \ldots,\ (a_n, b_n)$$

with end points on E and such that

$$\sum_{r=1}^{n} (b_r - a_r) \leqslant \delta.$$

Since $C_\lambda(F, a_r, x)$ is a continuous function of x for $x > a_r$, it follows that if $a_1, a_2, a_3, \ldots, a_n$ belong to E while $b_1, b_2, b_3, \ldots, b_n$ belong to $\bar{E}$, then

$$(4.1) \qquad \sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} |C_\lambda(F, a_r, x) - F(a_r)| \leqslant \eta,$$

provided that the intervals $\{(a_r, b_r)\}$ are non-overlapping, and are such that

$$\sum_{r=1}^{n} (b_r - a_r) < \delta.$$

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† $\bar{E}$ denotes the closed set consisting of the points of E together with the limit points of E.

If, on the other hand, $b_1, b_2, b_3, \ldots, b_n$ belong to E, while $a_1, a_2, a_3, \ldots, a_n$ belong to $\bar{E}$, it follows similarly that

$$(4.2) \qquad \sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} |C_\lambda(F, b_r, x) - F(b_r)| \leqslant \eta.$$

Now let

$$(c_1, d_1), (c_2, d_2), (c_3, d_3), \ldots, (c_m, d_m)$$

be any finite set of non-overlapping intervals with end points on $\bar{E}$ and such that

$$\sum_{r=1}^{m} (d_r - c_r) < \tfrac{1}{2}\delta.$$

These intervals can clearly be divided into two classes in such a way that the intervals of each class have no end points in common. We denote the intervals of one class by

$$(\gamma_1, \delta_1), (\gamma_2, \delta_2), (\gamma_3, \delta_3), \ldots, (\gamma_p, \delta_p).$$

Since $F(x)$ is C_λ-continuous at points of $\bar{E}$, there are positive numbers $\eta_1, \eta_2, \eta_3, \ldots, \eta_p$ such that

$$(4.3) \qquad |C_\lambda(F, \gamma_r, x) - F(\gamma_r)| \leqslant \eta/p,$$

whenever $0 < |x - \gamma_r| \leqslant \eta_r$.

Since $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p$ belong to $\bar{E}$ and the intervals $\{(\gamma_r, \delta_r)\}$ are non-overlapping and have no end points in common, we can find points $a_1, a_2, a_3, \ldots, a_p$ of E such that $a_r < \delta_r$ and

$$(4.4) \qquad |a_r - \gamma_r| < \tfrac{1}{2}\eta_r, \qquad \sum_{r=1}^{p} |a_r - \gamma_r| < \tfrac{1}{2}\delta,$$

while the intervals $\{(a_r, \delta_r)\}$ are non-overlapping. Also

$$\sum_{r=1}^{p} (\delta_r - a_r) < \delta,$$

since $$\sum_{r=1}^{p} (\delta_r - \gamma_r) < \tfrac{1}{2}\delta \quad \text{and} \quad \sum_{r=1}^{p} |a_r - \gamma_r| < \tfrac{1}{2}\delta.$$

It follows from Lemma III that

$$\sum_{r=1}^{p} |F(\gamma_r) - F(a_r)|$$

$$\leqslant H \sum_{r=1}^{p} \left\{ \overline{\text{bound}}_{x \in (a_r, \gamma_r)} |C_\lambda(F, a_r, x) - F(a_r)| + \overline{\text{bound}}_{x \in (a_r, \gamma_r)} |C_\lambda(F, \gamma_r, x) - F(\gamma_r)| \right\}$$

so that, in view of (4.1), (4.2), (4.3) and (4.4),

$$\sum_{r=1}^{p} |F(\gamma_r)-F(\alpha_r)| \leqslant 2H\eta. \tag{4.5}$$

We now show that

$$\sum_{r=1}^{p} \overline{\text{bound}}_{\gamma_r<x<\delta_r} |C_\lambda(F, \gamma_r, x)-F(\gamma_r)| < \tfrac{1}{2}\epsilon.$$

If $\gamma_r+\eta_r \geqslant \delta_r$, it follows from (4.3) that

$$\overline{\text{bound}}_{\gamma_r<x<\delta_r} |C_\lambda(F, \gamma_r, x)-F(\gamma_r)| \leqslant \eta/p. \tag{4.6}$$

It remains to consider the intervals (γ_r, δ_r) for which $\gamma_r+\eta_r < \delta_r$. If $\gamma_r+\eta_r < x < \delta_r$, then

$$\begin{aligned}
&|C_\lambda(F, \gamma_r, x)-F(\gamma_r)| \\
&\quad = \left| \frac{\lambda}{(x-\gamma_r)^\lambda} \int_{\gamma_r}^{\alpha_r} (x-t)^{\lambda-1} \{F(t)-F(\alpha_r)\}\, dt \right. \\
&\qquad\qquad \left. + \frac{\lambda}{(x-\gamma_r)^\lambda} \int_{\alpha_r}^{x} (x-t)^{\lambda-1} \{F(t)-F(\alpha_r)\}\, dt + F(\alpha_r)-F(\gamma_r) \right| \\
&\quad \leqslant \left| \frac{\lambda}{(x-\gamma_r)^\lambda} \int_{\gamma_r}^{\alpha_r} (x-t)^{\lambda-1} \{F(t)-F(\alpha_r)\}\, dt \right| \\
&\qquad\qquad + \left(\frac{x-\alpha_r}{x-\gamma_r}\right)^\lambda |C_\lambda(F, \alpha_r, x)-F(\alpha_r)| + |F(\alpha_r)-F(\gamma_r)| \\
&\quad \leqslant \left| \frac{\lambda}{(x-\gamma_r)^\lambda} \int_{\gamma_r}^{\alpha_r} (x-t)^{\lambda-1} \{F(t)-F(\alpha_r)\}\, dt \right| \\
&\qquad\qquad + 3^\lambda 2^{-\lambda} |C_\lambda(F, \alpha_r, x)-F(\alpha_r)| + |F(\alpha_r)-F(\gamma_r)|,
\end{aligned}$$

since $x-\gamma_r > \eta_r$ and hence, in view of (4.4),

$$(x-\alpha_r)/(x-\gamma_r) < \tfrac{3}{2}.$$

We now consider the first term in the two cases $\alpha_r < \gamma_r$ and $\alpha_r > \gamma_r$. If $\alpha_r < \gamma_r$, it follows from Lemma III that

$$\begin{aligned}
&\left| \frac{\lambda}{(x-\gamma_r)^\lambda} \int_{\gamma_r}^{\alpha_r} (x-t)^{\lambda-1} \{F(t)-F(\alpha_r)\}\, dt \right| \\
&\quad \leqslant \left(\frac{x-\alpha_r}{x-\gamma_r}\right)^\lambda H \left\{ \overline{\text{bound}}_{\alpha_r<x<\gamma_r} |C_\lambda(F, \alpha_r, x)-F(\alpha_r)| \right. \\
&\qquad\qquad \left. + \overline{\text{bound}}_{\alpha_r<x<\gamma_r} |C_\lambda(F, \gamma_r, x)-F(\gamma_r)| \right\},
\end{aligned}$$

while, if $\alpha_r > \gamma_r$, we write

$$\frac{\lambda}{(x-\gamma_r)^\lambda}\int_{\gamma_r}^{\alpha_r}(x-t)^{\lambda-1}\{F(t)-F(\alpha_r)\}\,dt$$

$$=\frac{\lambda}{(x-\gamma_r)^\lambda}\int_{\gamma_r}^{\alpha_r}(x-t)^{\lambda-1}\{F(t)-F(\gamma_r)\}\,dt$$

$$+\frac{\lambda}{(x-\gamma_r)^\lambda}\int_{\gamma_r}^{\alpha_r}(x-t)^{\lambda-1}\{F(\gamma_r)-F(\alpha_r)\}\,dt,$$

and find that

$$\left|\frac{\lambda}{(x-\gamma_r)^\lambda}\int_{\gamma_r}^{\alpha_r}(x-t)^{\lambda-1}\{F(t)-F(\alpha_r)\}\,dt\right|$$

$$\leqslant H\left\{\operatorname{bound}_{\gamma_r<x<\alpha_r}|C_\lambda(F,\alpha_r,x)-F(\alpha_r)|\right.$$

$$\left.+\overline{\operatorname{bound}}_{\gamma_r<x<\alpha_r}|C_\lambda(F,\gamma_r,x)-F(\gamma_r)|\right\}+|F(\gamma_r)-F(\alpha_r)|.$$

It therefore follows that

$$\overline{\operatorname{bound}}_{\gamma_r+\eta_r<x<\delta_r}|C_\lambda(F,\gamma_r,x)-F(\gamma_r)|$$

$$\leqslant 3^\lambda 2^{-\lambda}H\left\{\overline{\operatorname{bound}}_{x\,\epsilon\,(\alpha_r,\,\gamma_r)}|C_\lambda(F,\alpha_r,x)-F(\alpha_r)|+\overline{\operatorname{bound}}_{x\,\epsilon\,(\alpha_r,\,\gamma_r)}|C_\lambda(F,\gamma_r,x)-F(\gamma_r)|\right\}$$

$$+3^\lambda 2^{-\lambda}\overline{\operatorname{bound}}_{\alpha_r<x<\delta_r}|C_\lambda(F,\alpha_r,x)-F(\alpha_r)|+2|F(\gamma_r)-F(\alpha_r)|,$$

and hence, in view of (4.1), (4.2), (4.3), (4.4), (4.5) and (4.6),

$$\sum_{r=1}^{p}\overline{\operatorname{bound}}_{\gamma_r<x<\delta_r}|C_\lambda(F,\gamma_r,x)-F(\gamma_r)|\leqslant 3^\lambda 2^{-\lambda}H\,2\eta+3^\lambda 2^{-\lambda}\eta+4H\eta+\eta$$

$$<3^\lambda 2^{-\lambda}(2+6H)\eta$$

$$=\tfrac{1}{2}\epsilon.$$

The intervals of the second class can be considered in the same way, and hence

$$\sum_{r=1}^{m}\overline{\operatorname{bound}}_{c_r<x<d_r}|C_\lambda(F,c_r,x)-F(c_r)|<\epsilon.$$

By a similar method it can be shown that

$$\sum_{r=1}^{m}\overline{\operatorname{bound}}_{c_r<x<d_r}|C_\lambda(F,d_r,x)-F(d_r)|<\epsilon.$$

The function $F(x)$ is therefore AC^* (C_λ-sense) over $\bar{E}$.

THEOREM II. *For $F(x)$ to be AC^* (C_λ-sense) over a bounded closed set Q with complementary intervals $\{(a_n, b_n)\}$, it is necessary and sufficient that $F(x)$ should be absolutely continuous over Q and integrable $C_{\lambda-1}$-D in each interval (a_n, b_n), while*

$$(4.7) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_\lambda(F, a_n, x) - F(a_n)| < \infty,$$

$$(4.8) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_\lambda(F, b_n, x) - F(b_n)| < \infty.$$

It is easily seen that the conditions are necessary.

We therefore suppose that $F(x)$ is absolutely continuous over a closed set Q with end points a, b and complementary intervals $\{(a_n, b_n)\}$, and that (4.7) and (4.8) are satisfied.

It follows from property D for $r = \lambda - 1$ that $F(x)$ is integrable $C_{\lambda-1}$-D in (a, b).

Let ϵ be an arbitrary positive number. Then we can find δ such that

$$(4.9) \qquad \sum_{r=1}^{m} |F(\delta_r) - F(\gamma_r)| < \tfrac{1}{4}\epsilon$$

for all finite sets of non-overlapping intervals

$$(\gamma_1, \delta_1), (\gamma_2, \delta_2), (\gamma_3, \delta_3), \ldots, (\gamma_m, \delta_m),$$

with end points on Q and such that

$$\sum_{r=1}^{m} (\delta_r - \gamma_r) \leqslant \delta.$$

Also we can find ν such that

$$(4.10) \qquad \sum_{n=\nu+1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_\lambda(F, a_n, x) - F(a_n)| < \frac{\epsilon}{4H},$$

$$(4.11) \qquad \sum_{n=\nu+1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_\lambda(F, b_n, x) - F(b_n)| < \frac{\epsilon}{4H},$$

where H is the constant of Lemma III.

Let $\eta = \min(\delta, b_1 - a_1, b_2 - a_2, b_3 - a_3, \ldots, b_\nu - a_\nu)$,

and let $(c_1, d_1), (c_2, d_2), (c_3, d_3), \ldots, (c_m, d_m)$

be any finite set of non-overlapping intervals with end points on Q and such that

$$\sum_{r=1}^{m} (d_r - c_r) < \eta.$$

It is easily seen that no interval (c_r, d_r) can contain points of any of the intervals (a_1, b_1), (a_2, b_2), (a_3, b_3), ..., (a_ν, b_ν).

Suppose that $c_r < x < d_r$. Then, by property D for the $C_{\lambda-1}$-D integral,

$$C_\lambda(F, c_r, x) - F(c_r) = \frac{\lambda}{(x-c_r)^\lambda} \int_{Q_x} (x-t)^{\lambda-1} \{F(t) - F(c_r)\} dt$$
$$+ \Sigma' \frac{\lambda}{(x-c_r)^\lambda} \int_{a_n}^{b_n} (x-t)^{\lambda-1} \{F(t) - F(c_r)\} dt$$
$$+ \frac{\lambda}{(x-c_r)^\lambda} \int_{a_q}^{x} (x-t)^{\lambda-1} \{F(t) - F(c_r)\} dt,$$

where Q_x denotes the common part of Q and the interval (c_r, x), Σ' denotes that the summation refers to complementary intervals (a_n, b_n) for which $c_r \leqslant a_n < b_n \leqslant x$, and the last term is absent unless x lies in a complementary interval (a_q, b_q).

Now

$$\left| \frac{\lambda}{(x-c_r)^\lambda} \int_{Q_x} (x-t)^{\lambda-1} \{F(t) - F(c_r)\} dt \right| \leqslant \overline{\text{bound}}_{t \in Q_x} |F(t) - F(c_r)|.$$

Also, if $c_r \leqslant a_n < b_n \leqslant x$,

$$\left| \frac{\lambda}{(x-c_r)^\lambda} \int_{a_n}^{b_n} (x-t)^{\lambda-1} \{F(t) - F(c_r)\} dt \right|$$
$$\leqslant \left| \frac{\lambda}{(x-a_n)^\lambda} \int_{a_n}^{b_n} (x-t)^{\lambda-1} \{F(t) - F(a_n)\} dt \right|$$
$$+ |F(a_n) - F(c_r)| \frac{\lambda}{(x-c_r)^\lambda} \int_{a_n}^{b_n} (x-t)^{\lambda-1} dt,$$

while, if $c_r \leqslant a_q < x < b_q$,

$$\left| \frac{\lambda}{(x-c_r)^\lambda} \int_{a_q}^{x} (x-t)^{\lambda-1} \{F(t) - F(c_r)\} dt \right|$$
$$\leqslant \left| \frac{\lambda}{(x-a_q)^\lambda} \int_{a_q}^{x} (x-t)^{\lambda-1} \{F(t) - F(a_q)\} dt \right|$$
$$+ |F(a_q) - F(c_r)| \frac{\lambda}{(x-c_r)^\lambda} \int_{a_q}^{x} (x-t)^{\lambda-1} dt.$$

In view of Lemma III and the fact that

$$H \geqslant 2 \sum_{r=0}^{\lambda} {}^{\lambda}C_r(\lambda-r)^{\lambda} > 1$$

it follows that

$$|C_\lambda(F, c_r, x)-F(c_r)|$$

$$\leqslant H\Sigma' \left\{ \overline{\underset{a_n<t<b_n}{\text{bound}}} |C_\lambda(F, a_n, t)-F(a_n)| + \overline{\underset{a_n<t<b_n}{\text{bound}}} |C_\lambda(F, b_n, t)-F(b_n)| \right\}$$

$$+H|C_\lambda(F, a_q, x)-F(a_q)| + 2 \overline{\underset{t \in Q_x}{\text{bound}}} |F(t)-F(c_r)|,$$

so that, in view of (4.9), (4.10) and (4.11),

$$\sum_{r=1}^{m} \overline{\underset{c_r<x<d_r}{\text{bound}}} |C_\lambda(F, c_r, x)-F(c_r)| < \epsilon.$$

By a similar method it can be shown that

$$\sum_{r=1}^{m} \overline{\underset{c_r<x<d_r}{\text{bound}}} |C_\lambda(F, d_r, x)-F(d_r)| < \epsilon.$$

The function $F(x)$ is therefore AC^* (C_λ-sense) over Q.

THEOREM III. *If $F(x)$ is AC^* (C_λ-sense) over a measurable set E, then $C_\lambda D\,F(x)$ exists and is finite almost everywhere in E. Also $C_\lambda D\,F(x)$ is equal to the approximate derivative $ADF(x)$ almost everywhere in E, and is absolutely integrable over any bounded measurable sub-set of E.*

It is clearly sufficient to suppose that E is bounded.

Let ϵ be any positive number, and let Q be a closed set contained in E and such that

$$m\,Q > m\,E-\epsilon.$$

Since $F(x)$ is AC^* (C_λ-sense) over E, it is absolutely continuous over Q, and

$$\sum_{n=1}^{\infty} \overline{\underset{a_n<x<b_n}{\text{bound}}} |C_\lambda(F, a_n, x)-F(a_n)| < \infty, \tag{4.12}$$

$$\sum_{n=1}^{\infty} \overline{\underset{a_n<x<b_n}{\text{bound}}} |C_\lambda(F, b_n, x)-F(b_n)| < \infty, \tag{4.13}$$

where $(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3),\ \ldots,\ (a_n, b_n),\ \ldots,$

are the complementary intervals of Q.

If k is any positive integer, it follows from (4.12), (4.13) and Lemma II that

$$(4.14) \qquad \sum_{n=1}^{\infty} \left| \frac{k}{(b_n-a_n)^k} \int_{a_n}^{b_n} (b_n-t)^{k-1} F(t)\,dt - F(a_n) \right| < \infty,$$

$$(4.15) \qquad \sum_{n=1}^{\infty} \left| \frac{k}{(a_n-b_n)^k} \int_{b_n}^{a_n} (a_n-t)^{k-1} F(t)\,dt - F(b_n) \right| < \infty,$$

and

$$(4.16) \qquad \sum_{n=1}^{\infty} |F(b_n)-F(a_n)| < \infty.$$

Now let

$$M_n = \max \left\{ \overline{\operatorname{bound}}_{a_n<x<b_n} C_\lambda(F, a_n, x), \quad \overline{\operatorname{bound}}_{a_n<x<b_n} C_\lambda(F, b_n, x), \right.$$

$$\overline{\operatorname{bound}}_{k=1,2,3,\ldots,\lambda} \frac{k}{(b_n-a_n)^k} \int_{a_n}^{b_n} (b_n-t)^{k-1} F(t)\,dt,$$

$$\left. \overline{\operatorname{bound}}_{k=1,2,3,\ldots,\lambda} \frac{k}{(a_n-b_n)^k} \int_{b_n}^{a_n} (a_n-t)^{k-1} F(t)\,dt \right\},$$

and let m_n denote the minimum of the lower bounds of the same expressions.

We next define functions $M(x)$, $m(x)$ which coincide with $F(x)$ at points of Q, while

$$M(x) = M_n, \quad m(x) = m_n \quad (a_n < x < b_n).$$

Then $M(x)$ and $m(x)$ are absolutely continuous over Q and constant in each complementary interval (a_n, b_n), while, in view of (4.12), (4.13), (4.14), (4.15) and (4.16),

$$\sum_{n=1}^{\infty} |M_n - M(a_n)| < \infty, \quad \sum_{n=1}^{\infty} |M_n - M(b_n)| < \infty,$$

$$\sum_{n=1}^{\infty} |m_n - m(a_n)| < \infty, \quad \sum_{n=1}^{\infty} |m_n - m(b_n)| < \infty.$$

It follows that $M(x)$ and $m(x)$ are AC^* over Q, and hence† that $M'(x)$ and $m'(x)$ exist and are finite almost everywhere in Q. Since $M(x)$ and $m(x)$ coincide with $F(x)$ at points of Q,

$$(4.17) \qquad M'(x) = AD\,F(x) = m'(x)$$

almost everywhere in Q.

† *Cf.* S. Saks, **8**, 228–232.

Now suppose that ξ is a point of Q, and take $h>0$. It follows from property D for the $C_{\lambda-1}D$ integral that

$$(4.18)\quad \int_{\xi}^{\xi+h}(\xi+h-t)^{\lambda-1}F(t)\,dt$$
$$=\int_{Q_h}(\xi+h-t)^{\lambda-1}F(t)\,dt$$
$$+\Sigma'\int_{a_n}^{b_n}(\xi+h-t)^{\lambda-1}F(t)\,dt+\int_{a_q}^{\xi+h}(\xi+h-t)^{\lambda-1}F(t)\,dt,$$

where Q_h denotes the common part of Q and the interval $(\xi,\ \xi+h)$, Σ' denotes that the summation refers to complementary intervals $(a_n,\ b_n)$ for which $\xi\leqslant a_n<b_n\leqslant\xi+h$, and the last term is absent unless $\xi+h$ lies in a complementary interval $(a_q,\ b_q)$.

If $\xi\leqslant a_n<b_n\leqslant\xi+h$, then

$$\int_{a_n}^{b_n}(\xi+h-t)^{\lambda-1}F(t)\,dt=\sum_{k=1}^{\lambda}{}^{\lambda-1}C_{k-1}(\xi+h-b_n)^{\lambda-k}\int_{a_n}^{b_n}(b_n-t)^{k-1}F(t)\,dt$$
$$\leqslant\sum_{k=1}^{\lambda}{}^{\lambda-1}C_{k-1}(\xi+h-b_n)^{\lambda-k}(b_n-a_n)^k M_n/k,$$

in view of the definition of M_n, so that

$$\int_{a_n}^{b_n}(\xi+h-t)^{\lambda-1}F(t)\,dt\leqslant\sum_{k=1}^{\lambda}{}^{\lambda-1}C_{k-1}(\xi+h-b_n)^{\lambda-k}\int_{a_n}^{b_n}(b_n-t)^{k-1}M(t)\,dt$$
$$=\int_{a_n}^{b_n}(\xi+h-t)^{\lambda-1}M(t)\,dt.$$

Also, if $\xi\leqslant a_q<\xi+h<b_q$, then

$$\int_{a_q}^{\xi+h}(\xi+h-t)^{\lambda-1}F(t)\,dt\leqslant(\xi+h-a_q)^{\lambda}M_q/\lambda$$
$$=\int_{a_q}^{\xi+h}(\xi+h-t)^{\lambda-1}M(t)\,dt.$$

Since $F(x)$ coincides with $M(x)$ at points of Q, it therefore follows from (4.18) that

$$\int_{\xi}^{\xi+h}(\xi+h-t)^{\lambda-1}F(t)\,dt\leqslant\int_{Q_h}(\xi+h-t)^{\lambda-1}M(t)\,dt$$
$$+\Sigma'\int_{a_n}^{b_n}(\xi+h-t)^{\lambda-1}M(t)\,dt+\int_{a_q}^{\xi+h}(\xi+h-t)^{\lambda-1}M(t)\,dt$$
$$=\int_{\xi}^{\xi+h}(\xi+h-t)^{\lambda-1}M(t)\,dt,$$

so that

$$\frac{C_\lambda(F, \xi, \xi+h)-F(\xi)}{h/(\lambda+1)} \leqslant \frac{C_\lambda(M, \xi, \xi+h)-M(\xi)}{h/(\lambda+1)}.$$

Making h tend to zero, we see that

$$C_\lambda D^+ F(\xi) \leqslant C_\lambda D^+ M(\xi).$$

By similar methods it can be shown that

$$C_\lambda D^- F(\xi) \leqslant C_\lambda D^- M(\xi), \quad C_\lambda D_+ F(\xi) \geqslant C_\lambda D_+ m(\xi),$$

and

$$C_\lambda D_- F(\xi) \geqslant C_\lambda D_- m(\xi).$$

It therefore follows from (4.17) that

$$C_\lambda DF(x) = AD\,F(x)$$

almost everywhere in Q, and hence, since ϵ is arbitrary, almost everywhere in E.

Now $F(x)$, being AC^* (C_λ-sense) over E, is absolutely continuous over E. It follows that $ADF(x)$, and therefore $C_\lambda DF(x)$, is absolutely integrable over any bounded measurable subset of E.

THEOREM IV. *For the function $F(x)$ to be ACG^* (C_λ-sense) over a closed interval $[a, b]$, it is necessary and sufficient that*

(4.19) *$F(x)$ should be C_λ-continuous for $a \leqslant x \leqslant b$,*

(4.20) *$[a, b] = \sum_{n=1}^{\infty} Q_n$, where $\{Q_n\}$ is a sequence of closed sets over each of which $F(x)$ is AC^* (C_λ-sense).*

The sufficiency of the conditions is obvious, while the necessity follows from Theorem I.

THEOREM V. *For the function $F(x)$ to be ACG^* (C_λ-sense) over a closed interval $[a, b]$, it is necessary and sufficient for that $F(x)$ to be C_λ-continuous for $a \leqslant x \leqslant b$, while every closed subset of $[a, b]$ contains a portion† over which $F(x)$ is AC^* (C_λ-sense).*

The proof‡ is similar to that for a function ACG^* in the ordinary sense.

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† By a portion of a closed set Q is meant the common part of Q and an open interval, which contains points of Q.

‡ *Cf.* S. Saks, 8, 234.

THEOREM VI. *If $F(x)$ is ACG^* (C_λ-sense) over a closed interval $[a, b]$, whilst $C_\lambda DF(x) \geqslant 0$ almost everywhere in (a, b), then $F(x)$ increases for $a \leqslant x \leqslant b$.*

Let K be the set of points ξ of $[a, b]$ throughout no neighbourhood of which $F(x)$ increases. (If ξ coincides with a or b, instead of neighbourhoods we consider intervals having a or b as left or right-hand end point respectively.) It is easily seen that K is closed.

Suppose that $c < \gamma < \delta < d$, where (c, d) is any complementary interval of K. Then it can be shown by the method of repeated bisection that $F(x)$ increases for $\gamma \leqslant x \leqslant \delta$. Since $F(x)$ is C_λ-continuous for $c \leqslant x \leqslant d$, it follows that $F(x)$ increases and is continuous for $c \leqslant x \leqslant d$. The set K can therefore contain no isolated points, and must be perfect or null.

Suppose if possible that K is not null. By Theorem V, K contains a portion k over which $F(x)$ is AC^* (C_λ-sense). Let $\bar{k}$ have end points c, d and complementary intervals $\{(c_n, d_n)\}$. Then $F(x)$ increases and is continuous in each closed interval $[c_n, d_n]$, while, in view of Theorem I, $F(x)$ is AC^* (C_λ-sense) over $\bar{k}$ and hence absolutely continuous over $\bar{k}$. It follows that $F(x)$ is absolutely continuous for $c \leqslant x \leqslant d$. Hence $C_\lambda DF(x)$ is equal to the derivative $F'(x)$ almost everywhere in (c, d). Thus $F'(x) \geqslant 0$ almost everywhere in (c, d) and $F(x)$ increases for $c \leqslant x \leqslant d$.

This is impossible if c and d are the end points of the closure of a portion of K. The set K is therefore null, and $F(x)$ increases for $a \leqslant x \leqslant b$.

THEOREM VII. *If $F(x)$ is ACG^* (C_λ-sense) over a closed interval $[a, b]$ whilst $C_\lambda DF(x) = 0$ almost everywhere in (a, b), then $F(x)$ is constant for $a \leqslant x \leqslant b$.*

Theorem VII follows immediately from Theorem VI by considering the functions $F(x)$ and $\{-F(x)\}$.

5. *Definition of the special Cesàro-Denjoy integral.*

Definition 7. The function $f(x)$ is said to be integrable in the special Cesàro-Denjoy sense of order λ in (a, b), or integrable C_λ-D in (a, b), if there is a function $F(x)$ ACG^* (C_λ-sense) over the closed interval $[a, b]$, and such that $C_\lambda DF(x) = f(x)$ almost everywhere in (a, b). When this is the case, we describe $F(x)$ as an indefinite C_λ-D integral of $f(x)$ in (a, b), while $F(b) - F(a)$ is called the definite C_λ-D integral of $f(x)$ over (a, b), and is denoted by

$$(C_\lambda\text{-}D) \int_a^b f(t)\, dt.$$

Since C_λ-$D\,F(x)$ is a measurable function of x, any function $f(x)$ integrable C_λ-D in (a, b) must be measurable over (a, b). Also, in view of Theorem III, $f(x)$ must be finite almost everywhere in (a, b).

It follows that, if $F(x)$ and $G(x)$ are any two indefinite C_λ-D integrals of $f(x)$ in (a, b), then $C_\lambda D\{F(x)-G(x)\} = 0$ almost everywhere in (a, b). It can easily be seen that $\{F(x)-G(x)\}$ is ACG^* (C_λ-sense) over $[a, b]$. It therefore follows from Theorem VII that $\{F(x)-G(x)\}$ is constant for $a \leqslant x \leqslant b$. The definite C_λ-D integral of a function $f(x)$ integrable C_λ-D in (a, b) is therefore unique.

In order to complete the definition of the C_λ-D integral by induction, it remains to show that properties A, B, C (and hence D) are true for $r = \lambda$ (assuming, as before, that they are true for $r = \lambda-1$, λ being any positive integer).

Property A follows at once from the definition and Theorems II, III and IV.

PROPERTY B. *If $f(x)$ is absolutely integrable over a closed set Q with end points a, b and complementary intervals $\{(a_n, b_n)\}$, while $f(x)$ is integrable C_λ-D in each interval (a_n, b_n), with C_λ-D integral*

$$F_n(x) = C_\lambda\text{-}D \int_{a_n}^{x} f(t)\,dt,$$

and if

$$(5.1) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |\, C_\lambda(F_n, a_n, x)| < \infty,$$

$$(5.2) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |\, C_\lambda(F_n, b_n, x) - F_n(b_n)| < \infty,$$

then $f(x)$ is integrable C_λ-D in (a, b), and

$$(C_\lambda\text{-}D) \int_a^b f(t)\,dt = \int_Q f(t)\,dt + \sum_{n=1}^{\infty} (C_\lambda\text{-}D) \int_{a_n}^{b_n} f(t)\,dt.$$

We note that it follows from (5.1), (5.2) and Lemma III that $\Sigma F_n(b_n)$ or $\Sigma(C_\lambda\text{-}D) \int_{a_n}^{b_n} f(t)\,dt$ is absolutely convergent.

Let $$F(x) = \int_{Q_x} f(t)\,dt + \sum_{b_n \leqslant x} F_n(b_n) + F_q(x),$$

where Q_x denotes the common part of Q and the interval (a, x), and the last term is absent unless x lies in a complementary interval (a_q, b_q). We proceed to show that $F(x)$ is an indefinite C_λ-D integral of $f(x)$ in (a, b).

Since $\Sigma F_n(b_n)$ is absolutely convergent, it can easily be seen that $F(x)$ is absolutely continuous over Q. Also, if $a_n < x < b_n$,

$$C_\lambda(F, a_n, x) - F(a_n) = C_\lambda(F_n, a_n, x),$$

$$C_\lambda(F, b_n, x) - F(b_n) = C_\lambda(F_n, b_n, x) - F_n(b_n),$$

so that, in view of (5.1) and (5.2),

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_\lambda(F, a_n, x) - F(a_n)| < \infty,$$

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} |C_\lambda(F, b_n, x) - F(b_n)| < \infty.$$

The function $F(x)$ is therefore AC^* (C_λ-sense) over Q.

It follows that $F(x)$ is C_λ-continuous on the right (or left) at any point of Q which is not isolated on the right (or left, respectively). Also $F(x)$ is clearly ACG^* (C_λ-sense) over each closed interval $[a_n, b_n]$. Hence $F(x)$ must be ACG^* (C_λ-sense) over $[a, b]$.

It is easily seen that $C_\lambda DF(x) = f(x)$ at almost all points of the complement of Q with respect to (a, b). It therefore remains to show that $C_\lambda DF(x) = f(x)$ almost everywhere in Q.

Let $g(x)$ coincide with $f(x)$ at points of Q, whilst

$$g(x) = F_n(b_n)/(b_n - a_n)$$

if $a_n < x < b_n$. Then $g(x)$ is absolutely integrable in (a, b), and

$$\int_a^x g(t)\,dt$$

coincides with $F(x)$ at points of Q. It follows that

$$ADF(x) = g(x) = f(x)$$

almost everywhere in Q.

In view of Theorem III, $ADF(x) = C_\lambda DF(x)$ almost everywhere in Q. Hence

$$C_\lambda DF(x) = f(x)$$

almost everywhere in Q.

The function $f(x)$ is therefore integrable C_λ-D in (a, b), and

$$(C_\lambda\text{-}D)\int_a^b f(t)\,dt = F(b) - F(a)$$

$$= \int_Q f(t)\,dt + \sum_{n=1}^{\infty} (C_\lambda\text{-}D)\int_{a_n}^{b_n} f(t)\,dt.$$

It remains to prove property C. This appears as Theorems VIII and XI.

6. *The C_λ-D integral includes the C_λ-P integral.*

In order to prove that the C_λ-D integral includes the C_λ-P integral, further extensions of the idea of absolute continuity are introduced†.

Definitions 8, 9. The function $F(x)$ is said to be AC^* *below* (C_λ-sense) over a set E if it is integrable $C_{\lambda-1}$-D in an interval which contains E, and if to each positive number ϵ corresponds a number δ such that

$$(6.1) \qquad \sum_{r=1}^{n} \underline{\text{bound}}_{a_r<x<b_r} \{C_\lambda(F, a_r, x)-F(a_r)\} > -\epsilon,$$

$$(6.2) \qquad \sum_{r=1}^{n} \underline{\text{bound}}_{a_r<x<b_r} \{F(b_r)-C_\lambda(F, b_r, x)\} > -\epsilon,$$

for all finite sets of non-overlapping intervals

$$(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3),\ \ldots,\ (a_n, b_n)$$

with end points on E and such that

$$\sum_{r=1}^{n} (b_r-a_r) \leqslant \delta.$$

If (6.1) and (6.2) are replaced by

$$\sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} \{C_\lambda(F, a_r, x)-F(a_r)\} < \epsilon$$

and

$$\sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} \{F(b_r)-C_\lambda(F, b_r, x)\} < \epsilon,$$

the function $F(x)$ is said to be AC^* *above* (C_λ-sense) over E.

It is clear that a function AC^* (C_λ-sense) over a set is both AC^* below (C_λ-sense) and AC^* above (C_λ-sense) over the set.

Further

$$\sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} |C_\lambda(F, a_r, x)-F(a_r)|$$

$$= \Sigma' \overline{\text{bound}}_{a_r<x<b_r} \{C_\lambda(F, a_r, x)-F(a_r)\} - \Sigma'' \underline{\text{bound}}_{a_r<x<b_r} \{C_\lambda(F, a_r, x)-F(a_r)\},$$

† *Cf.* J. Ridder, 7, 251, for the definition of functions AC^* below and above (in the ordinary sense) over a set.

where Σ' refers to the intervals (a_r, b_r) for which

$$\overline{\text{bound}}_{a_r<x<b_r} |C_\lambda(F, a_r, x)-F(a_r)| = \overline{\text{bound}}_{a_r<x<b_r} \{C_\lambda(F, a_r, x)-F(a_r)\},$$

and Σ'' refers to the remaining intervals. In view of this result and the corresponding result for

$$\sum_{r=1}^{n} \overline{\text{bound}}_{a_r<x<b_r} |F(b_r)-C_\lambda(F, b_r, x)|,$$

it follows from the definitions that *if $F(x)$ is both AC^* below (C_λ-sense) and AC^* above (C_λ-sense) over a set E, then $F(x)$ is AC^* (C_λ-sense) over E.*

It also follows from the definitions that *if $F(x)$ is AC^* below (C_λ-sense) over a set E, then $\{-F(x)\}$ is AC^* above (C_λ-sense) over E.*

We now prove a lemma.

LEMMA IV. *If $C_\lambda D_* F(x) > -\infty$ at all points of a set E, then E is the sum of a denumerable number of sets over each of which $F(x)$ is AC^* below (C_λ-sense).*

Let E_n be the set of points x of E for which $F(t)$ is integrable $C_{\lambda-1}$-D in $(x-1/n, x+1/n)$, and

$$\frac{\lambda+1}{h} \{C_\lambda(F, x, x+h)-F(x)\} > -n$$

whenever $0 < |h| \leqslant 1/n$, and let $E_n{}^i$ denote the common part of E_n and the closed interval $[i/n, (i+1)/n]$. Then it is easily seen that

$$E = \sum_{n=1}^{\infty} \sum_{i=-\infty}^{\infty} E_n{}^i,$$

and it is therefore sufficient to prove that $F(x)$ is AC^* below (C_λ-sense) over each set $E_n{}^i$.

Let ϵ be an arbitrary positive number, and let

$$(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3),\ \ldots,\ (a_p, b_p)$$

be non-overlapping intervals whose end points belong to $E_n{}^i$, assuming that $E_n{}^i$ contains more than one point†. Then

$$\frac{\lambda+1}{x-a_r} \{C_\lambda(F, a_r, x)-F(a_r)\} > -n$$

† The case when $E_n{}^i$ consists of only one point is trivial.

whenever $a_r < x < b_r$, and hence

$$\underset{a_r<x<b_r}{\text{bound}} \{C_\lambda(F, a_r, x) - F(a_r)\} \geqslant -n(b_r - a_r)/(\lambda+1).$$

It follows that

$$\sum_{r=1}^{p} \underset{a_r<x<b_r}{\text{bound}} \{C_\lambda(F, a_r, x) - F(a_r)\} > -\epsilon,$$

provided that

$$\sum_{r=1}^{p} (b_r - a_r) < (\lambda+1)\,\epsilon/n.$$

By a similar method it can be shown that

$$\sum_{r=1}^{p} \underset{a_r<x<b_r}{\text{bound}} \{F(b_r) - C_\lambda(F, b_r, x)\} > -\epsilon,$$

provided that

$$\sum_{r=1}^{p} (b_r - a_r) < (\lambda+1)\,\epsilon/n.$$

The function $F(x)$, being integrable $C_{\lambda-1}$-D in $\{i/n, (i+1)/n\}$, is therefore AC^* below (C_λ-sense) over E_n^i, which is the result required.

We now prove the main theorem.

THEOREM VIII. *If $f(x)$ is integrable C_λ-P in (a, b), then $f(x)$ is integrable C_λ-D in (a, b), and*

$$C_\lambda\text{-}D \int_a^b f(t)\,dt = C_\lambda\text{-}P \int_a^b f(t)\,dt.$$

We suppose that $f(x)$ is integrable C_λ-P in (a, b), and write

$$F(x) = C_\lambda\text{-}P \int_a^x f(t)\,dt.$$

Let ϵ be an arbitrary positive number. Then $f(x)$ has a C_λ-major function† $M(x)$, C_λ-continuous for $a \leqslant x \leqslant b$ and such that

$$M(a) = 0, \quad C_\lambda D_* M(x) \geqslant f(x)$$

almost everywhere in (a, b), $C_\lambda D_* M(x) > -\infty$ whenever $a \leqslant x \leqslant b$, while

$$M(b) - F(b) \leqslant \tfrac{1}{2}\epsilon. \tag{6.3}$$

† *Cf.* J. C. Burkill, **5**, 548.

It follows from Lemma IV that

$$[a, b] = \sum_{n=1}^{\infty} E_n,$$

where $M(x)$ is AC^* below (C_λ-sense) over each set E_n. We proceed to prove that $F(x)$ is AC^* below (C_λ-sense) over E_n.

We can find δ_n such that

$$\sum_{r=1}^{m} \underline{\text{bound}}_{a_r<x<b_r} \{C_\lambda(M, a_r, x) - M(a_r)\} > -\tfrac{1}{2}\epsilon, \tag{6.4}$$

$$\sum_{r=1}^{m} \underline{\text{bound}}_{a_r<x<b_r} \{M(b_r) - C_\lambda(M, b_r, x)\} > -\tfrac{1}{2}\epsilon \tag{6.5}$$

for all finite sets of non-overlapping intervals

$$(a_1, b_1),\ (a_2, b_2),\ (a_3, b_3),\ \ldots,\ (a_m, b_m)$$

with end points on E_n and such that

$$\sum_{r=1}^{m} (b_r - a_r) \leqslant \delta_n.$$

Suppose that $a_r < x < b_r$. Then

$$\begin{aligned} &C_\lambda(F, a_r, x) - F(a_r) \\ &\quad = C_\lambda(M, a_r, x) - M(a_r) \\ &\qquad\qquad - \frac{\lambda}{(x-a_r)^\lambda} \int_{a_r}^{x} (x-t)^{\lambda-1} \{M(t) - F(t)\}\, dt + M(a_r) - F(a_r) \\ &\quad \geqslant C_\lambda(M, a_r, x) - M(a_r) - \{M(b_r) - F(b_r)\} + \{M(a_r) - F(a_r)\}, \end{aligned}$$

since $\{M(x) - F(x)\}$ increases in (a, b). It follows that

$$\begin{aligned} &\sum_{r=1}^{m} \underline{\text{bound}}_{a_r<x<b_r} \{C_\lambda(F, a_r, x) - F(a_r)\} \\ &\qquad\qquad \geqslant \sum_{r=1}^{m} \underline{\text{bound}} \{C_\lambda(M, a_r, x) - M(a_r)\} - \{M(b) - F(b)\} \\ &\qquad\qquad > -\epsilon, \end{aligned}$$

by (6.3) and (6.4), provided that the end points of the non-overlapping intervals $\{(a_r, b_r)\}$ belong to E_n, and

$$\sum_{r=1}^{m} (b_r - a_r) \leqslant \delta_n.$$

Again, if $a_r < x < b_r$, then

$$F(b_r) - C_\lambda(F, b_r, x)$$
$$= M(b_r) - C_\lambda(M, b_r, x)$$
$$+ \frac{\lambda}{(x-b_r)^\lambda} \int_{b_r}^{x} (x-t)^{\lambda-1} \{M(t) - F(t)\}\, dt - \{M(b_r) - F(b_r)\}$$
$$\geqslant M(b_r) - C_\lambda(M, b_r, x) + \{M(a_r) - F(a_r)\} - \{M(b_r) - F(b_r)\},$$

so that, in view of (6.3) and (6.5),

$$\sum_{r=1}^{m} \underline{\text{bound}}_{a_r < x < b_r} \{F(b_r) - C_\lambda(F, b_r, x)\} > -\epsilon,$$

provided that the end points of the non-overlapping intervals $\{(a_r, b_r)\}$ belong to E_n, and

$$\sum_{r=1}^{m} (b_r - a_r) \leqslant \delta_n.$$

The function $F(x)$ is thus AC^* below (C_λ-sense) over E_n.

Since the function $\{-f(x)\}$ is integrable C_λ-P in (a, b), with C_λ-P integral $\{-F(x)\}$, it follows that

$$[a, b] = \sum_{m=1}^{\infty} E_m',$$

where $\{-F(x)\}$ is AC^* below (C_λ-sense) over each set E_m'. The function $F(x)$ is therefore AC^* above (C_λ-sense) over E_m', and AC^* (C_λ-sense) over each set $E_n E_m'$.

Since $F(x)$ is C_λ-continuous for $a \leqslant x \leqslant b$, and

$$[a, b] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_m' E_n,$$

$F(x)$ is ACG^* (C_λ-sense) over $[a, b]$. Since $C_\lambda DF(x) = f(x)$ almost everywhere in (a, b), it follows that $f(x)$ is integrable C_λ-D in (a, b), and

$$C_\lambda\text{-}D \int_a^b f(t)\, dt = F(b) - F(a)$$
$$= C_\lambda\text{-}P \int_a^b f(t)\, dt.$$

7. *Some properties of the C_λ-P integral.* In order to prove that the C_λ-P integral includes the C_λ-D integral, various properties of the C_λ-P integral are required. These are given by Theorems IX and X.

THEOREM IX. *If $f(x)$ is integrable C_λ-P in the interval (α, β) whenever $a < \alpha < \beta < b$, and if*

$$(7.1) \qquad C_\lambda\text{-}\lim_{\substack{\alpha \to a+0 \\ \beta \to b-0}} \left\{ (C_\lambda\text{-}P) \int_\alpha^\beta f(t)\,dt \right\}$$

exists and is finite, then $f(x)$ is integrable C_λ-P in (a, b) to the value (7.1).

This result has been proved by Bosanquet†.

THEOREM X. *If $f(x)$ is absolutely integrable over a closed set Q with end points a, b and complementary intervals $\{(a_n, b_n)\}$, while $f(x)$ is integrable C_λ-P in each interval (a_n, b_n), and if*

$$(7.2) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n < x < b_n} |C_\lambda(F_n, a_n, x)| < \infty,$$

$$(7.3) \qquad \sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n < x < b_n} |C_\lambda(F_n, b_n, x) - F_n(b_n)| < \infty,$$

where

$$F_n(x) = (C_\lambda\text{-}P) \int_{a_n}^{x} f(t)\,dt \quad (a_n \leqslant x \leqslant b_n),$$

then $f(x)$ is integrable C_λ-P in (a, b) and

$$(C_\lambda\text{-}P) \int_a^b f(t)\,dt = \int_Q f(t)\,dt + \sum_{n=1}^{\infty} (C_\lambda\text{-}P) \int_{a_n}^{b_n} f(t)\,dt.$$

We first note that it follows from (7.2), (7.3) and Lemma III that $\Sigma F_n(b_n)$ or

$$\Sigma\, (C_\lambda\text{-}P) \int_{a_n}^{b_n} f(t)\,dt$$

is absolutely convergent.

Let $$f_1(x) = f(x)\,\psi_1(x), \quad f_2(x) = f(x)\,\psi_2(x),$$

where $\psi_1(x)$ and $\psi_2(x)$ are the characteristic functions of Q and of the complement of Q respectively.

Then $f_1(x)$ is absolutely integrable in (a, b), and

$$\int_a^b f_1(t)\,dt = \int_Q f(t)\,dt.$$

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† L. S. Bosanquet, **3**, 161–164.

Since $f(x) = f_1(x) + f_2(x)$, it will therefore be sufficient to show that $f_2(x)$ is integrable C_λ-P in (a, b), and that

$$(C_\lambda\text{-}P)\int_a^b f_2(t)\,dt = \sum_{n=1}^{\infty} (C_\lambda\text{-}P)\int_{a_n}^{b_n} f_2(t)\,dt.$$

Let ϵ be an arbitrary positive number. Then, corresponding to each complementary interval (a_n, b_n), $f_2(x)$ has a C_λ-major function $M_n(x)$, C_λ-continuous for $a_n \leqslant x \leqslant b_n$ and such that

$$M_n(a_n) = 0, \quad C_\lambda D_* M_n(x) \geqslant f_2(x)$$

almost everywhere in (a_n, b_n), $C_\lambda D_* M_n(x) > -\infty$ except at a denumerable set of points of (a_n, b_n), while

$$0 \leqslant M_n(b_n) - F_n(b_n) < 2^{-(n+1)}\epsilon. \tag{7.4}$$

It follows from (7.4) and the absolute convergence of $\Sigma F_n(b_n)$ that $\Sigma M_n(b_n)$ is absolutely convergent. Since $\{M_n(x) - F_n(x)\}$ increases in (a_n, b_n), it also follows that

$$F_n(x) \leqslant M_n(x) < F_n(x) + 2^{-(n+1)}\epsilon \quad (a_n \leqslant x \leqslant b_n),$$

and hence, in view of (7.2) and (7.3), that

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n < x < b_n} |C_\lambda(M_n, a_n, x)| < \infty, \tag{7.5}$$

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n < x < b_n} |C_\lambda(M_n, b_n, x) - M_n(b_n)| < \infty. \tag{7.6}$$

Corresponding to each positive integer n, we now proceed to construct a continuous increasing function $\omega_n(x)$, defined for $a_n \leqslant x \leqslant b_n$ and such that $\omega_n(a_n) = 0$, $\Sigma\omega(b_n)$ is convergent, and

$$M_n(b_n) + \omega_n(b_n) \geqslant 0, \tag{7.7}$$

$$\frac{k}{(b_n - a_n)^k}\int_{a_n}^{b_n} (b_n - t)^{k-1}\{M_n(t) + \omega_n(t)\}\,dt \geqslant 0 \quad (k = 1, 2, 3, \ldots, \lambda), \tag{7.8}$$

$$\frac{k}{(a_n - b_n)^k}\int_{b_n}^{a_n} (a_n - t)^{k-1}\{M_n(t) + \omega_n(t)\}\,dt \leqslant M_n(b_n) + \omega_n(b_n) \tag{7.9}$$

$$(k = 1, 2, 3, \ldots, \lambda),$$

$$C_\lambda(M_n + \omega_n, a_n, x) \geqslant 0 \qquad (a_n < x < b_n), \tag{7.10}$$

$$C_\lambda(M_n + \omega_n, b_n, x) \leqslant M_n(b_n) + \omega_n(b_n) \tag{7.11}$$

$$(a_n < x < b_n),$$

the integrals involved being $C_{\lambda-1}$-P integrals.

We first define subsidiary continuous increasing functions $\{u_n(x)\}$, such that $u_n(a_n) = 0$, $\Sigma u_n(b_n)$ is convergent, and

$$(7.12) \quad u_n(b_n) \geqslant |M_n(b_n)| + \sum_{k=1}^{\lambda} (k+1) \left| \frac{k}{(b_n - a_n)^k} \int_{a_n}^{b_n} (b_n - t)^{k-1} M_n(t)\, dt \right|,$$

$$(7.13) \quad u_n(x) \geqslant (\lambda+1) |C_\lambda(M_n, a_n, x)| \qquad (a_n < x < b_n),$$

(7.14) $u_n(x)/(x-a_n)$ is a decreasing function of x for $a_n < x \leqslant b_n$.

The function $u_n(x)$ is similar to a function defined by Bosanquet† in the proof of Theorem IX.

If we take

$$u_n(x) = \left(\frac{x-a_n}{b_n-a_n}\right) K_n + (\lambda+1)(x-a_n) \overline{\text{bound}}_{x \leqslant t \leqslant b_n} \left\{ \frac{1}{t-a_n} \overline{\text{bound}}_{a_n < v \leqslant t} |C_\lambda(M_n, a_n, v)| \right\},$$

where K_n denotes the expression on the right-hand side of (7.12), then $u_n(x)$ is an increasing function of x for $a_n \leqslant x \leqslant b_n$, $u_n(a_n) = 0$ and (7.12), (7.13) and (7.14) are satisfied. It follows from (7.5), (7.6) and Lemma III that

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\lambda} \left| \frac{k}{(b_n - a_n)^k} \int_{a_n}^{b_n} (b_n - t)^{k-1} M_n(t)\, dt \right|$$

is convergent. Hence, in view of (7.5), the continuity of $C_\lambda(M_n, a_n, x)$ for $x = b_n$ and the absolute convergence of $\Sigma M_n(b_n)$, $\Sigma u_n(b_n)$ is convergent.

By a similar method we can define continuous decreasing functions $\{v_n(x)\}$, such that $v_n(b_n) = 0$, $\Sigma v_n(a_n)$ is convergent, and

$$(7.15) \quad v_n(a_n) \geqslant \sum_{k=1}^{\lambda} (k+1) \left| \frac{k}{(a_n - b_n)^k} \int_{b_n}^{a_n} (a_n - t)^{k-1} M_n(t)\, dt \right|,$$

$$(7.16) \quad v_n(x) \geqslant (\lambda+1) |C_\lambda(M_n, b_n, x) - M_n(b_n)| \qquad (a_n < x < b_n),$$

(7.17) $v_n(x)/(b_n-x)$ is an increasing function of x for $a_n \leqslant x < b_n$.

If we write

$$\omega_n(x) = u_n(x) + v_n(a_n) - v_n(x) \quad (a_n \leqslant x \leqslant b_n),$$

then $\omega_n(x)$ has the properties stated.

It is easily seen that $\omega_n(x)$ is continuous and increasing for $a_n \leqslant x \leqslant b_n$, while $\omega_n(a_n) = 0$, $\Sigma \omega_n(b_n)$ is convergent and (7.7) is satisfied.

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† L. S. Bosanquet, 3, 163.

Now suppose that k is any positive integer not greater than λ. Since $\omega_n(x) \geqslant u_n(x)$ whenever $a_n \leqslant x \leqslant b_n$,

$$\frac{k}{(b_n-a_n)^k}\int_{a_n}^{b_n}(b_n-t)^{k-1}\{M_n(t)+\omega_n(t)\}\,dt$$

$$\geqslant \frac{k}{(b_n-a_n)^k}\int_{a_n}^{b_n}(b_n-t)^{k-1}\left\{M_n(t)+\frac{u_n(t)}{t-a_n}(t-a_n)\right\}dt$$

$$\geqslant \frac{k}{(b_n-a_n)^k}\int_{a_n}^{b_n}(b_n-t)^{k-1}M_n(t)\,dt+\frac{ku_n(b_n)}{(b_n-a_n)^{k+1}}\int_{a_n}^{b_n}(b_n-t)^{k-1}(t-a_n)\,dt,$$

by (7.14). Integrating by parts we find that

$$\frac{ku_n(b_n)}{(b_n-a_n)^{k+1}}\int_{a_n}^{b_n}(b_n-t)^{k-1}(t-a_n)\,dt=\frac{u_n(b_n)}{k+1}.$$

It therefore follows from (7.12) that (7.8) is satisfied.

In the same way it can be shown that

$$C_\lambda(M_n+\omega_n, a_n, x) \geqslant C_\lambda(M_n, a_n, x)+u_n(x)/(\lambda+1),$$

whenever $a_n < x < b_n$. It then follows from (7.13) that (7.10) is satisfied.

By a similar method the results (7.9) and (7.11) can be deduced from (7.15), (7.16) and (7.17).

Now let ν be such that

$$\sum_{n=\nu+1}^{\infty}\omega_n(b_n) < \tfrac{1}{2}\epsilon, \tag{7.18}$$

and define

$$G_n(x) = M_n(x) \qquad (a \leqslant x \leqslant b;\ n \leqslant \nu),$$

$$= M_n(x)+\omega_n(x) \quad (a \leqslant x \leqslant b;\ n > \nu).$$

Then $G_n(a_n) = 0$, $\Sigma G_n(b_n)$ is convergent and, in view of (7.7),

$$G_n(b_n) \geqslant 0 \quad (n > \nu). \tag{7.19}$$

We now define

$$\phi(x) = \sum_{b_n \leqslant x} G_n(b_n)+G_q(x),$$

the last term being absent unless x lies in a complementary interval (a_q, b_q), and proceed to prove that $\phi(x)$ is a C_λ-major function for $f_2(x)$ in (a, b).

Since $\Sigma G_n(b_n)$ is convergent, $\phi(x)$ is absolutely continuous over Q. Also, since $\omega_n(x)$ is an increasing function of x for $a_n \leqslant x \leqslant b_n$, it follows from (7.5), (7.6) and the convergence of $\Sigma \omega_n(b_n)$ that

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} | C_\lambda(\phi, a_n, x) - \phi(a_n) | < \infty,$$

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{a_n<x<b_n} | C_\lambda(\phi, b_n, x) - \phi(b_n) | < \infty.$$

The function $\phi(x)$ is therefore AC^* (C_λ-sense) over Q, and hence C_λ-continuous on the right (or left) at any point of Q which is not isolated on the right (or left respectively). Also $\phi(x)$ is clearly C_λ-continuous for $a_n \leqslant x \leqslant b_n$. It therefore follows that $\phi(x)$ is C_λ-continuous for $a \leqslant x \leqslant b$.

It is easily seen that $C_\lambda D_* \phi(x) \geqslant f_2(x)$ almost everywhere in (a_n, b_n), $C_\lambda D_* \phi(x) > -\infty$ except at a denumerable set of points of (a_n, b_n).

Now let x be any point of Q other than $a_1, a_2, a_3, \ldots, a_\nu$; and let η be such that the interval $(x, x+\eta)$ does not contain points of any of the complementary intervals (a_1, b_1), (a_2, b_2), (a_3, b_3), $\ldots$, (a_ν, b_ν).

Take h such that $0 < h \leqslant \eta$. It follows from property D for the $C_{\lambda-1}$-D or equivalent $C_{\lambda-1}$-P integral that

$$C_\lambda(\phi, x, x+h) - \phi(x)$$

$$= \frac{\lambda}{h^\lambda} \int_{Q_h} (x+h-t)^{\lambda-1} \{\phi(t) - \phi(x)\}\, dt + \Sigma' \int_{a_n}^{b_n} (x+h-t)^{\lambda-1} \{\phi(t) - \phi(x)\}\, dt$$

$$+ \int_{a_q}^{x+h} (x+h-t)^{\lambda-1} \{\phi(t) - \phi(x)\}\, dt,$$

where Q_h denotes the common part of Q and the interval $(x, x+h)$, Σ' denotes that the summation refers to complementary intervals (a_n, b_n) for which $x \leqslant a_n < b_n \leqslant x+h$, and the last term is absent unless x lies in a complementary interval (a_q, b_q).

If $x \leqslant t \leqslant x+h$ and t is a point of Q, it follows from (7.19) that $\{\phi(t) - \phi(x)\} \geqslant 0$, so that

$$\frac{\lambda}{h^\lambda} \int_{Q_h} (x+h-t)^{\lambda-1} \{\phi(t) - \phi(x)\} dt \geqslant 0.$$

Also, if $x \leqslant a_n < b_n \leqslant x+h$, $\phi(a_n) \geqslant \phi(x)$ and hence

$$\frac{\lambda}{h^\lambda}\int_{a_n}^{b_n}(x+h-t)^{\lambda-1}\{\phi(t)-\phi(x)\}\,dt$$

$$\geqslant \frac{\lambda}{h^\lambda}\int_{a_n}^{b_n}(x+h-t)^{\lambda-1}\{\phi(t)-\phi(a_n)\}\,dt$$

$$= \frac{\lambda}{h^\lambda}\sum_{k=1}^{\lambda} {}^{\lambda-1}C_{k-1}(x+h-b_n)^{\lambda-k}\int_{a_n}^{b_n}(b_n-t)^{k-1}\{M_n(t)+\omega_n(t)\}\,dt$$

$$\geqslant 0,$$

by (7.8).

Finally, if $x \leqslant a_q < x+h < b_q$,

$$\frac{\lambda}{h^\lambda}\int_{a_q}^{x+h}(x+h-t)^{\lambda-1}\{\phi(t)-\phi(x)\}\,dt \geqslant \frac{\lambda}{h^\lambda}\int_{a_q}^{x+h}(x+h-t)^{\lambda-1}\{\phi(t)-\phi(a_q)\}\,dt$$

$$= \frac{\lambda}{h^\lambda}\int_{a_q}^{x+h}(x+h-t)^{\lambda-1}\{M_q(t)+\omega_q(t)\}\,dt$$

$$\geqslant 0,$$

by (7.10).

It therefore follows that

$$C_\lambda(\phi, x, x+h)-\phi(x) \geqslant 0$$

whenever $0 < h \leqslant \eta$, and hence that

$$C_\lambda D_+\phi(x) \geqslant 0.$$

By a similar method it can be shown that

$$C_\lambda D_-\phi(x) \geqslant 0,$$

provided that x does not coincide with one of the points $b_1, b_2, b_3, \ldots, b_\nu$.

Thus $C_\lambda D_*\phi(x) \geqslant f_2(x)$ almost everywhere in Q, hence almost everywhere in (a, b), and $C_\lambda D_*\phi(x) > -\infty$ except at a denumerable set of points of (a, b). The function $\phi(x)$ is therefore a C_λ-major function for $f_2(x)$ in (a, b).

Since

$$\phi(b) = \sum_{n=1}^{\infty} G_n(b_n)$$

$$= \sum_{n=1}^{\infty} M_n(b_n) + \sum_{n=\nu+1}^{\infty} \omega_n(b_n),$$

it follows from (7.4) and (7.18) that

$$0 \leqslant \phi(b) - \sum_{n=1}^{\infty} F_n(b_n) < \epsilon.$$

By a similar method we can construct a C_λ-minor function $\psi(x)$ for $f_2(x)$ in (a, b), such that

$$0 \leqslant \sum_{n=1}^{\infty} F_n(b_n) - \psi(b) < \epsilon.$$

The function $f_2(x)$ is therefore integrable C_λ-P in (a, b), and

$$(C_\lambda\text{-}P)\int_a^b f_2(t)\,dt = \sum_{n=1}^{\infty} F_n(b_n) = \sum_{n=1}^{\infty} (C_\lambda\text{-}P)\int_{a_n}^{b_n} f_2(t)\,dt,$$

which is the result required.

8. *The C_λ-P integral includes the C_λ-D integral.*

THEOREM XI. *If $f(x)$ is integrable C_λ-D in (a, b), then $f(x)$ is integrable C_λ-P in (a, b), and*

$$(C_\lambda\text{-}P)\int_a^b f(t)\,dt = (C_\lambda\text{-}D)\int_a^b f(t)\,dt.$$

We suppose that $f(x)$ is integrable C_λ-D in (a, b), and write

$$F(x) = (C_\lambda\text{-}D)\int_a^x f(t)\,dt.$$

Let K be the set of points x of $[a, b]$ throughout no neighbourhood of which $F(x)$ is integrable C_λ-P (if x coincides with a or b, instead of neighbourhoods we consider intervals having a or b as left or right-hand end point respectively). It can easily be seen that K is closed. Further, if (a_r, b_r) is any complementary interval of K and $a_r < \alpha_r < \beta_r < b_r$, it can be shown by the method of repeated bisection that $f(x)$ is integrable C_λ-P in (α_r, β_r). It then follows from Theorem VIII that

$$(8.1) \qquad (C_\lambda\text{-}P)\int_{\alpha_r}^{\beta_r} f(t)\,dt = F(\beta_r) - F(\alpha_r).$$

Since the C_λ-D indefinite integral $F(x)$ is a C_λ-continuous function of x, $F(\beta_r)$ and $F(\alpha_r)$ tend to finite limits (C_λ-sense) as β_r, α_r tend to b_r, a_r respectively. It therefore follows from Theorem IX that (8.1) holds whenever $a_r \leqslant \alpha_r < \beta_r \leqslant b_r$. The set K can therefore contain no isolated points, and must be perfect or null.

Suppose if possible that K is not null.

It follows from property A (already proved for the C_λ-D integral) and Baire's theorem that K contains a portion π whose closure has end points c, d (say) and complementary intervals $\{c_n, d_n)\}$, such that

$$\int_\pi |f(t)|\,dt < \infty,$$

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{c_n<x<d_n} |C_\lambda(F, c_n, x) - F(c_n)| < \infty,$$

$$\sum_{n=1}^{\infty} \overline{\text{bound}}_{c_n<x<d_n} |C_\lambda(F, d_n, x) - F(d_n)| < \infty.$$

Hence, by Theorem X, $f(x)$ is integrable C_λ-P in (c, d). This is clearly impossible if c and d are the end points of the closure of a portion of K.

The set K must therefore be null and, in view of Theorem VIII,

$$(C_\lambda\text{-}P)\int_a^b f(t)\,dt = (C_\lambda\text{-}D)\int_a^b f(t)\,dt.$$

(*Added, October* 25, 1941. After the beginning of this paper had been printed, I noticed that the following emendations should be made:

On pp. 226 and 230, in the first lines of Theorems II and III, $F(x)$ should be restricted to be a C_λ-continuous function.

On p. 232, in the proof of Theorem VI, the absolute continuity of $F(x)$ over $[c, d]$ is a consequence of its being continuous and of bounded variation for $c \leqslant x \leqslant d$, and also satisfying the condition N over $[c, d]$.)

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GENERAL MERCERIAN THEOREMS (II)

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1. Suppose that $k(x)$ is of bounded variation in $(-\infty, \infty)$ and that $f(x)$ is bounded and continuous. Let†

$$g(x) = \int f(x-y)\,dk(y). \tag{1.1}$$

In a previous paper‡ (1) I have given conditions under which the equation (1.1) has a solution

$$f(x) = \int g(x-y)\,dp(y), \tag{1.2}$$

where $p(y)$ is of bounded variation. It is then almost trivial that $g(x) \to 0$ as $x \to \infty$ implies $f(x) \to 0$, and we called such results Mercerian theorems. The fundamental condition in such theorems was found to be the boundedness of $[K(\omega)]^{-1}$ in a certain strip $0 \leqslant \sigma \leqslant \sigma_1$ $(\omega = \sigma + it)$, $K(\omega)$ being defined in this strip by

$$K(\omega) = \int e^{-\omega x}\,dk(x). \tag{1.3}$$

In this paper we consider extensions of such theorems to kernels $k(x)$ for which $[K(\omega)]^{-1}$ is not bounded. The cases which we consider are (*a*) when $[K(it)]^{-1}$ is bounded except near a finite number of intervals (a_n, b_n) in which $K(it)$ may vanish, and (*b*) when $\sigma_1 > 0$ and $K(\omega)$ has a finite number of zeros in the strip $0 < \sigma \leqslant \sigma_1$, but $[K(\omega)]^{-1}$ is bounded away from these points. The two cases are essentially different, and we consider them separately. As in (1), we define V^* to be the class of functions of bounded variation in $(-\infty, \infty)$ which have no singular component. Such

† Integrals with unspecified limits are over $(-\infty, \infty)$.

‡ Numbers in heavy type refer to the bibliography at the end.

a function is the sum of an absolutely continuous function and a step function. We use the notation $k(x) \in V^*$ to mean that $k(x)$ belongs to V^*.

The principal result of (1), which we use subsequently, is as follows:

THEOREM A. *Suppose that* $\sigma_2 \geqslant \sigma_1 \geqslant 0$, *that* $k(x) \in V^*$ *and that*

$$K(\omega) = \int e^{-\omega y}\, dk(y)$$

converges absolutely† *for* $0 \leqslant \sigma \leqslant \sigma_2$. *Suppose also that* $[K(\omega)]^{-1}$ *is bounded in* $0 \leqslant \sigma \leqslant \sigma_1$, *and that* $f(x)$ *is continuous and either* (i) $\sigma_2 > \sigma_1 \geqslant 0$ *and* $f(x)e^{-\sigma x}$ *is bounded for each fixed* σ *in* $\sigma_1 < \sigma \leqslant \sigma_2$ *or* (ii) $\sigma_2 = \sigma_1 = 0$ *and* $f(x)$ *is bounded. Suppose also that*

$$g(x) = \int f(x-y)\, dk(y)$$

is bounded. Then

$$f(x) = \int g(x-y)\, dp(y),$$

where $p(y)$ *belongs to* V^* *and depends only on* $k(y)$.

2. We now show how the theorem can be modified when $\sigma_1 > 0$ and $K(\omega)$ has zeros in $0 < \sigma \leqslant \sigma_1$.

THEOREM 1.

HYPOTHESIS.

(I) $$\sigma_2 > \sigma_1 > 0,$$

(II) $$k(y) \in V^*,$$

and $$K(\omega) = \int e^{-\omega x}\, dk(x)$$

converges absolutely for $0 \leqslant \sigma \leqslant \sigma_2$,

(III) $K(\omega)$ *has zeros of order* a_n *at a finite set of points* ω_n *in* $0 < \sigma \leqslant \sigma_1$, *and, except in neighbourhoods of* ω_n, $[K(\omega)]^{-1}$ *is bounded in* $0 \leqslant \sigma \leqslant \sigma_1$,

(IV) $f(x)$ *is continuous and* $f(x)e^{-\sigma x}$ *is bounded for each fixed* σ *in* $\sigma_1 < \sigma \leqslant \sigma_2$,

(V) $g(x) = \int f(x-y)\, dk(y)$ *is bounded.*

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† That is, $\int e^{-\sigma x} |dk(x)| < \infty$.

CONCLUSION.

$$f(x) = \int g(x-y)\,dp(y) + \sum_{n=1}^{N} \sum_{\nu=1}^{a_n} A_{n\nu} e^{\omega_n x} x^{\nu-1},$$

where the $A_{n\nu}$ are constants and $p(x)$ belongs to V^* and depends only on $k(x)$.

This result may be stated in a slightly different form as follows:

If $g(x)$ is bounded and $k(x)$ satisfies the conditions of Theorem 1, *the only continuous solutions $f(x)$ of the equation*

$$g(x) = \int f(x-y)\,dk(y),$$

such that $f(x)e^{-\sigma x}$ is bounded for each fixed σ in $\sigma_1 < \sigma \leqslant \sigma_2$, are given by

$$f(x) = \int g(x-y)\,dp(y) + \sum_{n=1}^{N} \sum_{\nu=1}^{a_n} A_{n\nu} e^{\omega_n x} x^{\nu-1},$$

where the constants $A_{n\nu}$ are arbitrary and $p(y)$ belongs to V^* and depends only on $k(x)$.

THEOREM 2. *If the hypothesis of Theorem* 1 *is satisfied and also*

$$\text{(VI)} \qquad \varlimsup_{T\to\infty} \varlimsup_{x\to\infty} \left| \frac{1}{2T} \int_{x-T}^{x+T} f(y)\, e^{-\omega_n y}\, dy \right| = 0 \quad (n = 1, 2, \ldots, N),$$

then

$$f(x) = \int g(x-y)\,dp(y),$$

where $p(y)$ belongs to V^* and depends only on $k(x)$.

We need two lemmas, both contained in (1).

LEMMA 1. *Suppose that $F(\omega)$ is regular in $\sigma_1 < \sigma < \sigma_2$ and bounded and continuous in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. Let*

$$F(\sigma_1+it) = \int e^{-(\sigma_1+it)} f_1(x)\,dx, \quad F(\sigma_2+it) = \int e^{-(\sigma_2+it)} f_2(x)\,dx,$$

where $e^{-\sigma_1 x} f_1(x)$, $e^{-\sigma_2 x} f_2(x)$ belong to $L(-\infty, \infty)$. Then

$$f_1(x) = f_2(x) = f(x)$$

and

$$F(\omega) = \int e^{-\omega x} f(x)\,dx \quad (\sigma_1 \leqslant \sigma \leqslant \sigma_2).$$

This is substantially the Lemma† used in Theorem 4 of (1). We need only add that the equality of $f_1(x)$, $f_2(x)$ and $f(x)$ follows from the unique-

† The condition that $F(\omega)$ must be bounded was omitted by mistake in the original lemma in (1).

ness of representation of a function as the Fourier transform of an integrable function.

LEMMA 2. *Suppose that* $k(x) \in V^*$, *that*

$$K(\omega) = \int e^{-\omega x}\, dk(x)$$

converges absolutely for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ *and that* $[K(\omega)]^{-1}$ *is bounded in the same region. Then*

$$[K(\omega)]^{-1} = \int e^{-\omega x}\, dp(x) \quad (\sigma_1 \leqslant \sigma \leqslant \sigma_1),$$

where $p(x) \in V^*$ *and the integral is absolutely convergent.*

This is Theorem 4 of (1).

Proof of Theorem 1. We can define $\sigma_0 > 0$ so that the points ω_n all lie in $\sigma_0 < \sigma \leqslant \sigma_1$ and $[K(\omega)]^{-1}$ is bounded in $0 \leqslant \sigma \leqslant \sigma_0$. Then it follows from Lemma 2 that

$$[K(\omega)]^{-1} = \int e^{-\omega x}\, dp(x) \quad (0 \leqslant \sigma \leqslant \sigma_1), \tag{2.1}$$

where $p(x) \in V^*$ and the integral is absolutely convergent in this range. Now

$$|g(x)|\, e^{-\sigma_2 x} \leqslant \int |f(x-y)\, e^{-\sigma_2(x-y)} e^{-\sigma_2 y}\, dk(y)|,$$

so that $g(x)\, e^{-\sigma_2 x}$ is bounded. Since $g(x)$ is bounded, it follows that $g(x)\, e^{-\sigma x}$ is bounded for $0 \leqslant \sigma \leqslant \sigma_2$. Hence we can define

$$G(\omega) = \int e^{-\omega x} g(x)\, dx, \tag{2.2}$$

the integral being absolutely convergent for $0 < \sigma < \sigma_2$. It follows from this and (2.1) that, if $0 < \sigma \leqslant \sigma_0$,

$$G(\omega)\, [K(\omega)]^{-1} = \int e^{-\omega x} f_1(x)\, dx, \tag{2.3}$$

where

$$f_1(x) = \int g(x-y)\, dp(y). \tag{2.4}$$

Also, since $f(x)\, e^{-\sigma x}$ is bounded for $\sigma_1 < \sigma < \sigma_2$, we can write

$$F(\omega) = \int e^{-\omega x} f(x)\, dx, \tag{2.5}$$

and it follows easily from (V) that

$$G(\omega) = F(\omega)\, K(\omega) \quad (\sigma_1 < \sigma < \sigma_2).$$

We define $F(\omega)$ in the whole strip $0<\sigma<\sigma_2$ by this relation. Then $F(\omega)$ is regular in this region except at the points ω_n, and, since $K(\omega)$ has zeros of order a_n at these points, we can define constants $A_{n\nu}$ so that the function

$$Q(\omega)=F(\omega)-\sum_{n=1}^{N}\sum_{\nu=1}^{a_n}A_{n\nu}(\nu-1)!\,(\omega-\omega_n)^{-\nu}$$

is regular in $0<\sigma<\sigma_2$ and bounded in any interior strip. But

$$(\nu-1)!\,(\omega-\omega_n)^{-\nu}=\int_0^\infty e^{-(\omega-\omega_n)x}x^{\nu-1}\,dx \qquad (\sigma>\sigma_n),$$

$$=-\int_{-\infty}^0 e^{-(\omega-\omega_n)x}x^{\nu-1}\,dx \quad (\sigma<\sigma_n).$$

Hence, using (2.3) and (2.5), we have

$$Q(\omega)=\int e^{-\omega x}f(x)\,dx-\int_0^\infty e^{-\omega x}\left\{\sum_{n=1}^{N}\sum_{\nu=1}^{a_n}A_{n\nu}e^{\omega_n x}x^{\nu-1}\right\}dx \qquad (\sigma_1<\sigma<\sigma_2),$$

$$Q(\omega)=\int e^{-\omega x}f_1(x)\,dx+\int_{-\infty}^0 e^{-\omega x}\left\{\sum_{n=1}^{N}\sum_{\nu=1}^{a_n}A_{n\nu}e^{\omega_n x}x^{\nu-1}\right\}dx \quad (0<\sigma<\sigma_0).$$

Therefore, by Lemma 1,

$$f(x)=f_1(x)+\sum_{n=1}^{N}\sum_{\nu=1}^{a_n}A_{n\nu}e^{\omega_n x}x^{\nu-1}, \tag{2.6}$$

which is our conclusion.

Proof of Theorem 2. We have

$$|f_1(x)|\leqslant \underset{-\infty<x<\infty}{\text{u.b.}}\,|g(x)|\int|dp(x)|=C,$$

say, and therefore

$$\left|\frac{1}{2T}\int_{x-T}^{x+T}f_1(y)\,e^{-\omega_m y}\,dy\right|\leqslant\frac{C}{2T}\int_{x-T}^{x+T}e^{-\sigma_m y}=o(1)$$

as $x\to\infty$, since $\sigma_m>0$. Therefore, by (2.6) and (VI)

$$\varlimsup_{x\to\infty}\left|\frac{1}{2T}\int_{x-T}^{x+T}e^{-\omega_m y}\left\{\sum_{n=1}^{N}\sum_{\nu=1}^{a_n}A_{n\nu}e^{\omega_n y}y^{\nu-1}\right\}dy\right|=\eta(T), \tag{2.7}$$

where

$$\varlimsup_{T\to\infty}\eta(T)=0. \tag{2.8}$$

Now if σ' is the maximum of σ_n for $n = 1, 2, \ldots, N$, and $\omega_m = \sigma' + it_m$ in (2.7), the terms with $\sigma_n < \sigma'$ all vanish when $x \to \infty$, and we can write

$$\lim_{x\to\infty} \left| \frac{1}{2T} \int_{x-T}^{x+T} \left\{ \sum_n \sum_{\nu=1}^{a_n} A_{n\nu} e^{i(t_n - t_m)y} y^{\nu-1} \right\} dy \right| = \eta(T). \tag{2.9}$$

Now if $\alpha \neq 0$,

$$\begin{aligned} \frac{1}{2T} \int_{x-T}^{x+T} y^{\nu-1} e^{i\alpha y}\, dy &= \frac{x^{\nu-1} e^{i\alpha x}}{2T} \int_{-T}^{T} \left(1 + \frac{y}{x}\right)^{\nu-1} e^{i\alpha y}\, dy \\ &= x^{\nu-1} e^{i\alpha x} \left[1 + O\left(\frac{1}{x}\right)\right] \frac{1}{2T} \int_{-T}^{T} e^{i\alpha y}\, dy \\ &= x^{\nu-1} e^{i\alpha x} \frac{\sin T\alpha}{T\alpha} + O(x^{\nu-2}). \end{aligned} \tag{2.10}$$

Now let u be the maximum of a_n for values of n for which $\sigma_n = \sigma'$. Then unless all the $A_{n\nu}$ vanish, we can choose m so that

$$A_{mu} \neq 0. \tag{2.11}$$

Then (2.9) and (2.10) give

$$\lim_{x\to\infty} x^{u-1} \left| A_{mu} + \sum_n A_{nu} \frac{\sin (t_n - t_m) T}{|t_n - t_m|\, T} e^{i(t_n - t_m)x} \right| = \eta(T).$$

But the trigonometrical polynomial tends to 0 as $x \to \infty$ only if all its coefficients vanish. Therefore, since $\eta(T)$ is finite for sufficiently large T, we must have $u = 1$, so that

$$|A_{mu}| \leqslant \eta(T) + \sum_n \frac{|A_{nu}|}{|t_n - t_m|\, T},$$

and if we let $T \to \infty$, and use (2.8), we get $A_{mu} = 0$. This contradicts (2.11). Therefore all the $A_{n\nu}$ vanish and (2.6) reduces to

$$f(x) = \int g(x-y)\, dp(y).$$

3. Before considering kernels $k(x)$ for which $K(it)$ may vanish in intervals, we prove some results on the harmonic representation of functions. Except for the trivial change of sign of i, we use the notation of (2). There we defined $\mathfrak{B}_p$ $(1 \leqslant p \leqslant 2)$ to be the class of functions $f(x)$ for which

$$\sum_{n=-\infty}^{\infty} F_n{}^p (|n| + 1)^{-p} < \infty, \quad F_n = \int_n^{n+1} |f(x)|\, dx, \tag{3.1}$$

and $\mathfrak{B}$ to be the sum of the classes $\mathfrak{B}_p$ for $1 \leqslant p \leqslant 2$. In our application in this paper the function $f(x)$ is always bounded and so belongs to $\mathfrak{B}_p$ for any $p > 1$. It was shown† that, if $f(x) \epsilon \mathfrak{B}_p$, then

$$s_l(t) = \frac{i}{2\pi} \int_{-l}^{l} \frac{f(x)}{x} [e^{-ixt} - e^{-x^2}] dx \tag{3.2}$$

converges in mean of order‡ p' over any finite range of values of t to a function $s(t)$. This relation between $f(x)$ and $s(t)$ is expressed by

$$f(x) \sim \int e^{itx} ds(t). \tag{3.3}$$

LEMMA 3. *Suppose that $\Gamma(t)$ has a bounded third derivative and vanishes outside a finite interval. Then we can write*

$$\Gamma(t) = \int e^{-itx} \gamma(x) dx, \quad \gamma(x) = \frac{1}{2\pi} \int e^{itx} \Gamma(t) dt,$$

where $\gamma(x)$ and $x\gamma(x)$ both belong to L.

For, if we define $\gamma(x)$ by the second equation, we can integrate by parts three times and obtain

$$\gamma(x) = \frac{i}{2\pi x^3} \int e^{itx} \Gamma'''(t) dt,$$

which is sufficient for our conclusion, since $\gamma(x)$ is plainly bounded.

LEMMA 4. *If $1 \leqslant p \leqslant 2$ and $\phi(x) \epsilon L_p$, then*

$$\Phi_l(t) = \int_{-l}^{l} e^{-ixt} \phi(x) dx$$

converges in mean of order p' to a function $\Phi(t)$ of $L_{p'}$, and

$$\int |\Phi(t)|^{p'} dt \leqslant 2\pi \left[\int |\phi(x)|^p dx \right]^{p'-1}.$$

This is a well-known theorem due to Titchmarsh (4).

If $f(x)$ is bounded and $\Gamma'(t)$ is bounded we write

$$f(x) \sim \int e^{itx} \Gamma(t) ds(t)$$

if

$$f(x) \sim \int e^{itx} ds^*(t)$$

---

† (2), Lemma 1.

‡ $1/p + 1/p' = 1$, $p' = p/(p-1)$. When $p = 1$, convergence in mean of order p' is understood to mean that the essential upper bound of $s(t) - s_l(t)$ tends to 0.

and

$$(3.4) \qquad s^*(t)-s^*(u) = \Gamma(t)\,s(t)-\Gamma(u)\,s(u)-\int_u^t \Gamma'(\tau)\,s(\tau)\,d\tau$$

for almost all values of u and t.

THEOREM 3. *Suppose that $\Gamma(t)$ has a bounded third derivative and vanishes outside a finite set of intervals I_n, and that $\gamma(x)$ is defined by Lemma 3. Suppose that $f(x)$ is bounded and that*

$$f(x) \sim \int e^{itx}\,ds(t), \quad f^*(x) = \int f(x-y)\,\gamma(y)\,dy.$$

Then

$$f^*(x) = -\int e^{itx}[\Gamma'(t)+ix\,\Gamma(t)]\,s(t)\,dt \sim \int e^{itx}\,\Gamma(t)\,ds(t).$$

By (3.2), we have

$$(3.5) \qquad s(t) \equiv \underset{l\to\infty}{\text{l.i.m.}} \frac{i}{2\pi}\int_{-l}^{l} \frac{f(y)}{y}\,[e^{-iyt}-e^{-y}]\,dy.$$

Hence, using the fact that integrals of mean convergent functions converge, we have

$$(3.6) \qquad \int e^{itx}\,[\Gamma'(t)+ix\,\Gamma(t)]\,s(t)\,dt$$

$$= \frac{i}{2\pi}\int \frac{f(y)}{y}\,dy \int [\Gamma'(t)+ix\,\Gamma(t)]\,e^{itx}[e^{-iyt}-e^{-y}]\,dt$$

$$= \frac{i}{2\pi}\int \frac{f(y)}{y}\,dy \int [\Gamma'(t)+ix\,\Gamma(t)]\,e^{it(x-y)}\,dt,$$

since $\Gamma(t)$ vanishes outside a finite interval and so

$$\int [\Gamma'(t)+ix\,\Gamma(t)]\,e^{itx}\,dt = 0.$$

But

$$\gamma(x) = \frac{1}{2\pi}\int e^{itx}\,\Gamma(t)\,dt = \frac{i}{2\pi x}\int e^{itx}\,\Gamma'(t)\,dt,$$

and therefore

$$\frac{i}{2\pi}\int [\Gamma'(t)+ix\,\Gamma(t)]\,e^{it(x-y)}\,dt = (x-y)\,\gamma(x-y) - x\gamma(x-y)$$

$$= -y\gamma(x-y).$$

It follows now from (3.6) that

$$\int e^{itx}[\Gamma'(t)+ix\,\Gamma(t)]\,s(t)\,dt = -\int f(y)\,\gamma(x-y)\,dy = -f^*(x)$$

Next, since $f^*(x)$ is bounded, we can write

$$(3.7)\quad f^*(x) \sim \int e^{ilx}\,ds^*(x), \quad s^*(t) \equiv \underset{l\to\infty}{\text{l.i.m.}}\, \frac{i}{2\pi}\int_{-l}^{l} \frac{f^*(x)}{x}\,[e^{-ixl}-e^{-x^2}]\,dx.$$

It follows easily from (3.5) that

$$\Gamma(t)\,s(t) \equiv \int e^{-iyt}\gamma(y)\,dy\ \underset{l\to\infty}{\text{l.i.m.}}\, \frac{i}{2\pi}\int_{-l}^{l}\frac{f(x)}{x}\,[e^{-ixt}-e^{-x^2}]\,dx$$

$$\equiv \frac{i}{2\pi}\int \gamma(y)\,dy\ \underset{l\to\infty}{\text{l.i.m.}}\, \int_{-l}^{l}\frac{f(x)}{x}\,[e^{-i(x+y)t}-e^{-x^2-iyt}]\,dx.$$

Therefore, if c is any real number,

$$(3.8)\quad \Gamma(t+c)\,s(t+c)-\Gamma(t)\,s(t)$$

$$\equiv \frac{i}{2\pi}\int \gamma(y)\,dy\ \underset{l\to\infty}{\text{l.i.m.}}\, \int_{-l}^{l}\frac{f(x)}{x}\,[e^{-i(x+y)(t+c)}-e^{-i(x+y)t}$$

$$+e^{-x^2-iyt}-e^{-x^2-iy(t+c)}]\,dx.$$

Also, since $\Gamma'(t) = -i\int e^{-ity}\,y\gamma(y)\,dy,$

and $y\gamma(y) \in L$, we have

$$(3.9)\quad \int_t^{t+c}\Gamma'(\tau)\,s(\tau)\,d\tau$$

$$= -i\int_t^{t+c} s(\tau)\,d\tau \int e^{-i\tau y}\,y\gamma(y)\,dy$$

$$= -i\int y\gamma(y)\,dy\int_t^{t+c} e^{-i\tau y}\,s(\tau)\,d\tau$$

$$= \frac{1}{2\pi}\int y\gamma(y)\,dy\int_t^{t+c} e^{-i\tau y}\,d\tau\ \underset{l\to\infty}{\text{l.i.m.}}\, \int_{-l}^{l}\frac{f(x)}{x}\,[e^{-ix\tau}-e^{-x^2}]\,dx$$

$$= \frac{1}{2\pi}\int y\gamma(y)\,dy\ \underset{l\to\infty}{\text{l.i.m.}}\, \int_{-l}^{l}\frac{f(x)}{x}\,dx\int_t^{t+c} e^{-i\tau y}[e^{-ix\tau}-e^{-x^2}]\,d\tau,$$

since integrals of mean convergent functions also converge in mean. Now it is easy to verify that

$$e^{-i(x+y)(t+c)}-e^{-i(x+y)t}+e^{-x^2-iyt}-e^{-x^2-iy(t+c)}+iy\int_t^{t+c} e^{-i\tau y}(e^{-i\tau x}-e^{-x^2})\,d\tau$$

$$= \frac{x}{x+y}\,[e^{-i(t+c)(x+y)}-e^{-it(x+y)}],$$

and it follows from (3.8) and (3.9) that

$$\Gamma(t+c)\,s(t+c)-\Gamma(t)\,s(t)-\int_t^{t+c}\Gamma'(\tau)\,s(\tau)\,d\tau$$

$$\equiv\frac{i}{2\pi}\int\gamma(y)\,dy\ \underset{l\to\infty}{\text{l.i.m.}}\int_{-l}^{l}\frac{f(x)}{x+y}\,[e^{-i(t+c)(x+y)}-e^{-it(x+y)}]\,dx$$

$$\equiv\frac{i}{2\pi}\int\gamma(y)\,dy\ \underset{l\to\infty}{\text{l.i.m.}}\int_{-l}^{l}\frac{f(x-y)}{x}\,[e^{-i(t+c)x}-e^{-itx}]\,dx,$$

by Minkowski's inequality, since $\gamma(y)\in L$, $f(x)$ is bounded, and it is easy to show that the first limit in mean exists uniformly in y. Hence the expression on the left is equal almost everywhere to

$$\underset{l\to\infty}{\text{l.i.m.}}\frac{i}{2\pi}\int_{-l}^{l}\frac{e^{-i(t+c)x}-e^{-itx}}{x}\,dx\int f(x-y)\,\gamma(y)\,dy$$

$$\equiv\underset{l\to\infty}{\text{l.i.m.}}\frac{i}{2\pi}\int_{-l}^{l}\frac{f^*(x)}{x}\,[e^{i(t+c)x}-e^{-itx}]\,dx\equiv s^*(t+c)-s^*(t).$$

Since this is true for all values of c and almost all values of t, we have (3.4) for almost all values of u and t.

Next we prove

THEOREM 4. *Suppose that $1\leqslant p\leqslant 2$, that $f(x)\in\mathfrak{B}_p$ and that $f(x)\,e^{-\sigma x}$ is bounded for any fixed σ in $0<\sigma\leqslant\sigma_2$. Let*

$$F(\omega)=\int e^{-\omega x}f(x)\,dx,\quad f(x)\sim\int e^{itx}\,ds(t).$$

Then, if ω_0 is any point in the strip $0<\sigma<\sigma_2$ and $\sigma\to 0$,

$$F_1(\omega)=\int_{\omega_0}^{\omega}F(z)\,dz$$

converges in mean of order p' over any finite interval to the value $2\pi i s(t)+C$, C being a constant.

We have

$$F_1(\omega)=\int_{\omega_0}^{\omega}F(z)\,dz=\int_{\omega_0}^{\omega}dz\int e^{-zx}f(x)\,dx$$

$$=-\int\frac{f(x)}{x}\,[e^{-\omega x}-e^{-\omega_0 x}]\,dx=-\int\frac{f(x)\,e^{-\sigma x}}{x}\,[e^{-itx}-e^{-x^2+\sigma x}]\,dx+C(\sigma),$$

where $$C(\sigma)=-\int\frac{f(x)\,e^{-\sigma x}}{x}\,[e^{-x^2+\sigma x}-e^{-\omega_0 x}]\,dx.$$

If we define $s_l(t)$ by (3.2), we have

$$s(t) \equiv \underset{l\to\infty}{\text{l.i.m.}}\, s_l(t) \tag{3.10}$$

and

$$2\pi i s_l(t) - F_1(\sigma+it) \tag{3.11}$$

$$\equiv \int_{-l}^{l} \frac{f(x)}{x}\,[e^{-\sigma x}-1]\,e^{-ixt}\,dx$$

$$+\int_{-\infty}^{-l}+\int_{l}^{\infty}\frac{f(x)}{x}\,e^{-\sigma x-itx}\,dx-\int_{-\infty}^{-l}+\int_{l}^{\infty}\frac{f(x)}{x}\,e^{-x^2}\,dx-C(\sigma).$$

Since $f(x)\,e^{-\sigma x}\,\epsilon\, L$, the second term on the right converges in mean to 0 as $l\to\infty$, and the third term converges to 0 uniformly and therefore also in mean. Therefore, letting $l\to\infty$, we have

$$2\pi i s(t) - F_1(\sigma+it) \equiv P(\sigma,t) - C(\sigma), \tag{3.12}$$

where

$$P(\sigma,t) \equiv \underset{l\to\infty}{\text{l.i.m.}} \int_{-l}^{l} \frac{f(x)}{x}\,[e^{-\sigma x}-1]\,e^{-ixt}\,dx.$$

Now take $A>0$, and write

$$\phi(\sigma,x) = \frac{A}{2}\int_{x-A^{-1}}^{x+A^{-1}} \frac{f(u)}{u}\,[e^{-\sigma u}-1]\,du. \tag{3.13}$$

Then a straightforward calculation shows that

$$\frac{A}{t}\sin\frac{t}{A}\,P(\sigma,t) \equiv \underset{l\to\infty}{\text{l.i.m.}} \int_{-l}^{l}\phi(\sigma,x)\,e^{-ixt}\,dx. \tag{3.14}$$

Now, since $1-e^{-\sigma u}\leqslant \sigma u$ for $u\geqslant 0$, we have

$$1-e^{-\sigma u} \leqslant u\,\min\,[\sigma,\,1/u] \quad (u\geqslant 0),$$

and, if we suppose that $\sigma\leqslant 1$, then

$$\frac{1-e^{-\sigma u}}{u} \leqslant \frac{2}{1+u} \quad (u\geqslant 0).$$

Now let M be the upper bound of $|f(x)|\,e^{-\sigma_2 x}$. Then, if $\sigma\leqslant \frac{1}{2}\sigma_2$, $\sigma\leqslant 1$, and $u\leqslant 0$, we have

$$\left|\frac{f(u)\,[e^{-\sigma u}-1]}{u}\right| \leqslant M e^{\frac{1}{2}\sigma_2 u}\,\frac{(1-e^{\sigma u})}{-u} \leqslant 2M e^{\frac{1}{2}\sigma_2 u}.$$

Hence

$$|\phi(\sigma,x)| \leqslant A\int_{x-A^{-1}}^{x+A^{-1}}\left\{\frac{|f(u)|}{|u|+1}+Me^{-\frac{1}{2}\sigma_2|u|}\right\}du.$$

It follows easily from (3.1) that the function on the right belongs to L_p. Therefore, since $\phi(\sigma, x)$ is bounded by a function of L_p which is independent of σ for $\sigma \leqslant \min(\frac{1}{2}\sigma_2, 1)$, and since it is plain from (3.13) that

$$\lim_{\sigma \to 0} \phi(\sigma, x) = 0$$

for each fixed x, it follows that

$$\lim_{\sigma \to 0} \int |\phi(\sigma, x)|^p \, dx = 0. \tag{3.15}$$

Therefore, by (3.14) and Lemma 4,

$$\lim_{\sigma \to 0} \int \left| \frac{A}{t} \sin \frac{t}{A} P(\sigma, t) \right|^{p'} dt = 0.$$

Since A is an arbitrary positive number and

$$\frac{A}{t} \sin \frac{t}{A} \geqslant \tfrac{1}{2} \quad \text{for} \quad |t| \leqslant A,$$

it follows that over any finite interval

$$\text{l.i.m.}_{\sigma \to 0} \, P(\sigma, t) = 0. \tag{3.16}$$

Since

$$\lim_{\sigma \to 0} C(\sigma) = \lim_{\sigma \to 0} \int \frac{f(x)\, e^{-\sigma x}}{x} [e^{-\omega_0 x} - e^{-x^2 + \sigma x}] \, dx$$

$$= \int \frac{f(x)}{x} [e^{-\omega_0 x} - e^{-x^2}] \, dx = C,$$

say, our conclusion follows from (3.12) and (3.16).

4. We can now go on to consider the extensions of Theorem A in which $K(it)$ may vanish throughout a finite set of intervals (a_n, b_n).

To avoid repetition, we suppose that the intervals (a_n, b_n) are the same throughout this section. We suppose that the function $\Gamma_\epsilon(t)$ is defined for small positive values of ϵ to have the following properties:

$$\begin{cases} \Gamma_\epsilon'''(t) & \text{exists and is bounded,} \\ \Gamma_\epsilon(t) = 1 & (a_n - \frac{1}{2}\epsilon \leqslant t \leqslant b_n + \frac{1}{2}\epsilon, \ n = 1, 2, \ldots, N), \\ \Gamma_\epsilon(t) = 0 & \text{outside the intervals } (a_n - \epsilon, \ b_n + \epsilon). \end{cases} \tag{4.1}$$

Then it follows from Lemma 3 that

$$\Gamma_\epsilon(t) = \int e^{-itx} \gamma_\epsilon(x) \, dx, \tag{4.2}$$

where $\gamma_\epsilon(x) \in L$. We shall omit the ϵ from $\Gamma_\epsilon(t)$ and $\gamma_\epsilon(x)$ where there is no ambiguity.

Our next result is an extension of Theorem A (ii).

THEOREM 5.

HYPOTHESIS.

(I) $k(x) \in V^*$.

(II) $[K(it)]^{-1}$ *is bounded outside the finite set of intervals*

$$(a_n-\epsilon,\ b_n+\epsilon)$$

for any fixed positive ϵ.

(III) $f(x)$ *is bounded and continuous, and*

$$g(x)=\int f(x-y)\,dk(y).$$

(IV) $f(x)\sim\int e^{itx}\,ds(t)$.

CONCLUSION.

If $\epsilon>0$ *and* $\Gamma(t)$ *satisfies* (4.1), *then*

$$f(x)=\int g(x-y)\,dp(y)+f^*(x),$$

where $p(y)\in V^*$ *and*

$$f^*(x)\sim\int e^{itx}\,\Gamma(t)\,ds(t).$$

From this we deduce

THEOREM 6. (*a*) *Suppose that the hypothesis of Theorem* 5 *is satisfied and that for a system of functions* $\Gamma_\epsilon(t)$ *satisfying* (4.1) *we have*

$$\text{(V)}\qquad \varlimsup_{\epsilon\to 0}\ \overline{\lim_{y\to\infty}}\left|y\int e^{ity}\,s(t)\,\Gamma_\epsilon(t)\,dt\right|=0.$$

Let

$$G=\overline{\lim_{x\to\infty}}\,|g(x)|=0.$$

Then

$$F=\overline{\lim_{x\to\infty}}\,|f(x)|=0.$$

(*b*) *If instead of* (V) *we have*

$$\text{(VI)}\qquad \lim_{y\to\infty} y\int e^{ity}\,s(t)\,\Gamma_\epsilon(t)\,dt=0,$$

for some positive ϵ, *then*

$$f(x) = \int g(x-y)\,dp(y) + o(1) \quad (x \to \infty),$$

where $p(y) \in V^*$, *and* $F \leqslant C\,G$, *where* C *depends only on* ϵ *and* $k(x)$.

COROLLARY. *The conclusion of* (*b*) *holds if the conditions of Theorem* 5 *are satisfied and* $s(t)$ *is absolutely continuous* (*and*, a fortiori, *if it is constant*) *in each interval* $(a_n-\epsilon,\ b_n+\epsilon)$.

We need two lemmas on Fourier-Stieltjes integrals.

LEMMA 5. *Suppose that* $k(x) \in V^*$ *and that*

$$K(ia_n) \neq 0, \quad K(ib_n) \neq 0 \quad (n = 1, 2, \ldots, N).$$

Then we can define a function $h(x)$ *of* V^* *so that*

$$K(it) = H(t) = \int e^{-ixt}\,dh(x)$$

outside the intervals (a_n, b_n) *and* $H(t) \neq 0$ *inside* (a_n, b_n).

It is plainly sufficient to consider the case of a single interval (a, b). First we define a continuous function $H_1(t)$ which is equal to $K(it)$ outside (a, b) and does not vanish in the closed interval $a \leqslant t \leqslant b$. Then $|H_1(t)|$ has a positive lower bound δ in $a \leqslant t \leqslant b$, and we can define a function $P(t)$ with bounded second derivative which vanishes outside (a, b) and satisfies

$$|P(t)-H_1(t)+K(it)| < \tfrac{1}{2}\delta \quad (a \leqslant t \leqslant b).$$

Let $H(t) = K(it)+P(t)$. Then since $P(t) = 0$ outside (a, b), $H(t) = K(it)$. Moreover, for $a \leqslant t \leqslant b$,

$$\begin{aligned} |H(t)| &= |H_1(t)+[P(t)-H_1(t)+K(it)]| \\ &\geqslant \text{l.b.}\,|H_1(t)| - \text{u.b.}\,|P(t)-H_1(t)+K(it)| \geqslant \tfrac{1}{2}\delta. \end{aligned}$$

Finally, since $P(t)$ vanishes outside (a, b) and has a bounded second derivative, it is the Fourier transform of a function of L, as in Lemma 3, and the conclusion follows.

LEMMA 6. *Suppose that* $k(x) \in V^*$ *and that* $[K(it)]^{-1}$ *is bounded outside a finite set of intervals* $(a_n-\epsilon,\ b_n+\epsilon)$, *for any positive* ϵ. *Then, if* $\epsilon > 0$ *and* $\Gamma_\epsilon(t)$ *satisfies* (4.1),

$$[1-\Gamma_\epsilon(t)]\,[K(it)]^{-1} = \int e^{-itx}\,dp(x),$$

where $p(x) \in V^*$.

By Lemma 5, we can define $h(x)$ of V^* so that $H(t) = K(it)$ outside $(a_n - \frac{1}{2}\epsilon,\ b_n + \frac{1}{2}\epsilon)$ and $[H(t)]^{-1}$ is bounded. Then, since $\Gamma_\epsilon(t) = 1$ for $a_n - \frac{1}{2}\epsilon \leqslant t \leqslant b_n + \frac{1}{2}\epsilon$, we have

$$\frac{1-\Gamma_\epsilon(t)}{K(it)} = \frac{1-\Gamma_\epsilon(t)}{H(t)},$$

and the conclusion follows from Lemma 2 and straightforward calculation.

Proof of Theorem 5. If we define $\gamma(x)$ by Lemma 3, we have, by Theorem 3,

$$f^*(x) = \int \gamma(x-y) f(y)\, dy \sim \int e^{itx}\, \Gamma(t)\, ds(t). \tag{4.3}$$

By Lemma 6, we can write

$$[1-\Gamma(t)][K(it)]^{-1} = \int e^{-itx}\, dp(x),$$

where $p(x) \in V^*$, and therefore

$$1-\Gamma(t) = \int e^{-itx}\, dp(x) \int e^{-ity}\, dk(y) = \int e^{-itx}\, d\Big[\int p(x-y)\, dk(y)\Big].$$

But

$$1 = \int e^{-itx}\, d\lambda(x),$$

where $\lambda(x) = 0$ for $x < 1$, $\lambda(x) = 1$ for $x > 1$. Therefore, by the uniqueness theorem for Fourier-Stieltjes integrals,

$$\lambda(y) - \int_{-\infty}^{y} \gamma(u)\, du - \int p(y-u)\, dk(u)$$

is constant. Hence, using (4.3), we have

$$\begin{aligned} f(x) - f^*(x) &= \int f(x-y)\, d\lambda(y) - \int f(x-y)\, \gamma(y)\, dy \\ &= \int f(x-y)\, d\Big[\lambda(y) - \int_{-\infty}^{y} \gamma(u)\, du\Big] = \int f(x-y)\, d\Big[\int p(y-u)\, dk(u)\Big] \\ &= \int dp(y) \int f(x-y-u)\, dk(u) = \int dp(y) \int f(x-u-y)\, dk(u) \\ &= \int g(x-y)\, dp(y), \end{aligned}$$

which is what we require.

Theorem 6 (a) follows almost immediately from this. For, if $G = 0$, it is plain that

$$\lim_{x\to\infty} \int g(x-y)\, dp(y) = 0$$

for each positive ϵ, and therefore

$$F = \overline{\lim}\, |f^*(x)|. \tag{4.4}$$

But, by Theorem 3,

$$f^*(x) = \int e^{itx}[\Gamma_\epsilon'(t) + ix\Gamma_\epsilon(t)]\, s(t)\, dt$$

$$= \int e^{itx}\, \Gamma_\epsilon'(t)\, s(t)\, dt + ix \int e^{itx}\, \Gamma_\epsilon(t)\, s(t)\, dt.$$

Hence

$$F \leqslant \overline{\lim_{x\to\infty}} \left| \int e^{itx}\, \Gamma_\epsilon'(t)\, s(t)\, dt \right| + \overline{\lim_{x\to\infty}} \left| x \int e^{itx}\, \Gamma_\epsilon(t)\, s(t)\, dt \right|.$$

The first term on the right vanishes, by the Riemann-Lebesgue theorem, and the second is arbitrarily small by choice of ϵ. Hence $F = 0$.

In case (b) it is plain that, for some positive ϵ, $f^*(x) \to 0$ and

$$f(x) = \int g(x-y)\, dp(y) + o(1)$$

as $x \to \infty$, and therefore

$$F \leqslant G \int |dp(y)| = CG.$$

The same conclusion follows from Theorem 5 if $s(t)$ is absolutely continuous, because

$$f^*(x) = \int e^{itx}\, \Gamma_\epsilon(t)\, s'(t)\, dt = o(1)$$

as $x \to \infty$, by the Riemann-Lebesgue theorem.

5. Suppose that the conditions of Theorem 1 are satisfied, except that $[K(\omega)]^{-1}$ may be unbounded as $\sigma \to 0$. Then we can conclude that

$$f(x) = f^*(x) + \sum_{n=1}^{N} \sum_{\nu=1}^{a_n} A_{n\nu}\, e^{\omega_n x}\, x^{\nu-1}, \tag{5.1}$$

where $f^*(x)\, e^{-\sigma x}$ is bounded for any fixed value of σ in an interval $0 < \sigma \leqslant \sigma_0$. Also, since $K(\omega)$ has zeros of order a_n at ω_n,

$$K^{(q)}(\omega_n) = \left[\frac{d^q}{d\omega^q} K(\omega) \right]_{\omega=\omega_n} = 0 \quad (q = 1, 2, \ldots, a_n - 1).$$

Therefore

$$\int \left\{ \sum_{n=1}^{N} \sum_{\nu=1}^{a_n} A_{n\nu} e^{\omega_n(x-y)}(x-y)^{\nu-1} \right\} dk(y) = \sum_{n=1}^{N} \sum_{\nu=1}^{a_n} A_{n\nu} e^{\omega_n x} \int e^{-\omega_n y}(x-y)^{\nu-1} dk(y),$$

which is zero because it can be expressed as a sum of terms with coefficients $K^{(q)}(\omega_n)$ which all vanish. It follows then from (5.1) that

$$g(x) = \int f^*(x-y)\, dk(y). \tag{5.2}$$

The next theorem enables us to extend Theorem 1 to the case in which $K(\omega)$ has zeros in intervals on the line $\sigma = 0$ as well as isolated zeros in the strip $0 < \sigma \leqslant \sigma_1$. In virtue of (5.2), there is no loss of generality in considering only the function $f^*(x)$ (omitting the asterisk), which is $O(e^{\sigma x})$ for any fixed σ in the range $0 < \sigma \leqslant \sigma_1$, σ_1 being chosen so that none of the zeros of $K(\omega)$ lies in $0 < \sigma \leqslant \sigma_1$.

THEOREM 7.

HYPOTHESIS

(I) $\sigma_1 > 0$.

(II) $k(y) \in V^*$ *and* $K(\omega) = \int e^{-\omega y}\, dk(y)$ *converges absolutely for*

$$0 \leqslant \sigma \leqslant \sigma_1.$$

(III) $[K(\omega)]^{-1}$ *is bounded in* $0 \leqslant \sigma \leqslant \sigma_1$ *except near a finite set of intervals* $\sigma = 0$, $a_n \leqslant t \leqslant b_n$.

(IV) $f(x)$ *is continuous and* $f(x)e^{-\sigma x}$ *is bounded for any fixed* σ *in* $0 < \sigma < \sigma_1$.

(V) $g(x) = \int f(x-y)\, dk(y)$ *is bounded.*

(VI) $\Gamma(t)$ *satisfies* (4.1) *for some* $\epsilon > 0$,

and if

$$F(\omega) = \int e^{-\omega x} f(x)\, dx, \quad F_1(\omega) = \int_{\omega_0}^{\omega} F(z)\, dz \quad (\omega_0 \text{ in } 0 < \sigma < \sigma_1),$$

and $\sigma \to 0$, *then* $F_1(\sigma + it)$ *converges in mean of order* 1 *over each interval* $a_n - \epsilon \leqslant t \leqslant b_n + \epsilon$ *to a function* $F_1(it)$ *for which*

$$x \int e^{ixt} \Gamma(t) F_1(it)\, dt$$

is bounded.

CONCLUSION

$f(x)$ is bounded and

$$f(x) = \int g(x-y)\,dp(y) + f^*(x),$$

where $p(x) \in V^*$ and

$$f^*(x) \sim \frac{1}{2\pi i} \int e^{itx}\,\Gamma(t)\,dF_1(it).$$

We need certain elementary lemmas on Fourier-Stieltjes transforms. We write $F(t) \in \mathfrak{A}$ if

$$F(t) = \int e^{-iyt}\,df(y), \quad f(y) \in V^*,$$

and define

$$T\{F\} = \int |df(y)|.$$

The first two results are Lemmas 1 and 2 of (3).

LEMMA 7. *If $F_1(x)$, $F_2(x)$ belong to $\mathfrak{A}$, so do $c_1 F_1(x) + c_2 F_2(x)$ and $F_1(x)\,F_2(x)$, c_1 and c_2 being any real or complex numbers. Moreover,*

(a) $T\{c_1 F_1(x) + c_2 F_2(x)\} \leqslant |c_1|\,T\{F_1\} + |c_2|\,T\{F_2\}$,

(b) $T\{F_1 . F_2\} \leqslant T\{F_1\} . T\{F_2\}$.

LEMMA 8. *If $F(x) \in \mathfrak{A}$ and $T\{F\} = a < 1$, then $[1+F(x)]^{-1} \in \mathfrak{A}$ and*

$$T\{[1+F(x)]^{-1}\} \leqslant (1-a)^{-1}.$$

LEMMA 9. *Suppose that $k(x) \in V^*$ and that $[K(\omega)]^{-1}$ is bounded in $0 \leqslant \sigma \leqslant \sigma_1$, except near the finite set of intervals $\sigma = 0$, $a_n \leqslant t \leqslant b_n$. Then, if $0 \leqslant \sigma \leqslant \sigma_1$ and $\Gamma(t)$ satisfies (4.1),*

$$[1-\Gamma(t)][K(\sigma+it)]^{-1} = \int e^{-itx}\,dp(\sigma, x),$$

where $p(\sigma, x) \in V^*$ and $\int |dp(\sigma, x)|$ is bounded in $0 \leqslant \sigma \leqslant \sigma_1$.

Since $[K(it)]^{-1}$ is bounded outside $(a_n - \frac{1}{2}\epsilon,\ b_n + \frac{1}{2}\epsilon)$, it follows from Lemma 5 that we can define a function $h(x)$ of V^* so that

$$H(t) = \int e^{-ixt} h(x)\,dx = K(it) \quad \text{outside} \quad (a_n - \tfrac{1}{2}\epsilon,\ b_n + \tfrac{1}{2}\epsilon)$$

and $[H(t)]^{-1}$ is bounded. Then since $1-\Gamma(t) = 0$ for $a_n - \frac{1}{2}\epsilon \leqslant t \leqslant b_n + \frac{1}{2}\epsilon$, we have

$$(5.3) \quad \frac{1-\Gamma(t)}{K(\omega)} = \frac{1-\Gamma(t)}{H(t) + [K(\omega) - K(it)]} = \frac{1-\Gamma(t)}{H(t)} \left\{1 + \frac{K(\omega) - K(it)}{H(t)}\right\}^{-1}.$$

We know from Lemmas 2 and 6 that $[1-\Gamma(t)]\,[K(\omega)]^{-1} \in \mathfrak{A}$ for $0 \leqslant \sigma \leqslant \sigma_1$, and from Lemma 6 it is plain that $T\{[1-\Gamma_\epsilon(t)][K(\omega)]^{-1}\}$ is bounded in any interval $0 < \delta \leqslant \sigma \leqslant \sigma_1$. There is therefore no loss of generality in supposing σ_1 as small as we please. Now $[H(it)]^{-1} \in \mathfrak{A}$, by Lemma 2, and therefore $[K(\omega)-K(it)][H(it)]^{-1} \in \mathfrak{A}$ by Lemma 8. But

$$T\{K(\omega)-K(it)\} = \int |(e^{-\sigma x}-1)\,dk(x)| = o(1) \tag{5.4}$$

as $\sigma \to 0$. Therefore, if σ_1 is sufficiently small,

$$T\{[K(\omega)-K(it)][H(it)]^{-1}\} \leqslant a < 1$$

for $0 \leqslant \sigma \leqslant \sigma_1$, a being independent of σ. Our conclusion follows now from (5.3) and Lemmas 7 and 8.

Proof of Theorem 7. As in Theorem 1, we can show that $g(x)\,e^{-\sigma x}$ is bounded for $0 \leqslant \sigma \leqslant \sigma_1$, and define

$$G(\omega) = \int e^{-\omega x} g(x)\,dx \quad (0 < \sigma < \sigma_1).$$

From condition (V) we have

$$G(\omega) = K(\omega)\,F(\omega) \qquad (0 < \sigma < \sigma_1), \tag{5.5}$$

so that

$$\begin{aligned} F(\omega) &= F(\omega)\,\Gamma(t) + F(\omega)\,[1-\Gamma(t)] \\ &= F(\omega)\,\Gamma(t) + [1-\Gamma(t)]\,G(\omega)\,[K(\omega)]^{-1}. \end{aligned}$$

Hence, using Lemma 9, we have

$$\int e^{-\omega x} f(x)\,dx = F(\omega)\,\Gamma(t) + \int e^{-\omega x} g(x)\,dx \int e^{-itx}\,dp(\sigma, y), \tag{5.6}$$

where

$$\int |dp(\sigma, y)| \leqslant C \quad (0 \leqslant \sigma \leqslant \sigma_1). \tag{5.7}$$

It follows from (5.6), by the usual inversion formula, that

$$e^{-\sigma x} f(x) = \frac{1}{2\pi} \int e^{itx}\,F(\sigma+it)\,\Gamma(t)\,dt + \int g(x-y)\,e^{-\sigma(x-y)}\,dp(\sigma, y). \tag{5.8}$$

The first term on the right is

$$-\frac{i}{2\pi} \int F_1(\sigma+it)\,\frac{d}{dt}\,[e^{itx}\,\Gamma(t)]\,dt,$$

where

$$F_1(\omega) = \int_{\omega_0}^{\omega} F(z)\,dz,$$

and it follows from (VI) that, as $\sigma \to 0$, it converges to

$$\frac{1}{2\pi} \int F_1(it) \frac{d}{dt} [e^{itx} \Gamma(t)] dt.$$

Hence, if we let $\sigma \to 0$ in (5.8), and use (5.7), we have

$$(5.9) \quad |f(x)| \leqslant \left| \frac{1}{2\pi} \int F_1(it) \frac{d}{dt} [e^{itx} \Gamma(t)] dt \right| + \overline{\lim_{\sigma \to 0}} \left| \int g(x-y) \, dp(\sigma, y) \right|$$

$$= \left| \frac{1}{2\pi} \int F_1(it) \, e^{itx} [\Gamma'(t) + ix \Gamma(t)] dt \right| + \overline{\lim_{\sigma \to 0}} \left| \int g(x-y) \, dp(\sigma, y) \right|,$$

which is plainly bounded, by (VI). Hence, if

$$f(x) \sim \int e^{itx} \, ds(t),$$

then $2\pi i s(t) \equiv F_1(it)$ for $a_n - \epsilon \leqslant t \leqslant b_n + \epsilon$ by Theorem 4, and our conclusion follows from Theorem 5.

It is evident, of course, that we can deduce the stronger conclusion of Theorem 6 if $s(t)$ satisfies the appropriate conditions.

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ON GEOMETRY OF NUMBERS

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1. Minkowski stated his fundamental theorem concerning convex solids in relationship to a lattice as follows. Let the lattice in the n-dimensional space of vectors $\mathfrak{x} = (x_1, \ldots, x_n)$ consist of all vectors $\mathfrak{a}$ with integral components, and let $f(\mathfrak{x})$ be any "gauge function", *i.e.* any continuous function in vector space enjoying the following properties:

$$f(\mathfrak{x}) > 0 \text{ except for the origin } \mathfrak{x} = \mathfrak{o} = (0, \ldots, 0);$$

$$f(t\mathfrak{x}) = |t| . f(\mathfrak{x}) \text{ for any real factor } t;$$

$$f(\mathfrak{x}+\mathfrak{x}') \leqslant f(\mathfrak{x})+f(\mathfrak{x}').$$

Then $f(\mathfrak{x}) < 1$ defines an (open) convex solid $\mathfrak{K}$ with $\mathfrak{o}$ as its centre. V being the volume of $\mathfrak{K}$ and $2M$ the minimum of $f(\mathfrak{a})$ for all lattice vectors $\mathfrak{a} \neq \mathfrak{o}$, Minkowski's universal inequality is

$$M^n V \leqslant 1. \tag{1.1}$$

If $\mathfrak{S}$ is any set in vector space, we can place it round a given point a of the n-dimensional affine point space; the ensuing point set $\mathfrak{S}_a$ consists of all points x such that the vector $\mathfrak{x} = \overrightarrow{ax}$ is in $\mathfrak{S}$. Minkowski's argument is based on the simple remark that the convex solid

$$\mathfrak{K}(q): \ f(\mathfrak{x}) < q,$$

homothetic with $\mathfrak{K}$, when placed round each of the lattice points a, causes no overlapping so long as $q \leqslant M$, and hence up to that limit its volume $q^n V$ stays less than or equal to 1. With q growing beyond M, the solids of the lattice begin to overlap. Studying the portion of the space covered by

them, Minkowski arrives at the sharper inequality†

$$M_1 \dots M_n V \leqslant 1, \tag{1.2}$$

where the numbers M_k may be described thus: A bubble of shape $\mathfrak{K}$ while being blown up describes the series of figures $\mathfrak{K}(q)$ with increasing q; $q = 2M = 2M_1$ is the first moment when a lattice point $\mathfrak{d}_1 \neq \mathfrak{o}$ enters into the bubble, $2M_k$ that stage at which for the first time a lattice point $\mathfrak{d}_k$ enters off the $(k-1)$-dimensional linear manifold $[\mathfrak{d}_1, \dots, \mathfrak{d}_{k-1}]$ spanned by $\mathfrak{d}_1, \dots, \mathfrak{d}_{k-1}$. Clearly $M_1 \leqslant M_2 \leqslant \dots \leqslant M_n$. In other words, $\mathfrak{a} = \mathfrak{d}_k$ is that one among the lattice vectors $\mathfrak{a}$ outside the manifold $[\mathfrak{d}_1, \dots, \mathfrak{d}_{k-1}]$ for which $f(\mathfrak{a})$ assumes the least possible value $2M_k$. So long as $q \leqslant 2M_k$ the body $\mathfrak{K}(q)$ contains less than k linearly independent lattice vectors, while for $q > 2M_k$ their number is at least k.

The vectors

$$\pm\mathfrak{d}_1/2M_1, \quad \pm\mathfrak{d}_2/2M_2, \quad \dots, \quad \pm\mathfrak{d}_n/2M_n,$$

and hence their convex closure, are contained in $\mathfrak{K}$. If $P = |\mathfrak{d}_1 \dots \mathfrak{d}_n|$ is the volume of the parallelotope spanned by the lattice vectors $\mathfrak{d}_1, \dots, \mathfrak{d}_n$, the volume of that closure is

$$\frac{2^n}{n!} \frac{P}{2M_1 \dots 2M_2} = \frac{P}{n!} \frac{1}{M_1 \dots M_n}.$$

Therefore

$$\frac{P}{n!} \leqslant M_1 \dots M_n V \leqslant 1,$$

and thus the positive integer $P \leqslant n!$.

This note consists of two parts, the first stating Minkowski's problem in terms of characteristic functions ϕ which are capable of the two "truth values" 0 and 1 only, and then generalizing it to probability functions that move in the range $0 \leqslant \phi \leqslant 1$. Overlapping is not excluded, and is taken care of by using Boolean rather than ordinary sums. The substance of the argument remains unaltered, yet the essential points come out more clearly.

If P were equal to 1, the inequality (1.2) would tell an important fact about reduction under arithmetic equivalence, namely about normalization of the basis of our lattice in terms of a given convex solid $\mathfrak{K}$. As it stands, it points in a slightly different direction. However, by an almost trivial twist, one can make it tally with the problem of reduction. This is done in the second part, which concludes with some remarks about the special case of the reduction of quadratic forms.

† *Geometrie der Zahlen* (Leipzig, 1896, 1910), 211–218.

[*Added* January, 1940. Prof. L. J. Mordell, to whom I sent a copy of the MS., kindly pointed out to me two papers which I had overlooked; one by H. Davenport, *Quart. J. of Math.*, 10 (1939), 119–121, bears closely on Part I by containing another proof for the inequality (1.2), while the other by K. Mahler, *Quart. J. of Math.*, 9 (1938), 259–262, actually anticipates the main result of Part II, namely Theorem V. I am still surprised that the discovery of this fact in 1938 by Dr. Mahler, and of the simple remark on which it is based, had to wait for forty-two years after Minkowski established his inequality (1.2).]

I.

2. A set in the affine space of points $x = (x_1, \ldots, x_n)$ may be described by its characteristic function $\phi(x)$ which is equal to 1 inside the set and to 0 outside the set. The characteristic function ϕ for the union of two sets with the characteristic functions ϕ_1, ϕ_2 is their "Boolean" sum

$$\phi = \phi_1 + \phi_2 - \phi_1\phi_2,$$

which reduces to the ordinary sum only if the sets do not overlap. Any number a in the interval $0 \leqslant a \leqslant 1$ is said to be a *probability*. If a and b are probabilities of two statistically independent events A and B, then their Boolean sum

$$c = a \vee b = a + b - ab = 1 - (1-a)(1-b),$$

which again satisfies the inequality $0 \leqslant c \leqslant 1$, is the probability of the event "A or B". The Boolean sum is commutative and associative. $a' \leqslant a$ implies $(a' \vee b) \leqslant (a \vee b)$, as the expression $a \vee b = a(1-b)+b$ at once shows; and consequently

$$(2.1) \qquad a' \leqslant a, \quad b' \leqslant b \quad \text{imply} \quad (a' \vee b') \leqslant (a \vee b).$$

The Boolean sum of a finite number of probabilities $a_1, \ldots, a_h$ is

$$\begin{aligned} \mathop{\mathrm{S}}_{i} a_i &= a_1 \vee a_2 \vee \ldots \vee a_h = 1 - \prod_{i=1}^{h} (1-a_i) \\ &= \sum_{(i)} a_i - \sum_{(i,k)} a_i a_k + \sum_{(i,k,l)} a_i a_k a_l - \ldots \\ &= \frac{1}{1!} \sum_{i} a_i - \frac{1}{2!} \sum_{i,k} a_i a_k + \frac{1}{3!} \sum_{i,k,l} a_i a_k a_l - \ldots . \end{aligned}$$

Summation extends to all throws of 1 or 2 or 3 or ... different indices from the set $1, \ldots, h$, in the second line without regard to their order, in the

third line with regard to their order. Brackets as in (i, k, l) indicate the first kind of summation.

A function $\phi(x)$ in our space which satisfies the condition $0 \leqslant \phi \leqslant 1$ is called a probability function.

Let $\phi(x)$ now be an (integrable) probability function vanishing outside a finite region $\mathfrak{H}$ of the x-space which may be fixed as a parallelotope:

$$A_i < x_i < B_i \quad (i = 1, \ldots, n).$$

The lattice translation $\mathfrak{a} = (a_1, \ldots, a_n)$ with integral components a_i,

$$x_i' = x_i + a_i,$$

carries $\phi(x)$ into a function

$$\phi(x, \mathfrak{a}) = \phi(x_1 - a_1, \ldots, x_n - a_n).$$

In these circumstances, the Boolean sum

$$\psi(x) = \mathop{\mathbf{S}}_{\mathfrak{a}} \phi(x, \mathfrak{a})$$

extending to all lattice translations $\mathfrak{a}$ has a meaning, since for each x only a limited number of its terms do not disappear. $\psi(x)$ is a probability function with period 1 in each of the variables $x_1, \ldots, x_n$. We form the integral

$$J[\phi] = \int_E \psi(x)\, dx \quad [dx = dx_1\, dx_2 \ldots dx_n]$$

extending over the fundamental mesh of the lattice

$$E: \quad 0 < x_1 < 1, \ldots, 0 < x_n < 1.$$

Obviously $0 \leqslant \psi(x) \leqslant 1$ entails

$$0 \leqslant J[\phi] \leqslant 1. \tag{2.2}$$

Moreover, for two probability functions ϕ_2 and $\phi_1 \leqslant \phi_2$ we have $\psi_1 \leqslant \psi_2$, because of the law of monotony (2.1) for Boolean sums, and hence

$$J[\phi_1] \leqslant J[\phi_2]. \tag{2.3}$$

The explicit expression for ψ is

$$\psi(x) = 1 - \prod_{\mathfrak{a}} \{1 - \phi(x, \mathfrak{a})\}$$

$$= \frac{1}{1!} \sum_{\mathfrak{a}} \phi(x, \mathfrak{a}) - \frac{1}{2!} \sum_{\mathfrak{a}, \mathfrak{b}} \phi(x, \mathfrak{a})\, \phi(x, \mathfrak{b}) + \frac{1}{3!} \sum_{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}} \phi(x, \mathfrak{a})\, \phi(x, \mathfrak{b})\, \phi(x, \mathfrak{c}) - \ldots$$

[$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ lattice vectors].

In substituting $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a}+\mathfrak{c}$, ... for $\mathfrak{b}$, $\mathfrak{c}$, ... we see that $\psi(x)$ may be written as an ordinary sum

$$\psi(x) = \sum_{\mathfrak{a}} \phi^*(x, \mathfrak{a}),$$

where

$$\phi^*(x) = \phi(x)\left\{\frac{1}{1!} - \frac{1}{2!}\sum_{\mathfrak{b}} \phi(x, \mathfrak{b}) + \frac{1}{3!}\sum_{\mathfrak{b}, \mathfrak{c}} \phi(x, \mathfrak{b})\,\phi(x, \mathfrak{c}) - \ldots\right\}$$

[$\mathfrak{b}$, $\mathfrak{c}$, ..., lattice vectors $\neq \mathfrak{o}$].

This trick works because the lattice translations form a *group*. When we return to "combinations" without regard to order, the second factor on the right in the last formula is

$$\phi^{**}(x) = 1 - \tfrac{1}{2}\sum_{(\mathfrak{a})} \phi(x, \mathfrak{a}) + \tfrac{1}{3}\sum_{(\mathfrak{a}, \mathfrak{b})} \phi(x, \mathfrak{a})\,\phi(x, \mathfrak{b}) - \ldots. \tag{2.4}$$

[$\mathfrak{a}$, $\mathfrak{b}$, ... lattice vectors $\neq \mathfrak{o}$].

Take a real parameter t ranging from 0 to 1, and introduce the product

$$\prod_{\mathfrak{a}\neq\mathfrak{o}} \{1 - t\phi(x, \mathfrak{a})\} = 1 - t\sum_{(\mathfrak{a})} \phi(x, \mathfrak{a}) + t^2 \sum_{(\mathfrak{a}, \mathfrak{b})} \phi(x, \mathfrak{a})\,\phi(x, \mathfrak{b}) - \ldots.$$

Its integration from $t = 0$ to 1 leads to (2.4). Hence our final definitions are:

$$\phi^*(x) = \phi(x)\,.\,\phi^{**}(x), \qquad \phi^{**}(x) = \int_0^1 \prod_{\mathfrak{a}\neq\mathfrak{o}} \{1 - t\phi(x, \mathfrak{a})\}\,dt; \tag{2.5}$$

$$J[\phi] = \int_{\infty} \phi^*(x)\,dx. \tag{2.6}$$

∞ indicates that the integral extends over the whole x-space. Notice that $0 < \phi^{**}(x) \leqslant 1$ and $\phi_1^{**} \geqslant \phi_2^{**}$ if $\phi_1 \leqslant \phi_2$. (2.6) proves $J[\phi]$ to be invariant under translation, namely

$$J[\phi'] = J[\phi] \quad \text{for} \quad \phi'(x) = \phi(x - x^0). \tag{2.7}$$

If $0 \leqslant \phi_1 \leqslant \phi$ and $\phi - \phi_1 \leqslant \epsilon$, we find that

$$\phi^*(x) = \phi(x)\,.\,\phi^{**}(x) \leqslant \phi(x)\,.\,\phi_1^{**}(x)$$

and thus, from (2.6), derive the result

$$J[\phi] - J[\phi_1] \leqslant \int_{\mathfrak{H}} \{\phi(x) - \phi_1(x)\}\,\phi_1^{**}(x)\,dx \leqslant H\epsilon, \tag{2.8}$$

where H is the volume of $\mathfrak{H}$. We summarize:

THEOREM I. *The integral* (2.6) *is less than or equal to* 1 *for any probability function* ϕ *which vanishes outside a finite region.* $J[\phi]$ *is*

invariant under translation, and

$$J[\phi_1] \leqslant J[\phi_2] \quad \textit{if} \quad \phi_1(x) \leqslant \phi_2(x).$$

Specilization of the theorem for the characteristic function ϕ of an (open) set $\mathfrak{S}$ of definite Jordan volume V yields the simple Minkowski-Blichfeldt-Scherrer theorem: If $\mathfrak{S}_0$ has no point in common with the sets $\mathfrak{S}_a$ placed round the other lattice points $a \neq 0$, or, what is the same thing: if $\mathfrak{S}_0$ never contains both end points x and $x+a$ of a lattice vector $a \neq 0$, then its volume is less than or equal to 1. I consider Theorem I to be the natural general form of this principle, and our proof is nothing but a neat analytical form of the simplest argument by which that principle has been proved†.

[Our affine x-space could be replaced by any differentiable manifold U of points x which carries an infinitesimal volume measure, and the group of lattice translations by any group of differentiable volume-preserving one-to-one transformations $x \to x^s$ of U which

(1) is discrete in the strictest sense, so that the set of points x^s, equivalent to any given point x, never has a condensation point,

(2) has a compact fundamental domain; *i.e.*, the manifold, arising from U by identification of equivalent points, is supposed to be compact.]

3. With Siegel‡ we introduce the Fourier coefficients of the periodic function $\psi(x)$:

$$J_{\mathfrak{l}} = \int_E \psi(x) \,.\, e(l_1 x_1 + \ldots + l_n x_n)\, dx$$

corresponding to the various lattice-vectors $\mathfrak{l} = (l_1, \ldots, l_n)$ in the dual space, with $e(x)$ as an abbreviation for $e^{2\pi i x}$, and formulate the Parseval equation

$$\int_E \big(\psi(x)\big)^2 dx = \sum_{\mathfrak{l}} |J_{\mathfrak{l}}|^2.$$

The zero coefficient $J_{\mathfrak{o}}$ is our $J = J[\phi]$. Making use of the equation

$$\int_E \psi(x) \,.\, \omega(x)\, dx = \int_\infty \phi^*(x) \,.\, \omega(x)\, dx$$

† G. Hajós, *Act. Litt. Sci., Szeged*, 6 (1934), 224–225. H. F. Blichfeldt, *Trans. American Math. Soc.*, 15 (1914), 227–235. W. Scherrer, *Math. Annalen*, 86 (1922), 99–107.

‡ *Acta Math.*, 65 (1935), 307–323.

which holds for any periodic function $\omega(x)$, we get

$$J_l = \int_\infty \phi^*(x) \cdot e(l_1 x_1 + \ldots + l_n x_n)\, dx \tag{3.1}$$

and

$$\int_E \big(\psi(x)\big)^2 dx = \int_\infty \phi^*(x) \cdot \psi(x)\, dx.$$

We find that

$$\phi^* \psi = \phi^* \Big\{1 - \prod_{\mathfrak{a}} \big(1 - \phi(x, \mathfrak{a})\big)\Big\}$$

$$= \phi^* - \phi^*(1-\phi) \cdot \prod_{\mathfrak{a} \neq \mathfrak{o}} \{1 - \phi(x, \mathfrak{a})\}$$

and thus

$$\phi^* \psi = \phi^* - \phi(1-\phi)\, \phi_* \phi^{**} \quad \text{with} \tag{3.2}$$

$$\phi_*(x) = \prod_{\mathfrak{a} \neq \mathfrak{o}} \{1 - \phi(x, \mathfrak{a})\}.$$

THEOREM II. *If ϕ_* and ϕ^{**} are defined by* (3.2), (2.4) *and J_l denotes the integral* (3.1), *then the following equation holds for $J = J[\phi]$:*

$$J(1-J) = \int_\infty \phi(1-\phi) \cdot \phi_* \phi^{**}\, dx + \sum_{l \neq \mathfrak{o}} |J_l|^2.$$

It shows very clearly how $0 \leqslant \phi \leqslant 1$ leads to $0 \leqslant J \leqslant 1$.

[This argument works on any differentiable manifold U on which we have a transitive group Γ of differentiable one-to-one transformations, the lattice consisting of a discrete subgroup of Γ whose fundamental domain is compact. The harmonics replacing $e(l_1 x_1 + \ldots + l_n x_n)$ are obtained by a slight modification of the process described by the author, *Annals of Math.*, 35 (1934), 486–499.]

4. We now introduce an assumption of *convexity* by supposing $\{\phi(x) \geqslant r\}_x$, *i.e.* the locus $\mathfrak{F}_r$ of the points x where $\phi(x) \geqslant r$, to be a convex set for any $r > 0$. The relations $\phi(x) \geqslant r$ and $\phi(x') \geqslant r$ then imply

$$\phi\big(tx + (1-t)x'\big) \geqslant r \tag{4.1}$$

for any number t between 0 and 1. *A function ϕ having this property in our n-dimensional space preserves it on any linear subspace.* The "terrace" $\{r \leqslant \phi(x) < r'\}_x$ between the levels r and $r' > r$ has a Jordan volume because

it is the difference between the two convex bodies $\mathfrak{F}_{r'}$ and $\mathfrak{F}_r$. Let

$$0 = r_0 < r_1 < r_2 < \ldots < r_h = 1 \tag{4.2}$$

be any division of the interval $0 \leqslant r \leqslant 1$ into subintervals of lengths $r_{i+1} - r_i \leqslant \epsilon$, and let $\phi_\Delta(x)$, $\phi^\Delta(x)$ be the corresponding step functions which approximate ϕ from below and above with an error less than or equal to ϵ. They vanish outside $\mathfrak{H}$ and are defined inside $\mathfrak{H}$ by

$$\phi_\Delta(x) = r_i \text{ and } \phi^\Delta(x) = r_{i+1} \text{ if } r_i \leqslant \phi(x) < r_{i+1}.$$

The Riemann integrals of ϕ_Δ and ϕ^Δ clearly differ by not more than $H\epsilon$; this proves the Riemann integrability of ϕ. Moreover, by (2.3) and (2.8),

$$J[\phi_\Delta] \leqslant J[\phi] \leqslant J[\phi^\Delta] \quad \text{and} \quad J[\phi^\Delta] - J[\phi_\Delta] \leqslant H\epsilon,$$

and these inequalities could also serve to define $J[\phi]$ by mean of the Riemann integrals $J[\phi_\Delta]$ and $J[\phi^\Delta]$ of the approximate terraced functions.

We repeat our assumptions (A) about ϕ:

(i) $0 \leqslant \phi(x) \leqslant 1$;

(ii) $\phi(x) = 0$ outside a finite region $\mathfrak{H}$;

(iii) the sets $\mathfrak{F}_r$: $\{\phi(x) \geqslant r\}_x$ are convex.

Minkowski's second step deals with the process of dilation as defined by

$$\phi_q(x_1, \ldots, x_n) = \phi\left(\frac{x_1}{q}, \ldots, \frac{x_n}{q}\right) \tag{4.3}$$

and, properly generalized, leads to

THEOREM III. *Under the assumptions* (A),

$$J[\phi_q] \geqslant J[\phi] \text{ for } q \geqslant 1. \tag{4.4}$$

Proof. Suppose that we have a point x^0 where $\phi(x)$ takes on its maximum value. If x is any point and $\phi(x) = r$, we then have $\phi(x^0) \geqslant r$, and, by (4.1),

$$\phi\left(\frac{1}{q}\, x + \left(1 - \frac{1}{q}\right) x^0\right) \geqslant r = \phi(x) \text{ for any } q \geqslant 1. \tag{4.5}$$

For the moment denote the function on the left-hand side by $\phi_q'(x)$. Formula (4.5) entails

$$J[\phi_q'] \geqslant J[\phi],$$

and thus, since ϕ_q' arises from ϕ_q by a translation,

$$J[\phi_q] = J[\phi_q'] \geqslant J[\phi].$$

Since we have neither assumed $\phi(x)$ to be continuous nor the sets $\mathfrak{F}_r$ to be closed, the existence of a maximum is doubtful. But the following slight modification will do. Let r_i in the finite sequence (4.2) be the highest level for which $\mathfrak{F}_{r_i}$ is not empty. We cut off the top of the mountain at the altitude r_i, *i.e.* we form

$$\hat{\phi}(x) = \min\left(r_i, \phi(x)\right)$$

and choose x^0 as a point in $\mathfrak{F}_{r_i}$. Then we get

$$\phi_q'(x) \geqq \hat{\phi}(x) \quad (q \geqq 1)$$

instead of (4.5), and hence

$$J[\phi_q] \geqq J[\hat{\phi}].$$

However, according to (2.8)

$$J[\hat{\phi}] \geqq J[\phi] - H\epsilon,$$

and by driving ϵ towards 0 the inequality (4.4) is re-established. (Instead of the decapitated mountain, we might have used the terraced mountain ϕ_Δ.)

Since $$(\phi_{q'})_q = \phi_{q'q},$$

we may claim Theorem III to assert that

$$J[\phi_q] = J(q)$$

is a monotonically increasing function of the parameter q of dilation.

5. Our next move is to modify the procedure by taking into consideration only the lattice vectors

(5.1) $$(a_1, \ldots, a_k, 0, \ldots, 0) \quad (a_i \text{ integers}, \ k \geqq 1)$$

in the linear subspace

$$x_{k+1} = \ldots = x_n = 0.$$

$\phi^* = \phi^{(k)}$ is now defined by the last equation (2.5) with the product ranging over the vectors $\mathfrak{a} \neq \mathfrak{o}$ of this k-dimensional lattice, and

$$J^{(k)}[\phi] = \int_\infty \phi^{(k)}(x)\, dx.$$

The integration is then conveniently carried out in two steps, first with respect to $x_1, \ldots, x_k$ for any constant $x_{k+1}, \ldots, x_n$, and secondly with respect to $x_{k+1}, \ldots, x_n$. Thus we find that

$$\phi_1 \leqslant \phi_2 \quad \text{implies} \quad J^{(k)}[\phi_1] \leqslant J^{(k)}[\phi_2],$$

while the inequality

$$J^{(k)}[\phi] \leqslant J^{(k)}[\phi_1] + H\epsilon,$$

holding under the condition $0 \leqslant \phi - \phi_1 \leqslant \epsilon$, follows by the same argument as (2.8), without a break in integration.

Like ϕ itself, the function

$$\Phi(x_1 \ldots x_k) = \phi(x_1 \ldots x_k, x_{k+1}^0 \ldots x_n^0)$$

of k variables satisfies the assumptions (A), and therefore

$$J[\Phi_q] \geqslant J[\Phi] \quad \text{for} \quad q \geqslant 1. \tag{5.2}$$

Notice that $$\Phi_q(x_1 \ldots x_k) = \bar{\phi}_q(x_1 \ldots x_k, x_{k+1}^0 \ldots x_n^0)$$

with $$\bar{\phi}_q(x) = \phi\left(\frac{x_1}{q}, \ldots, \frac{x_k}{q}, x_{k+1}, \ldots, x_n\right).$$

Thus, by integrating (5.2) with respect to $x_{k+1}, \ldots, x_n$, we get

$$J^{(k)}[\bar{\phi}_q] \geqslant J^{(k)}[\phi].$$

The substitution

$$\frac{x_{k+1}}{q} = x'_{k+1}, \ \ldots, \ \frac{x_n}{q} = x'_n \tag{5.3}$$

performed on the last $n-k$ variables results in the equation

$$J^{(k)}[\phi_q] = q^{n-k} \cdot J^{(k)}[\bar{\phi}_q].$$

Hence $$J^{(k)}[\phi_q] \geqslant q^{n-k} \cdot J^{(k)}[\phi] \quad \text{for} \quad q \geqslant 1.$$

For our Riemann integrals the performance of the integration in two steps $x_1 \ldots x_k | x_{k+1} \ldots x_n$ and of the substitution (5.3) offers no difficulty. But maybe it is preferable first to argue for the step function ϕ_Δ, thus avoiding the difficulty about the maximum of ϕ mentioned above, and then to combine the ensuing inequality

$$J^{(k)}[\phi_q] \geqslant q^{n-k} \cdot J^{(k)}[\phi_\Delta]$$

with $$J^{(k)}[\phi_\Delta] \geqslant J^{(k)}[\phi] - H\epsilon.$$

For $k = 0$ we define $\phi^{(0)} = \phi$ and therefore

$$J^{(0)}[\phi_q] = q^n \cdot \int_\infty \phi(x)\, dx.$$

Any k-dimensional discrete lattice $\mathfrak{L}_k$ may serve here instead of (5.1), since the first k coordinate vectors $e_1, \ldots, e_k$ may be so chosen as to span the pre-assigned lattice (that is, so that the lattice consists of the vectors $a_1 e_1 + \ldots + a_k e_k$ with integral components a_i).

THEOREM IV. *By means of a k-dimensional lattice $\mathfrak{L}_k$ we define*

$$\phi^{(k)}(x) = \phi(x) \cdot \int_0^1 \prod_{\mathfrak{a}} \{1 - t\phi(x, \mathfrak{a})\}\, dt,$$

the product extending to all vectors $\mathfrak{a} \neq 0$ of the lattice $\mathfrak{L}_k$, and

$$J^{(k)}[\phi] = \int_\infty \phi^{(k)}(x)\, dx, \qquad J_k(q) = J^{(k)}[\phi_q].$$

Then $J_k(q)/q^{n-k}$ is an increasing function of q. In particular,

$$J_0(q)/q^n = \int_\infty \phi(x)\, dx.$$

The application to the situation considered by Minkowski is immediate. ϕ is now the characteristic function of the convex solid $\mathfrak{K}$: $f(x) < 1$. Let $\mathfrak{L}_k$ consist of the lattice vectors in $[\mathfrak{d}_1, \ldots, \mathfrak{d}_k]$. So long as $q \leqslant M_{k+1}$, we have

$$\phi_q^{(k)} = \phi_q{}^*, \quad J_k(q) = J(q).$$

Indeed, the solids $\mathfrak{K}(q)$ and $\mathfrak{K}_a(q)$ do not overlap if a is a lattice point outside $[\mathfrak{g}_1, \ldots, \mathfrak{g}_k]$ and $q \leqslant M_{k+1}$. We thus find that

$$J(q) = q^n \cdot V \quad \text{for} \quad q \leqslant M_1,$$

in particular $$J(M_1) = M_1{}^n \cdot V.$$

Then

$$J(q) \geqslant \left(\frac{q}{M_1}\right)^{n-1} \cdot J(M_1) = M_1 q^{n-1} \cdot V \quad \text{for} \quad M_1 \leqslant q \leqslant M_2,$$

and hence $$J(M_2) \geqslant M_1 M_2^{n-1} \cdot V.$$

Continuing in the same manner, we get

$$J(q) \geqslant M_1 \ldots M_{k-1} q^{n-k+1} \cdot V \quad \text{for} \quad M_{k-1} \leqslant q \leqslant M_k$$

$$(k = 1, \ldots, n)$$

and ultimately

$$J(q) \geqq M_1 \dots M_n . V \quad \text{for} \quad q \geqq M_n.$$

Since $J(q)$ always remains less than or equal to 1, Minkowski's inequality (1.2) is proved.

II.

6. Contrary to the procedure in I, yielding the vectors $\mathfrak{d}_k$ and ratios of dilation $2M_k$, the *problem of reduction* requires the determination of vectors $\mathfrak{e}_k$ and ratios N_k according to the following inductive rule:

Let $\mathfrak{a}$ range over all lattice vectors outside the $(k-1)$-dimensional manifold $[\mathfrak{e}_1 \dots \mathfrak{e}_{k-1}]$ such that every lattice vector in the k-dimensional manifold $[\mathfrak{e}_1 \dots \mathfrak{e}_{k-1} \mathfrak{a}]$,

$$\mathfrak{x}_{k-1} + x\mathfrak{a} \text{ with } \mathfrak{x}_{k-1} \text{ in } [\mathfrak{e}_1 \dots \mathfrak{e}_{k-1}],$$

has an integral $\mathfrak{a}$-component x. $\mathfrak{a} = \mathfrak{e}_k$ is a vector within this range $\mathfrak{R}_k$ for which $f(\mathfrak{a})$ takes on the least possible value N_k.

As the first step, we may take $\mathfrak{e}_1 = \mathfrak{d}_1$, so that $N_1 = 2M_1$. By induction we readily conclude that $\mathfrak{e}_1, \dots, \mathfrak{e}_k$ span (*i.e.* form a basis of) the whole lattice in the manifold $[\mathfrak{e}_1 \dots \mathfrak{e}_k]$; hence $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ span the whole n-dimensional lattice. Relative to this coordinate system, the lattice vectors

$$\mathfrak{x} = x_1 \mathfrak{e}_1 + \dots + x_n \mathfrak{e}_n = (x_1, \dots, x_n)$$

are again described by integral components x_i, and our reduction is equivalent to the following conditions:

$$f(x_1, x_2, \dots, x_n) \geqq f(\delta_1^k, \dots, \delta_n^k) = N_k \tag{6.1}$$

which hold provided that $x_1, x_2, \dots, x_n$ are integers and $x_k, \dots, x_n$ are without common divisor. The δ_i^k are the Kronecker δ's and thus $\mathfrak{e}_k = (\delta_r^k, \dots, \delta_n^k)$, the k-th unit vector.

This characterization (6.1) obviously entails the inequalities

$$N_1 \leqq N_2 \leqq \dots \leqq N_n.$$

We have normalized the basis of our n-dimensional lattice relative to the convex solid by means of the n sets (6.1) of infinitely many inequalities, and this is what the problem of reduction demands. The question arises whether one can establish a universal upper bound for $N_1 \dots N_n V$. I now prove

THEOREM V. *If a reduced lattice basis $\mathfrak{e}_k$ is so chosen as to satisfy the conditions* (6.1), *then*

$$N_1 \dots N_n V \leqq \mu_n, \tag{6.2}$$

where

$$\mu_n = 2^n \cdot (\tfrac{3}{2})^{\frac{1}{2}(n-1)(n-2)}. \tag{6.3}$$

Proof. One of the vectors $\mathfrak{d}_1, \ldots, \mathfrak{d}_k$, say $\mathfrak{d}$, lies outside the linear manifold

$$E_{k-1} = [\mathfrak{e}_1 \ldots \mathfrak{e}_{k-1}].$$

In

$$E_k^* = [\mathfrak{e}_1 \ldots \mathfrak{e}_{k-1}\, \mathfrak{d}]$$

we can find a lattice vector $\mathfrak{e}^*$ which, together with $\mathfrak{e}_1, \ldots, \mathfrak{e}_{k-1}$, spans the whole lattice in E_k^*. Out of the finite number of lattice vectors of the form

$$x_1\mathfrak{e}_1 + \ldots + x_{k-1}\mathfrak{e}_{k-1} + y\mathfrak{d} \tag{6.4}$$

with

$$-\tfrac{1}{2} < x_i \leqslant \tfrac{1}{2} \quad (i = 1, \ldots, k-1), \quad 0 < y \leqslant 1,$$

select one, say $\mathfrak{e}^*$, with the lowest y. Since $\mathfrak{d}$ itself must be an integral combination of $\mathfrak{e}_1, \ldots, \mathfrak{e}_{k-1}, \mathfrak{e}^*$, this y is the reciprocal $1/e$ of a positive integer e. If $e = 1$, $\mathfrak{e}^*$ coincides with $\mathfrak{d}$. Our $\mathfrak{e}^*$ is in $\mathfrak{R}_k$ and thus in the competition for the lowest value N_k of $f(\mathfrak{a})$. But, since $f(\mathfrak{d})$, as one of the numbers

$$2M_1, \ldots, 2M_k,$$

is certainly less than or equal to $2M_k$, we infer from these remarks and (6.4) that

$$f(\mathfrak{e}^*) \leqslant 2M_k \text{ if } e = 1,$$

$$f(\mathfrak{e}^*) \leqslant \frac{2M_k}{e} + \tfrac{1}{2}(N_1 + \ldots + N_{k-1}) \leqslant M_k + \tfrac{1}{2}(N_1 + \ldots + N_{k-1}) \text{ if } e \geqslant 2.$$

Hence N_k cannot exceed the larger of the two numbers

$$M_k + \tfrac{1}{2}(N_1 + \ldots + N_{k-1}) \quad \text{and} \quad 2M_k. \tag{6.5}$$

This allows us to establish universal inequalities of the kind

$$N_i \leqslant 2\theta_i M_i. \tag{6.6}$$

Indeed (6.6) is true for $i = 1$ with $\theta_1 = 1$. Once it holds good for

$$i = 1, \ldots, k-1,$$

the bound (6.5) for N_k yields a similar inequality for $i = k$ with θ_k as the greater of the two numbers

$$1 \quad \text{and} \quad \tfrac{1}{2}(1 + \theta_1 + \ldots + \theta_{k-1}).$$

This determines the factors θ_k by recursion, and we readily verify that

$$\theta_1 = 1, \quad \theta_k = (\tfrac{3}{2})^{k-2} \quad \text{for} \quad k \geqslant 2.$$

Therefore (1.2) implies

$$N_1 \dots N_n V \leqslant 2^n . \theta_1 \dots \theta_n = \mu_n .$$

It would be an interesting problem to ascertain the lowest constant μ_n for which the relation (6.2) holds.

Fuller exploitation of the method yields the following

GENERALIZED THEOREM V. *Let $q_1, \dots, q_n$ be given positive numbers greater than or equal to* 1. *If the inequality*

$$f(x_1, x_2, \dots, x_n) \geqslant \frac{1}{q_k} . f({\delta_1}^k, \dots, {\delta_n}^k)$$

holds whenever $x_1, \dots, x_n$ are integers and $x_k, \dots, x_n$ have no common factor $(k = 1, \dots, n)$, then the values

$$N_k = f({\delta_1}^k, \dots, {\delta_n}^k)$$

satisfy the relations $\quad q_k N_{k+1} \geqslant N_k \quad (k = 1, \dots, n-1)$

and

$$N_1 \dots N_n V \leqslant 2q_1 \dots q_n (1+q_1)^{n-1} . (1+\tfrac{1}{2}q_2)^{n-2} \dots (1+\tfrac{1}{2}q_{n-1})^1 .$$

The proof is practically the same as before. The recursive equations for the θ_k are now

$$\theta_1 = q_1 \quad \text{and} \quad \frac{2\theta_k}{q_k} = 1+\theta_r+\dots+\theta_{k-1} \quad (k \geqslant 2)$$

i.e.

$$\frac{2\theta_2}{q_2} = 1+q_1, \quad \frac{2\theta_k}{q_k}+\theta_k = \frac{2\theta_{k+1}}{q_{k+1}} \qquad (k \geqslant 2),$$

with the solution

$$\theta_k = \tfrac{1}{2}q_k(1+q_1) . (1+\tfrac{1}{2}q_2) \dots (1+\tfrac{1}{2}q_{k-1}) \quad (k \geqslant 2).$$

7. Here is another similar proposition:

THEOREM VI. *Put*

$$h_\nu = \nu^2 - \tfrac{1}{2}\nu - 1 \quad (\nu = 1, 2, \dots)$$

and

$$\rho_n = 2^{2\nu-n} . (2\nu+2)(2\nu+3) \dots (\nu+n+1) \quad \textit{for} \quad h_\nu \leqslant n \leqslant h_{\nu+1} .$$

We can determine a lattice basis $\mathfrak{e}_1^*, \dots, \mathfrak{e}_n^*$ such that the distances $f(\mathfrak{e}_k^*) = M_k^*$ satisfy the inequality

(7.1) $\qquad M_1^* \dots M_n^* V \leqslant \rho_n .$

This constant ρ_n is much lower than the one in Theorem V. Its increase with n is best judged by the recurrence formulae

$$\rho_n = \frac{n+\nu+1}{2}\,\rho_{n-1} \quad \text{if} \quad 1+h_\nu \leqslant n \leqslant h_{\nu+1},$$

and $$\rho_n = \frac{(n+\nu)(n+\nu+1)}{2(n+\nu)+1}\,\rho_{n-1} \quad \text{for odd } \nu \text{ and } n = h_\nu + \tfrac{1}{2}.$$

Proof. Out of the vector basis $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$, we construct a lattice basis $\mathfrak{e}_1^*, \ldots, \mathfrak{e}_n^*$ according to the recurrent equations

$$\mathfrak{e}_k^* = \frac{1}{e_k}\,\mathfrak{d}_k + a_{k,k-1}\,\mathfrak{d}_{k-1} + \ldots + a_{k,1}\,\mathfrak{d}_1 \tag{7.2}$$

with the conditions

$$-\frac{1}{2e_i} < a_{k,i} \leqslant \frac{1}{2e_i} \quad (i<k),$$

the e_k being positive integers. If $e_k = 1$, all the coefficients $a_{k,k-1}, \ldots, a_{k,1}$ vanish, and $f(\mathfrak{e}_k^*) = 2M_k$. However, if $e_k \geqslant 2$,

$$M_k^* = f(\mathfrak{e}_k^*) \leqslant \frac{2}{e_k}\,M_k + \sum_{i=1}^{k-1} M_i/e_i \leqslant \eta_k M_k$$

with $$\eta_k = \frac{2}{e_k} + \left(\frac{1}{e_{k-1}} + \ldots + \frac{1}{e_1}\right) \quad (e_k \geqslant 2).$$

To cover both cases we put $\eta_k = 2$ for $e_k = 1$. From (1.2) we now get

$$M_1^* \ldots M_n^* V \leqslant \eta_1 \ldots \eta_n,$$

and it remains to discuss the product $\eta_1 \ldots \eta_n$.

We put

$$\delta_k = 2 \text{ or } \delta_k = 1 \quad \text{according as} \quad e_k = 1 \text{ or } e_k \geqslant 2;$$

$$\eta_k' = 2 \text{ if } \delta_k = 2, \quad \text{and} \quad \eta_k' = \delta_k + \tfrac{1}{2}(\delta_{k-1} + \ldots + \delta_1) \text{ if } \delta_k = 1.$$

Then $$\eta_k \leqslant \eta_k' \quad \text{and} \quad \eta_1 \ldots \eta_n \leqslant \eta_1' \ldots \eta_n'.$$

Among the 2^n values of the product

$$\eta_1' \ldots \eta_n' \tag{7.3}$$

resulting from the 2^n possible sequences $\delta_1, \ldots, \delta_n$ of 1's and 2's, we have to ascertain the largest. An interchange of two consecutive figures 1 and 2 in this sequence whereby

$$\delta_k = 1,\ \delta_{k+1} = 2 \quad \text{is changed into} \quad \delta_k = 2,\ \delta_{k+1} = 1$$

(all other figures remaining unaltered) affects merely the two factors $\eta_k' \eta_{k+1}'$ in (7.3), turning their values

$$[1+\tfrac{1}{2}(\delta_{k-1}+\ldots+\delta_1)].2 \quad \text{into} \quad 2.[2+\tfrac{1}{2}(\delta_{k-1}+\ldots+\delta_1)],$$

and so increasing (7.3). For a given number ν of 2's in the sequence, the maximum value is thus attained when they come first ($\delta_1, \ldots, \delta_\nu$), and then (7.3) is equal to

$$2^\nu \cdot \frac{2\nu+2}{2} \cdot \frac{2\nu+3}{2} \cdots \frac{2\nu+1+\mu}{2} \quad (\mu = n-\nu)$$

i.e. $$2^{2\nu-n} \cdot (2\nu+2)(2\nu+3) \ldots (\nu+1+n) = w_\nu.$$

Let ν pass through the values $\nu = 0, 1, \ldots, n$. Since

$$w_\nu = \frac{4(\nu+1+n)}{2\nu(2\nu+1)} \cdot w_{\nu-1},$$

w_ν is on the upgrade ($w_\nu > w_{\nu-1}$ or $w_\nu \geqslant w_{\nu-1}$) so long as

$$\nu(2\nu+1) < 2(\nu+1+n) \quad \text{or} \quad \nu(2\nu+1) \leqslant 2(\nu+1+n),$$

and then goes down. Hence the maximum is attained for the last ν for which $\nu(\nu-\frac{1}{2})$ does not exceed $n+1$.

Here is a table for the lowest values of ρ_n:

$$n = 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad \ldots.$$

$$\rho_n = 2, \quad 4, \quad 12, \quad 42, \quad 168, \quad 756, \quad 3960, \quad \ldots,$$

Higher bounds, for instance

$$\rho_n = (n+1)! \quad \text{or} \quad \rho_n = 2.n!,$$

can even more easily be obtained. Again we have the question of the best value of ρ_n in the inequality (7.1).

8. A positive-definite *quadratic form*

$$f(x) = \Sigma g_{ik} x_i x_k \quad (i, k = 1, \ldots, n) \tag{8.1}$$

is said to be *reduced* if it satisfies the inequalities

$$f(x) \geqslant g_{kk} \quad \text{for every } x \text{ in } \mathfrak{R}_k \quad \text{and} \quad k = 1, \ldots, n, \tag{8.2}$$

where $\mathfrak{R}_k$ is the set of all lattice points $x = (x_1, \ldots, x_n)$ for which $x_k, \ldots, x_n$ are without common divisor. While for every positive form of discriminant $D = \det(g_{ik})$,

$$g_{11} \cdots g_{nn} \geqslant D,$$

we have found that for a reduced form

$$\lambda_n g_{11} \dots g_{nn} \leqslant D \quad \text{with} \quad \lambda_n = (\omega_n/\mu_n)^2. \tag{8.3}$$

Here ω_n denotes the volume of the n-dimensional unit sphere, most handily described by the recurrent relation

$$\omega_n = \frac{2\pi}{n} \cdot \omega_{n-2} \quad \text{together with} \quad \omega_0 = 1, \ \omega_1 = 2.$$

Indeed, with $\sqrt{f}$ as gauge function, the convex body $\mathfrak{K}$: $f < 1$, becomes an ellipsoid of volume $\omega_n/\sqrt{D}$, as is readily proved by linear transformation of $f(x)$ into the normal form $x_1^2+\dots+x_n^2$.

The linear substitutions S of the variables x_i with integral coefficients s_{ik} of determinant $|s_{ik}| = \pm 1$ (or only those of determinant 1) form a group $\{S\}$. Two quadratic forms f and Sf which arise from each other by a substitution of the group are said to be *equivalent*. Every positive form is equivalent to a reduced form.

All quadratic forms $f = \{g_{ik}\}$ with real coefficients $g_{ik} = g_{ki}$ form an $N = \frac{1}{2}n(n+1)$-dimensional linear space R, the positive ones an open convex region G in R, the reduced forms a convex subset Z of G which is closed relative to G. Both G and Z are cones and may, therefore, be considered as sets in the $(N-1)$-dimensional sphere of rays in R issuing from the origin. A convex cone H is a set having these three properties:

(i) $f = 0$ is not in H;

(ii) if f is in H, so is tf, where t is any positive factor;

(iii) with f and f' in H, the sum $f+f'$ is in H.

We operate throughout in G; all topological words referring to subcones of G are meant relative to G (not R).

Each of the inequalities (8.2) for the variables g_{ij} is of the form

$$\sum_{i,j} a_{ij} g_{ij} \geqslant 0 \tag{8.4}$$

with integral coefficients a_{ij}. We expressly exclude from the set $\mathfrak{Z}$ of inequalities (8.2) those for which all coefficients a_{ij} vanish, namely (8.2) with $x = \pm(\delta_1^k, \dots, \delta_n^k)$.

Each individual inequality (8.4) of $\mathfrak{Z}$ defines a half-space bounded by a plane

$$\sum_{i,j} a_{ij} g_{ij} = 0. \tag{8.5}$$

A substitution S carries the central cell Z into an equivalent cell Z_S which is also convex. The equivalent cells jointly cover the whole space G. Z_J coincides with Z for any of the 2^n substitutions

$$J: \quad x_1 \to \pm x_1, \ \ldots, \ x_n \to \pm x_n,$$

and S and SJ have the same effect upon Z.

From (8.3) Minkowski deduced two important theorems of finiteness:

I. *A finite number of the inequalities* $\mathfrak{Z}$ *suffices to define the cell* Z.

II. *There is only a finite number of substitutions* S *capable of carrying a point of* Z *into a point of* Z (worded more precisely, the demand on the element S of $\{S\}$ which limits it to a finite set is to the effect that there should exist two points f, f' in Z such that $f' = Sf$).

Minkowski has a straightforward algebraic proof of II, in § 7 of his paper "Diskontinuitätsbereich für arithmetische Äquivalenz"†. His proof of I is involved because he establishes I simultaneously with an inequality of the nature of (8.3) by a complicated induction. This entanglement was dissolved in a joint paper by L. Bieberbach and I. Schur‡ (oh tempi passati!). They improve on I by breaking it up into the two theorems:

I*. *There is only a finite number of lattice points* x *in* $\mathfrak{R}_k$ *for which there exists a reduced form* $f_0 = \{g^0_{ij}\}$ *satisfying the equation*

$$f_0(x) = g^0_{kk}. \tag{8.6}$$

I**. *If we substitute for* x *in succession each of the lattice points assigned by Theorem* I*, *then the corresponding inequalities*

$$f(x) \geqq g_{kk} \tag{8.7}$$

for $k = 1, \ldots, n$ *completely close in the cell* Z *in* G.

While the proof of I* again is a straightforward algebraic affair based on (8.3), I** depends on the topological argument that a straight line joining a point inside to a point outside a convex region must cross the border. I explain the Bieberbach-Schur argument for I** in this simplified form.

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† *Journal für Math.*, 129 (1905), 220–274; *Gesammelte Abhandlungen* 2 (Leipzig, 1911), 53–100.

‡ *Sitzungsber. Berlin Akad.*, (1928), 519–523; (1929), 508.

I say that those points f (in G) for which all the inequalities $\mathfrak{Z}$ hold with the sign $>$ rather than $\geqslant$ form the *core* of Z. Any diagonal form

$$g_1 x_1^2 + \ldots + g_n x_n^2 \quad \text{with} \quad 0 < g_1 < g_2 < \ldots < g_n$$

evidently belongs to the core of Z. So do the inner points of Z. I maintain that conversely every point of the core is an inner point of Z. Indeed if f is a positive form, U_ϵ a sufficiently small neighbourhood of f and A any positive number, then the points x of the x-space such that $f_\epsilon(x) \leqslant A$ holds for at least one f_ϵ in U_ϵ belong to a limited region. Therefore all lattice points x with a finite number of exceptions satisfy the inequality $f_\epsilon(x) > A$ for every f_ϵ in U_ϵ. This follows readily from Jacobi's step transformation of f_ϵ into a square sum. Hence we have a finite subset $\mathfrak{Z}^0$ of $\mathfrak{Z}$ such that every form f_ϵ in U_ϵ satisfies all inequalities of $\mathfrak{Z} - \mathfrak{Z}^0$. Provided that the inequalities $\mathfrak{Z}^0$ hold for f with the $>$ sign, we may shrink U_ϵ so as to have the inequalities $\mathfrak{Z}^0$ satisfied throughout U_ϵ. This establishes I** by the topological argument mentioned above.

Let f_0 again be a point on the boundary of Z and let x range over the finite number of lattice points x in $\mathfrak{R}_k$ which satisfy (8.6). Studying the convex wedge defined by the corresponding simultaneous inequalities (8.7), we find† that those inequalities $\mathfrak{Z}$ suffice which are satisfied with the $=$ sign for $N-1$ linearly independent points f in Z. They are at the same time indispensable, and the corresponding equations describe the $(N-1)$-dimensional planes which contain the faces of the pyramidal cell Z in G.

Another way of constructing these faces would be by determining the common boundaries of Z with the adjacent equivalent cells Z_S. But Z and Z_S have a point f_0 in common only if both f_0 and $S^{-1}f_0$ are in Z. Hence the finiteness of the number of plane faces must also be a consequence of II, so that I ought to be deducible from II. Let us examine the situation more carefully.

Only the substitutions J are capable of transforming a point in the core of Z into a point of Z. Hence the two convex cells Z and Z_S have no inner point in common provided that S is not a J. The points which they have in common, if any, form a convex cone W in a linear submanifold of $N-1$ or ... or 1 dimensions. If $N-1$ is the correct number of dimensions we speak of a wall separating Z from Z_S. Choose f_0 as an inner point of W in its $(N-1)$-dimensional plane. It satisfies one of the inequalities $\mathfrak{Z}$, say (8.4), with the sign $=$ while every point of W satisfies it with the $\geqslant$ sign. Therefore the whole W clearly lies in the plane (8.5), and thus I is again established, *provided that we can show that every point f on the boundary of Z*

† For an elementary treatment see H. Weyl, *Comm. Math. Helvet.*, 7 (1934–5), 290–306.

lies on a wall separating Z from an adjacent cell Z_S. We deduce this from the fact, important in itself, that the equivalent cells cluster only towards the boundary of G.

On closer examination Minkowski's proof of II reveals that there is not more than a finite number of S capable of carrying a point of Z into a point of $G(\rho, \mu)$. Here ρ is any number greater than 1 and μ is any positive number. If D_k denotes the determinant of the form $f(x_1 \dots x_k\, 0 \dots 0)$, the subcone $G(\rho, \mu)$ of G is defined by the following simultaneous inequalities:

$$\begin{aligned} D_k &\geqslant \frac{\lambda_k}{\rho}\, g_{11} \cdots g_{kk}, \\ g_{kk} &\geqslant \frac{1}{\rho}\, g_{k-1,\,k-1} \quad [g_{00} = 0], \\ f(x_1, \dots, x_{k-1}, 1, 0, \dots, 0) &\geqslant g_{kk} - \mu g_{11} \end{aligned} \qquad (k = 1, \dots, n)$$

for any integers $x_1, \dots, x_{k-1}$.

All points of Z are *inner* points of $G(\rho, \mu)$. This is more than sufficient to settle our question. Choose a definite $\rho > 1$ and $\mu > 0$. Let f be a point on the boundary of Z and let $f_1, \dots, f_\nu, \dots$ be a sequence of points outside Z approaching f. We may assume that they lie in $G(\rho, \mu)$. A certain substitution S_ν^{-1} of $\{S\}$ will carry f_ν into a point $f_\nu' = S_\nu^{-1} f_\nu$ of Z. In consequence of our statement about $G(\rho, \mu)$, a certain S recurs infinitely often among the S_ν. This S is no J. The points f_ν of the corresponding selected sequence all lie in Z_S, and so does their limit f. Consequently any point f on the boundary of Z belongs to the common boundary of Z and an equivalent Z_S. The domain $G(\rho, \mu)$ increases when ρ and μ increase, and with $\rho \to \infty$, $\mu \to \infty$ exhausts the whole G. At every stage $G(\rho, \mu)$ has points in common with only a finite number of cells Z_S. An elementary argument like the one applied before shows that among the common boundaries $Z \mid Z_S$ (which exist in finite number only) none but the "walls", the $(N-1)$-dimensional ones, matter. The substitutions S which carry Z into cells Z_S bordering on Z along a wall, together with the J, generate the entire group $\{S\}$.

The Generalized Theorem V proves that all our statements about $G(\rho, \mu)$ remain true if we define it by the following *linear* inequalities:

$$f(x_1, \dots, x_n) \geqslant \frac{1}{\rho}\, g_{kk}$$

whenever the x_i are integers and $x_k, \dots, x_n$ have no common factor,

$$f(x_1, \dots, x_n) \geqslant g_{kk} - \mu g_{11}$$

whenever the x_i are integers and $(x_k, x_{k+1}, \ldots, x_n) = (1, 0, \ldots, 0)$. Incidentally the estimates may be improved considerably if we carry out the comparison of the coefficients g_{kk}, g'_{kk} of two equivalent forms f, f' in Z and $G(\rho, \mu)$ respectively for any convex bodies rather than ellipsoids by the procedure of § 6.

Throughout we have studied Z relative to G; this is really the more natural standpoint. However, the results I and I** are true even relative to the space R of all quadratic forms. Z is not closed in R; there may be boundary points f of Z relative to R which are not positive forms. I maintain that for such a form the equation $g_{11} = 0$ holds. Since f is non-negative, it can be brought into the form

$$\xi_1^2 + \ldots + \xi_m^2 \quad (m < n)$$

by a transformation with real coefficients. We can determine a lattice point $x \neq 0$ for which the m linear forms $\xi_1, \ldots, \xi_m$ are in absolute value less than any preassigned positive number ϵ, either by Minkowski's theorem (1.1) for parallelotopes, or by a simple application of Dirichlet's principle of the distribution of $\nu+1$ objects over ν boxes. Hence $g_{11} > 0$ would contradict the inequality $f(x) \geqq g_{11}$ which holds for all lattice points $x \neq 0$, and Z relative to R is defined by the inequalities described under I* and I** together with $g_{11} \geqq 0$. But even the addition of $g_{11} \geqq 0$ becomes superfluous if $n \geqq 2$. The substitution

$$x_1 = 1, \quad x_2 = \pm 1, \quad x_3 = \ldots = x_n = 0$$

yields the inequality

$$g_{11} \pm 2g_{12} + g_{22} \geqq g_{22}$$

for any reduced form, or $2|g_{12}| \leqq g_{11}$. The same holds for $g_{13}, \ldots, g_{1n}$. Hence the boundary form f even satisfies the n equations

$$g_{11} = g_{12} = \ldots = g_{1n} = 0. \tag{8.8}$$

The fact that Z borders on the plane $g_{11} = 0$ only along this linear variety (8.8) of n dimensions less, proves our point.

Minkowski's investigations set out from the reduction of quadratic forms. Owing to some "unexpected difficulties" the second half of his book *Geometrie der Zahlen* never appeared, and fourteen years after the discovery of the basic idea of the geometry of numbers he published his theory of reduction in a strictly arithmetical form, without a hint of geometric approach. What was the snag he struck? After having taken the main hurdle with the general inequality (1.2), did he fail to notice the simple remark that led us on to (6.2), or was he unable to unravel the

inductive kink by which in his final publication the establishment of the inequality (6.2) is tied to the selection of a finite number of inequalities $\mathfrak{Z}$? It is now idle to speculate. But of him it might be said as of Saul that he went out to look after his father's asses and found a kingdom.

For quadratic forms there is a simple algebraic method of deriving (1.2) from (1.1). Minkowski describes it in *Geometrie der Zahlen*, § 51; and very likely this suggested to him the general inequality (1.2). Hence every improvement of the constant γ_n in the inequality for positive forms

$$M^n \leqslant \gamma_n D$$

in which M is the minimum of $f(x)$ for all lattice points $x \neq 0$ entails a proportional improvement in (1.2) and for the constants μ_n and ρ_n in the estimates (6.2), (7.1) for ellipsoids. Our value was $\gamma_n = (2^n/\omega_n)^2$. Blichfeldt† succeeded in improving it by the factor

$$(1+\tfrac{1}{2}n)^2 \cdot 2^{-n}.$$

This leads to the following λ_n:

$$1/\lambda_n = 2^n(1+\tfrac{1}{2}n)^2\,\omega_n^{-2} \cdot (\tfrac{9}{4})^{\frac{1}{4}(n-1)(n-2)}. \tag{8.9}$$

Remak recently obtained a better estimate‡ which differs from (8.9) essentially by having $\frac{5}{4}$ instead of $\frac{9}{4}$. However, the chief interest of our constant μ_n in (6.2) lies in its validity for all convex solids whatsoever.

Theorem VI together with Blichfeldt's improved γ_n shows that every positive form has an equivalent one f satisfying the relation

$$\lambda^*{}_n\, g_{11} \cdots g_{nn} \leqslant D,$$

where

$$1/\lambda_n{}^* = (1+\tfrac{1}{2}n)^2\, 2^{-n}\, \omega_n^{-2}\, \rho_n{}^2.$$

Such an estimate is best suited to prove that there is only a finite number of classes of equivalent positive forms with integral coefficients.

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† *Trans. American Math. Soc.*, 15 (1914), 227–235; *cf.* R. Remak, *Math. Zeitschrift*, 26 (1927), 694–699, and Blichfeldt, *Math. Annalen*, 101 (1929), 605–608.

‡ *Compositio Math.*, 5 (1938), 368–391.

ON THE CONVERGENCE OF FOURIER-BESSEL SERIES

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1. *Introduction.* This paper concerns the Fourier-Bessel series of a function $f(x)$ defined for $0 \leqslant x \leqslant 1$, or, more precisely, the Fourier expansion of the function $x^{\frac{1}{2}}f(x)$ in terms of the system of normal and orthogonal functions

$$\sqrt{(2x)}\, J_\nu(j_m x)/J_{\nu+1}(j_m), \tag{1.1}$$

where, in the notation of Watson's book on Bessel functions†, the expression

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \tag{1.2}$$

denotes the Bessel function of order ν of the first kind, and j_m is its m-th positive zero in ascending order of magnitude. The index ν is fixed in (1.1) and we suppose, with Watson‡, that

$$\nu+\tfrac{1}{2} \geqslant 0. \tag{1.3}$$

It is convenient to stipulate further, unless the contrary is explicitly stated, that we have

$$x^{\frac{1}{2}}f(x) = 0 \text{ *when* } x = 0 \text{ *and when* } x = 1, \tag{1.4}$$

in view of the fact that the functions (1.1) vanish at the end points 0 and 1.

The problem of the convergence of our series in the partial interval $\epsilon \leqslant x \leqslant 1-\epsilon$, where $\epsilon > 0$, may be regarded as settled by the memoir of

† Watson, **1**, [p. 40 (8)], hereafter cited as W.
‡ W., 582.

W. H. Young†, written some twenty years ago, in which it is reduced to the problem for trigonometric Fourier series. There remain, however, the problems of convergence for $0 \leqslant x \leqslant \epsilon$ and for $1-\epsilon \leqslant x \leqslant 1$, and it is with these that the present paper is primarily concerned. A discussion of these two problems is given by Watson in his book‡. He supposes that the function $x^{\frac{1}{2}}f(x)$ is of bounded variation and that it satisfies

(i) at the point 0 a *Lipschitz condition*, Lip $(\nu+\frac{1}{2})$, which requires $x^{-\nu}f(x)$ to tend to 0 with x,

(ii) at the point 1 a mere *continuity condition*.

It is the object of the present note to extend Watson's results, first by weakening the condition (i) to a mere continuity condition and thus restoring the *symmetry* between (i) and (ii), and secondly by interpreting the *notion of bounded variation* in a much more general sense, by means of the notions of p-th power and exponential variation, which have already proved important in the convergence theory for trigonometric Fourier series§.

We state and prove our main results for the whole interval $0 \leqslant x \leqslant 1$, rather than for the parts $0 \leqslant x \leqslant \epsilon$ and $1-\epsilon \leqslant x \leqslant 1$. This is because those of our results which bear on the remaining part $\epsilon \leqslant x \leqslant 1-\epsilon$ appear to follow most directly from the fundamental inequalities for higher variations and Stieltjes integrals‖ without employing the corresponding results which I have established previously for trigonometric Fourier series.

2. *Notation and relevant formulae.* The sum to n terms of the Fourier expansion of $x^{\frac{1}{2}}f(x)$, in the generalized sense, as a series of the normal and orthogonal functions (1.1), is $x^{\frac{1}{2}}s_n(x)$, where

$$(2.1) \qquad s_n(x) = \sum_{m=1}^{n} J_\nu(j_m x) \frac{\int_0^1 tf(t)\, J_\nu(j_m t)\, dt}{\frac{1}{2}J_{\nu+1}^2(j_m)}$$

$$= \int_0^1 tf(t)\, T_n(t, x)\, dt,$$

† W. H. Young, **2**.
‡ W., 594 and 615.
§ L. C. Young, **4**. We do not use these notions until § 7.
‖ L. C. Young, **1, 2, 3**.

and it is shown in Watson's book† that

$$(2.2)\quad T_n(t,x)=\sum_{m=1}^{n}\frac{2J_\nu(j_m x)\,J_\nu(j_m t)}{J_{\nu+1}^2(j_m)}$$

$$=\frac{1}{\pi i(t^2-x^2)}\int_{A_n-i\infty}^{A_n+i\infty}\{tJ_\nu(xw)\,J_{\nu+1}(tw)-xJ_\nu(tw)\,J_{\nu+1}(xw)\}\frac{dw}{J_\nu^2(w)},$$

provided that we choose A_n so that $j_n<A_n<j_{n+1}$; moreover‡

$$(2.3)\quad \int_0^t t^{\nu+1}T_n(t,x)(t^2-x^2)\,dt$$

$$=\frac{t^{\nu+1}}{\pi i}\int_{A_n-i\infty}^{A_n+i\infty}\{tJ_\nu(xw)\,J_{\nu+2}(tw)-xJ_{\nu+1}(xw)\,J_{\nu+1}(tw)\}\frac{dw}{wJ_\nu^2(w)}.$$

We shall take over, from the convergence theory for the partial interval $\epsilon\leqslant x\leqslant 1-\epsilon$, the following results which are obtained by choosing two particular functions§ of bounded variation $x^{\frac{1}{2}}f(x)$:

(2.4) *As $n\to\infty$, we have, for $0<x<1$,*

$$\int_0^x t^{\nu+\frac{1}{2}}\sqrt{(xt)}\,T_n(t,x)\,dt\to\tfrac{1}{2}x^{\nu+\frac{1}{2}},$$

and we have uniformly, for $\epsilon\leqslant x\leqslant 1-\epsilon$,

$$\int_0^1 t^{\nu+\frac{1}{2}}\sqrt{(xt)}\,T_n(t,x)\,dt\to x^{\nu+\frac{1}{2}}.$$

Actually, in Watson's book, these results are proved directly‖.

Finally, we quote for reference some simple consequences of the asymptotic expansion of a Bessel function¶. Here and in the rest of the present paper, K denotes a positive constant, whose precise value does not interest us and is not necessarily the same in different formulae. Also $R(z)$ and $I(z)$ denote the real and imaginary parts of a complex variable z.

† W., 581–585.

‡ See the preceding footnote.

§ The first of these, strictly, depends on an additional parameter: we choose $t^{\frac{1}{2}}f(t)$ to be $t^{\nu+\frac{1}{2}}$ for $0<t<x$ and to be 0 elsewhere. The second is simply the function $t^{\frac{1}{2}}f(t)=t^{\nu+\frac{1}{2}}$ for $0<t<1$ and 0 for $t=0$ and for $t=1$.

‖ W., 585, § 18.22.

¶ W., 199 (1).

We have

$$(2.5) \qquad \left| z^{\frac{1}{2}} J_\nu(z) - \sqrt{\left(\frac{2}{\pi}\right)} \cos\left[z - \tfrac{1}{2}(\nu+\tfrac{1}{2})\pi\right] \right| < \frac{K}{|z|} \exp\left(|I|(z)\right)$$

for large $|z|$ provided that $R(z) > 0$ (or, more generally, provided that $|\arg z| < \pi - \epsilon$). Moreover

$$(2.6) \qquad j_m = m\pi + \tfrac{1}{2}(\nu - \tfrac{1}{2})\pi + o(1), \quad J^2_{\nu+1}(j_m) \sim 1/j_m,$$

and in (2.2) and (2.3) we may take

$$(2.7) \qquad A_n = n\pi + \tfrac{1}{2}(\nu+\tfrac{1}{2})\pi.$$

With this value for A_n, we have, on account of (2.5), the following two inequalities†, valid for $0 \leqslant t \leqslant 1$ and $R(w) = A_n$, *uniformly* when n is sufficiently large,

$$(2.8) \qquad |\sqrt{(tw)}\, J_\nu(tw)| < K \exp(|I(tw)|), \quad |w^{\frac{1}{2}} J_\nu(w)| > K \exp\left(I\,|(w)|\right).$$

3. *Some estimates of magnitude.* An essential part will be played by the following rough estimates of magnitude, which are valid uniformly for $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 1$:

$$(3.1) \qquad \sqrt{(xt)}\,|T_n(t, x)| \leqslant Kn,$$

$$(3.2) \qquad \sqrt{(xt)}\,|T_n(t, x)| \leqslant K/|t-x|,$$

$$(3.3) \qquad \left| \int_0^t t^{\nu+1} T_n(t, x)(t^2 - x^2)\, dt \right| \leqslant K \frac{(t+x)\, t^{\nu+1}}{n\sqrt{(xt)}}.$$

The corresponding inequalities given by Watson‡ are just not good enough for our purpose: they are based only on (2.8) and so contain factors which the present more detailed calculations show to be superfluous.

Proof of (3.1). This inequality is obvious when we use the expression, given in (2.2), of $T_n(t, x)$ as a *sum* of n terms. For the modulus of each term is less than $K/\sqrt{(xt)}$ on account of (2.6), since $z^{\frac{1}{2}} J_\nu(z)$ is bounded for real positive z.

Proof of (3.2). We now employ the *integral* representation of $T_n(t, x)$, also given in (2.2), and we choose A_n as in (2.7). We may suppose, in view of (3.1), that $|t-x| > 1/n$. This implies, by (2.7),

$$(3.4) \qquad (2-x-t)\, A_n > K,$$

† W., 584 (9).
‡ W., 585 (10) and 585 (11).

since we have $2-x-t \geqslant 1-x \geqslant t-x$ and $2-x-t \geqslant 1-t \geqslant x-t$. It is, moreover, easy to establish (3.2) in the case in which one of the numbers x and t does not exceed $\frac{1}{2}$; for in that case we have $2-x-t \geqslant \frac{1}{2} > K$ and so

$$\pi|(t^2-x^2)\,T_n(t,x)|$$

$$\leqslant t\int_{An-i\infty}^{An+i\infty} |J_\nu(xw)\,J_{\nu+1}(tw)|\,\frac{|dw|}{|J_\nu^2(w)|} + x\int_{An-i\infty}^{An+i\infty} |J_\nu(tw)\,J_{\nu+1}(xw)|\,\frac{|dw|}{|J_\nu^2(w)|}$$

$$\leqslant (t+x)\,\frac{K}{\sqrt{(xt)}}\int_0^\infty e^{-(2-x-t)v}\,dv \leqslant K\,(t+x)/\sqrt{(xt)},$$

by a straightforward application of (2.8).

In addition to (3.4), we may therefore assume that

$$x > \tfrac{1}{2} \quad \textit{and} \quad t > \tfrac{1}{2}. \tag{3.5}$$

This being so, we write

$$T_n(t,x) = T'_n(t,x) + T_n''(t,x),$$

where $T_n'(t,x)$ is obtained by substituting, with suitable square root factors, the cosines occurring in the appropriate forms of (2.5), for the Bessel functions which involve x or t, in the integral representing $T_n(t,x)$. Then

$$T_n'(t,x) = \frac{1}{2\pi i(t^2-x^2)}\int_{An-i\infty}^{An+i\infty} \frac{2}{\pi}\,g'(w,x,t)\,\frac{dw}{\sqrt{(xt)}\,J_\nu^2(w)},$$

where $g'(w,x,t)$ denotes the expression

$$2t\cos[xw-\tfrac{1}{2}(\nu-\tfrac{1}{2})\pi]\cos[tw-\tfrac{1}{2}(\nu+\tfrac{3}{2})\pi]$$

$$-2x\cos[tw-\tfrac{1}{2}(\nu+\tfrac{1}{2})\pi]\cos[xw-\tfrac{1}{2}(\nu+\tfrac{3}{2})\pi],$$

which may be written simply

$$(t+x)\cos[(t-x)w-\tfrac{1}{2}\pi]-(t-x)\cos[(t+x)w-\nu\pi].$$

Hence, from (2.8) and the corresponding trivial inequalities for the cosines, together with the inequalities $|t-x|<1$ and $|t-x|<2-t-x$, we find that

$$2\pi|(t^2-x^2)\,T_n'(t,x)| \leqslant \frac{K}{\sqrt{(xt)}}\left\{(t+x)\int_0^\infty e^{(|t-x|-2)v}\,dv + |t-x|\int_0^\infty e^{(t+x-2)v}\,dv\right\}$$

$$\leqslant \frac{K}{\sqrt{(xt)}}\left\{2\int_0^\infty e^{-v}\,dv + (2-x-t)\int_0^\infty e^{-(2-x-t)v}\,dv\right\}$$

and, in view of (3.5), we at once deduce that

$$T_n'(t, x) \leqslant \frac{K}{|t-x|\sqrt{(xt)}}. \tag{3.6}$$

On the other hand, we have

$$T_n''(t, x) = \frac{1}{2\pi i(t^2-x^2)} \int_{A_n-i\infty}^{A_n+i\infty} \frac{g''(w, x, t)}{\sqrt{(xt)}\, w} \frac{dw}{J_\nu^2(w)},$$

where $g''(w, x, t)$ denotes the difference

$$2\sqrt{(xt)}\, w\{tJ_\nu(xw)\, J_{\nu+1}(tw) - xJ_\nu(tw)\, J_{\nu+1}(xw)\} - \frac{2}{\pi} g'(w, x, t),$$

so that, by (2.5) and (3.5),

$$|g''(w, x, t)| < \frac{K}{|w|} \exp[(x+t)|I(w)|].$$

Hence, by (2.8),

$$|(t^2-x^2)\, T_n''(t, x)| \leqslant K \int_{A_n-i\infty}^{A_n+i\infty} \frac{\exp[(x+t-2)|I(w)|]}{\sqrt{(xt)}\,|w|} |dw|$$

$$\leqslant K \int_0^\infty \frac{e^{-(2-x-t)v}\, dv}{A_n \sqrt{(xt)}}$$

and, in view of (3.4) and (3.5), this gives

$$|T_n''(t, x)| < \frac{K}{|t-x|\sqrt{(xt)}}, \tag{3.7}$$

so that, by adding the inequalities (3.6) and (3.7), we obtain (3.2) and the proof is complete.

Proof of (3.3). We denote by Φ the expression on the right of (2.3) and we argue with it as with $T_n(t, x)$ in the proof of (3.2), except for a trivial simplification.

If at least one of the numbers x, t does not exceed $\frac{1}{2}$, we have, by (2.8),

$$|\Phi| \leqslant Kt^{\nu+1} \int_{A_n-i\infty}^{A_n+i\infty} (t+x) \frac{\exp[-(2-x-t)|I(w)|]}{\sqrt{(xt)}} \frac{|dw|}{|w|}$$

$$\leqslant \frac{K(t+x)\, t^{\nu+1}}{\sqrt{(xt)}\, A_n} \int_0^\infty e^{-(2-x-t)v}\, dv$$

and this last integral may be absorbed into the K.

We may, therefore, suppose, as before, that x and t are both greater than $\frac{1}{2}$, in which case, by (2.5), we may write

$$\Phi = \frac{t^{\nu+1}}{2\pi i}\int_{A_n-i\infty}^{A_n+i\infty} \frac{2}{\pi}\{h'(w, x, t)+h''(w, x, t)\}\frac{dw}{\sqrt{(xt)}\, w^2 J_\nu^2(w)},$$

where $h'(w, x, t)$ denotes the expression

$$2t\cos[xw-\tfrac{1}{2}(\nu+\tfrac{1}{2})\pi]\cos[tw-\tfrac{1}{2}(\nu+\tfrac{5}{2})\pi]$$
$$-2x\cos[xw-\tfrac{1}{2}(\nu+\tfrac{3}{2})\pi]\cos[tw-\tfrac{1}{2}(\nu+\tfrac{3}{2})\pi],$$

which we write in the more convenient form

$$(x-t)\cos[(x+t)w-(\nu+\tfrac{1}{2})\pi]-(t+x)\cos[(x-t)w],$$

and

$$|h''(w, x, t)| \leqslant \frac{\exp[(x+t)|I(w)|]}{|w|}.$$

From this, by majorizing crudely and by applying (2.8), it is now easy to see that $|\Phi|$ cannot exceed a constant multiple of the following combination of three integrals

$$|x-t|\int_0^\infty e^{-(2-x-t)v}\frac{dv}{\sqrt{(A_n^2+v^2)}}+(x+t)\int_0^\infty e^{-(2-|x-t|)v}\frac{dv}{\sqrt{(A_n^2+v^2)}}$$
$$+\int_0^\infty e^{-(2-x-t)v}\frac{dv}{A_n^2+v^2},$$

and this clearly cannot exceed

$$\frac{|x-t|}{(2-x-t)A_n}+\frac{K}{A_n}+\int_0^\infty \frac{dv}{A_n^2+v^2};$$

here the three terms are at most K/A_n, which may also be written as

$$\frac{K(x+t)t^{\nu+1}}{A_n\sqrt{(xt)}},$$

since x and t are supposed to exceed $\frac{1}{2}$. This completes the proof of (3.3).

4. *First properties of the integrated Fourier-Bessel kernels* $g_n(t)$. We are not interested in the expressions (2.2) and (2.3) for themselves, but in the related expression

$$g_n(t) = \int_0^t \sqrt{(xt)}\, T_n(t, x)\, dt \tag{4.1}$$

considered as a function of t depending on the further parameters n and x. We call $g_n(t)$ the "integrated Fourier-Bessel kernel."

The central idea of the present mode of attack on the Fourier-Bessel convergence problem consists in deriving as much information as we can

about $g_n(t)$ merely from the rough estimates of the magnitude of the expressions (2.2) and (2.3), obtained in the preceding paragraph.

We begin by establishing the results which constitute a basis for a Fourier-Bessel *localization principle.*

(4.2) *Given* $\delta > 0$, *we have uniformly in* n, t, x:

(4.3) $|g_n'(t)| < K(\delta)$ *when* $|t-x| \geqslant \delta$, *where* $K(\delta)$ *is a positive number depending on* δ *but independent of* n, t, x. Moreover

$$\underset{t}{\text{Osc}}\,[g_n(t);\ 0 \leqslant t \leqslant x-\delta] < K(\delta)\frac{\log n}{n}, \tag{4.4}$$

$$\underset{t}{\text{Osc}}\,[g_n(t);\ x+\delta \leqslant t \leqslant 1] < \frac{K(\delta)}{n}. \tag{4.5}$$

Of these results, (4.3) and (4.5) are straightforward, but (4.4), which already differs in form, by the intrusion of a logarithm, from the corresponding result for ordinary trigonometric Fourier kernels, requires a slight device which will play an even more important part later.

Proof of (4.3). It is sufficient to remark that this follows at once from (3.2), since the derivative $g_n'(t)$ of $g_n(t)$ is simply $\sqrt{(xt)}T_n(t, x)$.

Proof of (4.5). The factors

$$t^{-(\nu+\frac{1}{2})} \quad \text{and} \quad (t^2-x^2)^{-1}$$

are monotonic and do not exceed $K(\delta)$ when t lies in the interval

$$x+\delta \leqslant t \leqslant 1.$$

Hence, by the second mean-value theorem, for any two values t', t'', where $t' < t''$, in this interval, we have

$$|g_n(t'')-g_n(t')| = \frac{t'^{-(\nu+\frac{1}{2})}x^{\frac{1}{2}}}{t'^2-x^2}\int_{t'}^{t''} t^{\nu+1}T_n(t,x)(t^2-x^2)\,dt \qquad (\text{where } t' < t''' < t'')$$

$$\leqslant \frac{K(\delta)}{n},$$

by (3.3). The inequality (4.5) follows at once.

Proof of (4.4). Instead of the oscillation of $g_n(t)$, we may, by the second mean value theorem, consider that of the function of t

$$g_n^*(t) = \int_0^t \sqrt{(xt)}\,T_n(t,x)(t-x)\,dt$$

since $x-t$ is monotonic and exceeds $K(\delta)$ for $0 \leqslant t \leqslant x-\delta$.

Now in any interval of the form $\frac{1}{2}u \leqslant t \leqslant u$, by the second mean value theorem applied to the factor $(t+x)^{-1}t^{-(\nu+\frac{1}{2})}$, the oscillation of $g_n^*(t)$ cannot exceed

$$K(u+2x)^{-1}u^{-(\nu+\frac{1}{2})}x^{\frac{1}{2}} \operatorname*{Osc}_t \left\{\int_0^t t^{\nu+1}T_n(t,x)(t^2-x^2)\,dt\right\}$$

and therefore, by (3.3), cannot exceed K/n. Hence, choosing successively $u = x-\delta$, $u = \frac{1}{2}(x-\delta)$, ..., $u = 2^{-(p-1)}(x-\delta)$ and adding the corresponding oscillations, we see that

$$\operatorname*{Osc}_t [g_n^*(t);\ 2^{-p}(x-\delta) \leqslant t \leqslant x-\delta] \leqslant Kp/n. \tag{4.6}$$

Moreover, by (3.2), since the oscillation cannot exceed the total variation,

$$\operatorname*{Osc}_t [g_n(t);\ 0 \leqslant t \leqslant 2^{-p}(x-\delta)] < K2^{-p}/\delta. \tag{4.7}$$

Now it follows, from statements already made, that in (4.6) we may replace $g_n^*(t)$ by $g_n(t)$ provided that we replace on the right K by $K(\delta)$. Hence, by choosing $p = \log n$ and combining (4.6) with (4.7), we obtain (4.4) as desired.

5. *Existence of a bound for the $g_n(t)$.* We now prove that

(5.1) *There exists a positive constant K independent of n, x, t such that*

$$|g_n(t)| < K. \tag{5.2}$$

It is clearly sufficient to prove, instead of (5.2), the inequality

$$\operatorname*{Osc}_t [g_n(t);\ 0 \leqslant t \leqslant 1] < K, \tag{5.3}$$

which is equivalent to (5.2) since $g_n(0) = 0$.

Again, instead of (5.3) it is clearly sufficient to prove the three inequalities

$$\begin{cases} \operatorname*{Osc}_t [g_n(t);\ 0 \leqslant t \leqslant \frac{2}{3}x] < K, \\ \operatorname*{Osc}_t [g_n(t);\ \frac{2}{3}x \leqslant t \leqslant \frac{4}{3}x] < K, \\ \operatorname*{Osc}_t [g_n(t);\ \frac{4}{3}x \leqslant t \leqslant 1] < K. \end{cases} \tag{5.4}$$

The first of these three inequalities is trivial. In fact, by (3.2),

$$\int_0^{\frac{2}{3}x} |g_n'(t)|\,dt < K. \tag{5.5}$$

In order to establish the second inequality (5.4), we write, as in the proof of (4.4) in the preceding paragraph,

$$g_n^*(t) = \int_0^t \sqrt{(xt)}\, T_n(t,x)\,(t-x)\,dt.$$

Evidently, by the second mean-value theorem applied to the factor $t^{-(\nu+\frac{1}{2})}(x+t)^{-1}$, (3.3) implies that

$$\operatorname{Osc}\,[g_n^*(t);\ \tfrac{2}{3}x \leqslant t \leqslant \tfrac{4}{3}x] < K/n.$$

Consequently, if $0 < u < \frac{1}{3}x$, the same theorem gives

$$(5.6)\qquad \begin{cases} \operatorname{Osc}\,[g_n(t);\ x-u \leqslant t \leqslant x-\frac{1}{2}u] < K/(nu), \\ \operatorname{Osc}\,[g_n(t);\ x+\frac{1}{2}u \leqslant t \leqslant x+u] < K/(nu). \end{cases}$$

If we now add the inequalities obtained from (5.6) by choosing successively for u those values of the form $\frac{1}{3}x2^{-r}$ which exceed $1/n$, where r denotes a non-negative integer, we see that the sum of the right-hand sides is less than a fixed constant K, since†

$$\sum_{u>1/n}{}' \frac{1}{nu} < \sum_{m=0}^{\infty} 2^{-m} < K.$$

The second inequality (5.4) now follows at once, if we observe that, by (3.1), the oscillation of $g_n(t)$ in the remaining part of $\frac{2}{3}x \leqslant t \leqslant \frac{4}{3}x$ is trivial, since we actually have

$$(5.7)\qquad \int_{x-1/n}^{x+1/n} |g_n'(t)|\,dt < K.$$

In order, finally, to establish the third inequality (5.4), we first remark that, in any sub-interval of $\frac{4}{3}x \leqslant t \leqslant 1$, the oscillation of the function

$$\bar{g}_n(t) = \int_0^t t^{\nu+2\frac{1}{2}} \sqrt{(xt)}\, T_n(t,x)\,dt$$

cannot exceed a certain fixed constant K multiplied by the corresponding oscillation of the function

$$\int_0^t t^{\nu+1} x^{\frac{1}{2}}\, T_n(t,x)(t^2-x^2)\,dt;$$

this is an immediate consequence of the second mean value theorem applied with the factor $t^2/(t^2-x^2)$ to this last integral. Hence, for $\frac{8}{3}x \leqslant u \leqslant 1$,

† The index m of the second summation is connected with our former index r by the equation $r+m=N$, where N is the number of relevant values of u.

we have, by (3.3),

$$\text{Osc}\,[\bar{g}_n(t);\ \tfrac{1}{2}u \leqslant t \leqslant u] < Ku^{\nu+\frac{3}{2}}/n,$$

and therefore

$$\text{Osc}\,[g_n(t);\ \tfrac{1}{2}u \leqslant t \leqslant u] < K/(nu), \tag{5.8}$$

since, by the second mean-value theorem, the left-hand side of (5.8) is at most $Ku^{-(\nu+2\frac{1}{2})}$ times the corresponding oscillation of $\bar{g}_n(t)$.

If we now add the inequalities obtained from (5.8) by choosing successively $u = 2^{-r}$, where r denotes a non-negative integer and where $u \geqslant a = \max(1/n, \frac{8}{3}x)$, the sum of the right-hand sides is less than a fixed constant K, as before. The third inequality (5.4) now follows at once, if we observe that the inequalities (5.7) and

$$\int_{\frac{3}{8}x}^{\frac{8}{3}x} |g_n'(t)|\,dt < K, \tag{5.9}$$

which is an immediate consequence of (3.2), show that the oscillation of $g_n(t)$ for $\frac{4}{3}x \leqslant t \leqslant a$ is trivial.

6. *First applications.* We now establish the following result.

(6.1) *Let $x^{\frac{1}{2}}f(x)$ denote a function of bounded variation in $0 \leqslant x \leqslant 1$, vanishing for $x = 0$ and for $x = 1$. Then* (i), as $n \to \infty$, *we have, for $0 < x < 1$,*

$$x^{\frac{1}{2}}s_n(x) \to x^{\frac{1}{2}}\left\{\frac{f(x+0)+f(x-0)}{2}\right\}$$

boundedly; moreover (ii), *if $x^{\frac{1}{2}}f(x)$ is, furthermore, continuous, we have, uniformly in x, for $0 \leqslant x \leqslant 1$,*

$$x^{\frac{1}{2}}s_n(x) \to x^{\frac{1}{2}}f(x).$$

To prove this, we require a few simple preliminary lemmas.

(6.2) *"Localization" principle. Given any fixed $\delta > 0$, the integrals*

$$\int_{0 \leqslant t \leqslant x-\delta} t^{\frac{1}{2}}f(t)\sqrt{(xt)}\,T_n(t, x)\,dt \quad \text{and} \quad \int_{x+\delta \leqslant t \leqslant 1} t^{\frac{1}{2}}f(t)\sqrt{(xt)}\,T_n(t, x)\,dt$$

tend to zero uniformly in x as $n \to \infty$ for $0 \leqslant x \leqslant 1$, provided that $t^{\frac{1}{2}}f(t)$ is Lebesgue integrable.

This follows at once from (4.2) by a classical Lebesgue theorem†.

(6.3) *With the hypotheses of* (6.1) (i), *the expression*‡

$$x^{\frac{1}{2}}s_n(x)-x^{\frac{1}{2}}f(x-0)\,g_n(x)-x^{\frac{1}{2}}f(x+0)[g_n(1)-g_n(x)]$$

tends to 0 *boundedly as* $n\to\infty$, *for* $0\leqslant x\leqslant 1$.

The modulus of the expression in question may be written successively

$$\left|\int_0^x\{t^{\frac{1}{2}}f(t)-x^{\frac{1}{2}}f(x-0)\}\,dg_n(t)+\int_x^1\{t^{\frac{1}{2}}f(t)-x^{\frac{1}{2}}f(x+0)\}\,dg_n(t)\right|$$

$$=\left|\int_{x-\delta}^x\{t^{\frac{1}{2}}f(t)-x^{\frac{1}{2}}f(x-0)\}\,dg_n(t)+\int_x^{x+\delta}\{t^{\frac{1}{2}}f(t)-x^{\frac{1}{2}}f(x+0)\}\,dg_n(t)+o(1)\right|$$

$$\leqslant K\tau(\delta;\ x)+K\tau(\delta;\ x)+o(1),$$

by the localization principle and by (5.1), where $\tau(\delta;\ x)$ denotes the total variation§ of $t^{\frac{1}{2}}f(t)$ in $x-\delta\leqslant t<x$ and $x<t\leqslant x+\delta$, which is bounded in x, δ, and, for fixed x, arbitrarily small with δ; and $o(1)$ denotes an expression which tends uniformly to 0 as $n\to\infty$, for fixed δ.

(6.4) *With the hypotheses of* (6.1) (ii), *the expression*

$$x^{\frac{1}{2}}s_n(x)-x^{\frac{1}{2}}f(x)\,g_n(1)$$

tends to 0 *uniformly, as* $n\to\infty$, *for* $0\leqslant x\leqslant 1$.

It is sufficient to observe that in the preceding proof $\tau(\delta;\ x)$ is now small with δ uniformly in x, and that $f(x+0)=f(x-0)=f(x)$‖.

To establish (6.1) it is now sufficient, in view of (5.1), to prove that, for $0<x<1$, $g_n(x)\to\frac{1}{2}$ as $n\to\infty$, and that, for $\epsilon\leqslant x\leqslant 1-\epsilon$, $g_n(1)\to 1$ *uniformly* as $n\to\infty$. This is a variant of (2.4) which might have been taken over from the convergence theory for the partial interval $\epsilon\leqslant x\leqslant 1-\epsilon$,

† Hobson, **1**, II, 422, Theorem I.

‡ The functions $t^{\frac{1}{2}}f(t)$ and $g_n(t)$ are supposed constant for $t\leqslant 0$ and for $t\geqslant 1$, if necessary, and we have $f(-0)=f(1+0)=0$.

§ We use the inequality (valid for $a\leqslant\xi\leqslant b$)

$$\int_a^b F(t)\,dG(t)\leqslant\{|F(\xi)|+V_1(F(t),\ a\leqslant t\leqslant b)\}\ \text{osc}\,[G(t);\ a\leqslant t\leqslant b],$$

where $V_1(F(t),\ a\leqslant t\leqslant b)$ is the total variation of $F(t)$ in $a\leqslant t\leqslant b$.

‖ At the origin these equations must be supposed multiplied through by $x^{\frac{1}{2}}$, since $f(+0)$ by itself need not even exist.

in which case (2.4), which is formally more complicated, could have been dispensed with. With the notation of (2.4) this variant is:

(6.5) *As* $n \to \infty$, *we have*, for $0 < x < 1$,

$$\int_0^x \sqrt{(xt)}\, T_n(t, x)\, dt \to \tfrac{1}{2},$$

and we have uniformly, for $\epsilon \leqslant x \leqslant 1-\epsilon$,

$$\int_0^1 \sqrt{(xt)}\, T_n(t, x)\, dt \to 1.$$

It is easy to deduce the variant from (2.4) itself, by using (6.3) and (6.4) with two particular functions $t^{\frac{1}{2}}f(t)$. Thus, taking $t^{\frac{1}{2}}f(t) = t^{\nu+\frac{1}{2}}$ for $0 < t < x$ and 0 for $t \geqslant x$, (6.3) asserts that $x^{\frac{1}{2}}s_n(x) - x^{\nu+\frac{1}{2}}g_n(x) \to 0$, and (2.4) that $x^{\frac{1}{2}}s_n(x) \to \frac{1}{2}x^{\nu+\frac{1}{2}}$ for $0 < x < 1$. This gives the first of the relations (6.5). Again, taking $t^{\frac{1}{2}}f(t) = t^{\nu+\frac{1}{2}}$ for $0 < t < 1$, (6.4) asserts that $x^{\frac{1}{2}}s_n(x) - x^{\nu+\frac{1}{2}}g_n(1) \to 0$ and (2.4) that $x^{\frac{1}{2}}s_n(x) \to x^{\nu+\frac{1}{2}}$ both these relations being certainly uniform in x for $\epsilon \leqslant x \leqslant 1-\epsilon$. This gives the second of the relations (6.5).

To establish (6.1) it is sufficient to observe that, by (6.5) and (5.1), combined with the conditions $f(0) = f(1) = 0$, the expressions

$$x^{\frac{1}{2}}f(x-0)\, g_n(x) \quad \text{and} \quad x^{\frac{1}{2}}f(x+0)[g_n(1) - g_n(x)]$$

tend boundedly to

$$\tfrac{1}{2}x^{\frac{1}{2}}f(x-0) \quad \text{and} \quad \tfrac{1}{2}x^{\frac{1}{2}}f(x+0),$$

and that, if $x^{\frac{1}{2}}f(x)$ is continuous at 0 and 1,

$$x^{\frac{1}{2}}f(x)\, g_n(1) \to x^{\frac{1}{2}}f(x)$$

uniformly in x for $0 \leqslant x \leqslant 1$.

7. *Further properties of the* $g_n(t)$. The results so far obtained are still very crude. Nevertheless, the considerations of the preceding paragraphs, for whose length and elementary character we must ask the reader to be indulgent, contain in germ all that is necessary to enable us to apply modern ideas to study the convergence problem more in detail.

In § 5, in particular, in establishing the existence of a bound for the $g_n(t)$, we established incidentally a property almost akin to bounded variation: we proved that, for a particular division of $0 \leqslant t \leqslant 1$ into partial intervals, the oscillations of the $g_n(t)$ are majorized by the terms of three convergent geometric series of the form $\sum_m K2^{-m}$. We might also have

remarked, as is at once obvious from the inequality (3.2), that *in each of the partial intervals in question*

$$\int |g_n'(t)|\,dt < K. \tag{7.1}$$

From these two properties above it is possible to deduce at once, by convexity in p of the logarithm of p-th power variation (or mean variation of order p)† that in the same partial intervals the numbers $V_p(g_n)$ are majorized by K times the $[(p-1)/p]$-th powers of the terms of our three geometric series and therefore are majorized for every choice of $p > 1$, by the terms of three other convergent geometric series. And hence, by repeated application of the formula

$$V_p(f;\ a, b) \leqslant V_p(f;\ a, \xi) + V_p(f;\ \xi, b) \tag{7.2}$$

(which is an easy consequence of Minkowski's inequality), we see that, in the whole interval $0 \leqslant t \leqslant 1$, $V_p(g_n)$ is less than a fixed constant K provided that we fix $p > 1$.

We can go further still. For a geometric series not only remains convergent when its terms are raised to any fixed power (in this case, the $[(p-1)/p]$-th power), but also when we pass to suitable powers of their logarithms.

This line of thought, which makes it necessary to avoid using (7.2) owing to the fact that Minkowski's inequality is restricted to powers, leads to the following theorem:

(7.3) *Let $\Psi(u)$ denote either the function u^q where $q > 1$ or the function $u\{\log [3M/u]\}^{-q}$ [defined for $u \leqslant 2M$, where M is the common upper bound of the $g_n(t)$ which exists by (5.1)], where $q > 1$. Then the total Ψ-variations of the $g_n(t)$ in the interval $0 \leqslant t \leqslant 1$ are bounded, uniformly in n and x, for fixed q.*

By the total Ψ-variation‡

$$V_\Psi(f;\ a, b)$$

of a function $f(t)$ in an interval $a \leqslant t \leqslant b$ we mean the upper bound, for every choice of a finite system e of non-overlapping intervals Δ contained in $0 \leqslant t \leqslant 1$, of the sum

$$\sum_e \Psi(|\Delta f|),$$

† L. C. Young, **1**, § 8 [*cf.* in particular, 259, (8.2*a*)].
‡ L. C. Young, **2**.

where Δf denotes the difference of the values of $f(t)$ at the ends of Δ. The classical total variation is the case $\Psi(u) = u$; the total p-th power variation is the case $\Psi(u) = u^p$; and so on. The function $\Psi(u)$ is generally supposed increasing from $\Psi(0) = 0$ to $\Psi(\infty) = \infty$ with u, but in our case only the values $u \leqslant 2M$ are relevant and we may suppose $\Psi(u)$ suitably modified for $u > 2M$.

The general lines of the proof of (7.3) have already been indicated. We may, therefore, limit ourselves to the case

$$\Psi(u) = u\{\log [3M/u]\}^{-q}$$

where $q > 1$.

Let us denote by E the finite set of points of the form $t = x \pm u$, where† $u = 2^{-r}x/3 > 1/(2n)$, together with those of the form $t = 2^{-s} > 4x/3$ for which, moreover, $t > 1/n$ (r and s are non-negative integers).

This being so, let e denote any finite set of non-overlapping intervals in $0 \leqslant t \leqslant 1$. We have to show that

$$\sum_e \Psi(|\Delta g_n|) < K, \tag{7.4}$$

where K is independent of e, n, x. We denote by e_1 the set of the intervals of e which contains at least one point of E. Clearly, in view of the remarks already made, it follows from § 5 that

$$\sum_{e_1} |\Delta g_n| < K,$$

and therefore that

$$\sum_{e_1} \Psi(|\Delta g_n|) < K. \tag{7.5}$$

The remaining intervals of e can be ranged in groups e_2 such that each group e_2 consists of intervals between the same pair of points t', t'' of E. For each such group e_2 we have

$$\begin{aligned}\sum_{e_2} \Psi(|\Delta g_n|) &\leqslant [\log(3M/\max|\Delta g_n|)]^{-q} \sum_{e_2} |\Delta g_n| \\ &\leqslant [\log(3M/\operatorname{Osc} g_n)]^{-q} \int_{t'}^{t''} |g_n'(t)|\,dt \\ &\leqslant K[\log(3M/\operatorname{Osc} g_n)]^{-q}.\end{aligned}$$

Now we have seen that the oscillations of $g_n(t)$ in the partial intervals determined by the consecutive pairs of points t', t'' of E are majorized by

† The symbol u has now no connection with the variable occurring in $\Psi(u)$.

the terms of three geometric series $\sum_m K2^{-m}$. It follows that the groups of terms

$$\sum_{e_2} \Psi(|\Delta g_m|)$$

are majorized by single terms of three series $\sum_m Km^{-q}$. Hence, adding the contributions of the various groups e_2 to that of e_1 which is estimated in (7.5), we obtain (7.4) as required.

[*Note.*—If we write $g_n(t)$ as a function of t, x in the form $g_n(t, x)$ say, it is natural to ask whether the p-th power, and other, variations in x for constant t are subject to inequalities at all similar to those established here for the corresponding variations in t for constant x. The analogy with trigonometric Fourier series rather suggests this, but there appear to be serious formal difficulties in the way.

One type of conclusion which could be drawn from such results (if true) concerns the existence of a bound for certain power variations of the functions $x^{\frac{1}{2}}s_n(x)$ in the case in which $x^{\frac{1}{2}}f(x)$ is of a corresponding bounded power variation. This follows from an inequality for the "Stieltjes Faltung" which has been given in a recent paper† in these *Proceedings*.]

8. *Extension of the convergence criteria.* We now come to our main theorem which asserts the following:

(8.1) *In* (6.1) *the term "function of bounded variation" may, without affecting the validity of the theorem, be interpreted to mean either* (*a*) *"function of bounded p-th power variation", or* (*b*) *"function of bounded Φ-variation" where, for the relevant‡ u,*

$$\Phi(u) = \exp(-u^{-c}), \quad 0 < c < \tfrac{1}{2}. \tag{8.2}$$

Proof of (8.1). Denoting by $\mu = \mu(\delta; x)$ the oscillation of the function $t^{\frac{1}{2}}f(t)$ in $x-\delta \leqslant t < x$ or in $x < t \leqslant x+\delta$ (whichever is the larger), we observe that, by the results proved in § 6, it is sufficient to show that the integrals

$$\int_{x-\delta}^{x} \{t^{\frac{1}{2}}f(t) - x^{\frac{1}{2}}f(x-0)\}\,dg_n(t) \quad \text{and} \quad \int_{x}^{x+\delta} \{t^{\frac{1}{2}}f(t) - x^{\frac{1}{2}}f(x+0)\}\,dg_n(t) \tag{8.3}$$

(in which the functions concerned are supposed constant for $t \leqslant 0$ and $t \geqslant 1$ if necessary) are less in absolute value than an expression of the form

† L. C. Young, **3**.

‡ We suppose that Φ increases to ∞ with u. This implies that the function concerned is bounded. But once we know the function to be bounded, the large values of u are irrelevant.

$K\epsilon(\mu)$, where $\epsilon(\mu)$ is a bounded function of μ (for the relevant values of μ) and tends to 0 as $\mu \to 0$. It is sufficient to prove this in the case of the former of the two integrals (8.3), since the proof for the second integral proceeds on symmetrical lines.

Now it is easy to see that for any $p_1 > p$ in case (a), or for any c_1 such that $c < c_1 < \frac{1}{2}$ in case (b) and for the corresponding function $\Phi_1(u)$ obtained by replacing c by c_1 in the definition of $\Phi(u)$, (8.2), we have

$$(8.4) \qquad (a) \quad V_{p_1}(F) < \epsilon(\mu) \quad \text{or} \quad (b) \quad V_{\Phi_1}(F) < \epsilon(\mu),$$

where $F(t)$ denotes the function defined for $x-\delta \leqslant t \leqslant x$ and equal to $t^{\frac{1}{2}}f(t)$ except perhaps† at $t = x$, where it has the value $x^{\frac{1}{2}}f(x-0)$.

Moreover we have

$$(8.5) \qquad (a) \quad V_{q_1}(g_n) < K \quad \text{and} \quad (b) \quad V_{\Psi_1}(g_n) < K,$$

when we choose q_1 so that $1 < q_1 < p_1/(p_1-1)$ and

$$\Psi_1(u) = u/[\log(3M/u)]^{1/(2c_1)}.$$

This being so, we apply the fundamental inequalities for Stieltjes integrals of the author's papers in the *Acta Mathematica* and in the *Mathematische Annalen*‡. We obtain for the absolute value of the integral

$$\int_{x-\delta}^{x} \{F(t)-F(x)\}\, dg_n(t),$$

which is equal to the first integral (8.3), the majorant§

$$\epsilon(\mu) \,.\, K$$

in case (a) since $(1/p_1)+(1/q_1) > 1$, in view of (8.4) (a) and (8.5) (a); and the majorant‖

$$(8.6) \qquad K \sum_m \phi_1\left(\frac{\epsilon(\mu)}{m}\right) \psi_1\left(\frac{K}{m}\right)$$

in case (b), where $\phi_1(u)$, $\psi_1(u)$ denote the inverse functions of $\Phi_1(u)$, $\Psi_1(u)$, in view of (8.4) (b) and (8.5) (b). This completes the proof in case (a), while in case (b) it is now only necessary to verify that the series

$$(8.7) \qquad \sum_m \phi_1\left(\frac{1}{m}\right) \psi_1\left(\frac{1}{m}\right)$$

† We suppose the function $t^{\frac{1}{2}}f(t)$ continued for $t \leqslant 0$ and for $t \geqslant 1$ if necessary, by defining it to be zero.

‡ L. C. Young, **1**, **2**.

§ L. C. Young, **1**, 266 (10.9).

‖ L. C. Young, **2**, 597, Theorem (5.1)(*i*).

converges, since, if (8.7) converges, (8.6) split up at $m=N$ is at most $KN\phi_1[\epsilon(\mu)]+\epsilon_N$, where $\epsilon_N \to 0$ independently of μ as $N \to \infty$, so that (8.6) is at most another $\epsilon(\mu)$. This completes the proof since the convergence of (8.7) is an immediate consequence of the following relations, valid for sufficiently small $\epsilon > 0$,

$$\phi_1(u) = [\log(1/u)]^{-1/c_1}, \qquad \psi_1(u) < Ku[\log(K/u)]^{(1+\epsilon)/(2c_1)}$$

and $$\phi_1(u)\psi_1(u) < Ku[\log(K/u)]^{-(1-\epsilon)(2c_1)} < Ku[\log(K/u)]^{-(1+\epsilon)}.$$

As a particular case of the theorem just proved, we may mention that *if the function* $F(x) = \sqrt{x}f(x)$ *satisfies uniformly in* x *the Lipschitz condition*

$$|F(x+h)-F(x)| < K[\log(1/h)]^{-(2+\epsilon)},$$

where $\epsilon > 0$, *then it is uniformly in* x *the sum of its orthogonal and normalized Fourier-Bessel series, i.e. the limit of* $x^{\frac{1}{2}}s_n(x)$.

Actually the analogy with ordinary Fourier series at once suggests that the exponent $-(2+\epsilon)$ in this last result may be replaced by -1.

The present writer is unable to state definitely whether this extension is possible and whether it has been made. What is certain is that the problem of a corresponding extension of the range of c in (8.1) (*b*) is unsolved.

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ON RIESZ SUMMABILITY OF FOURIER SERIES*

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1. *Introduction.*

In this paper we begin by proving in § 2 some general theorems concerning the summability of Fourier series by Riesz means of certain types. In § 3 we solve a problem of Hardy and Littlewood concerning their criterion for the convergence of a Fourier series†. In § 4 we apply Riesz summability to obtain further convergence criteria for Fourier series. The reader may begin at § 3 or § 4 if he wishes.

Suppose that $a_0+a_1+a_2+\ldots$ is a given series, and let $\lambda(\omega)$ be continuous and monotone increasing in $(0, \infty)$, and tend to infinity as $\omega\to\infty$. Let

$$c_\kappa(\omega) = \sum_{n\leqslant\omega} \{\lambda(\omega)-\lambda(n)\}^\kappa a_n,$$

where $\kappa \geqslant 0$. Then Σa_n *is said to be summable* (λ, κ) *to sum* s *if* $c_\kappa(\omega)/\{\lambda(\omega)\}^\kappa$ *tends to* s *as* $\omega\to\infty$.

It is well known that a series which is summable (λ, κ) is also summable (λ, κ') to the same sum, for $\kappa' > \kappa$, while it is summable (ν, κ) if ν is a logarithmico-exponential function of λ such that $\nu = o(\lambda^\Delta)$, for some $\Delta > 0$.

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* This paper was received on 2 February, 1939. Owing to the difficulties of communication with China, certain modifications which seemed necessary have been made by Dr. L. S. Bosanquet. A number of proofs have been simplified and improved; and Dr. Bosanquet has also extended Theorem 10 from the case of summability of type $e^{(\log\omega)^\alpha}$ $(1<\alpha<2)$ to the case $\alpha=2$, and reduced the order of magnitude of the Fourier coefficients in the example for Theorems 6 and 11 from $O(n^{-1}\log^3 n)$ to $o(n^{-1}\log^2 n)$. In Theorem 1, the restriction that κ should be an integer has been removed by a suggestion of Dr. G. W. Morgan.

† Hardy and Littlewood, **2, 3.**

Condition A. Suppose that

$$\lambda \equiv \lambda(\omega) = e^{\mu(\omega)} \equiv \mu,$$

where μ satisfies the following conditions:

(I) $\mu(\omega)$ is differentiable and monotone increasing in $(0, \infty)$, and $\mu(\omega) \to \infty$ as $\omega \to \infty$,

(II) $\mu'(\omega)$ is monotone decreasing for $\omega > A$, and $\mu'(\omega) \to 0$ as $\omega \to \infty$,

(III) $\{\mu'(\omega)\}^2 \geqslant -\mu''(\omega) \geqslant 0$ for $\omega > A$.

Condition III implies that $\lambda'(\omega) = e^{\mu(\omega)}\mu'(\omega)$ is monotone increasing for $\omega > A$. For example, $\mu(\omega)$ may have any of the following forms:

$$\text{(i)} \quad \frac{\omega}{(\log \omega)^\alpha} \text{ for } \omega > e^\alpha \ (\alpha > 0), \qquad \text{(ii)} \quad \omega^\alpha \ (0 < \alpha < 1),$$

$$\text{(iii)} \quad (\log \omega)^\alpha \text{ for } \omega > 1 \ (\alpha \geqslant 1).$$

If μ satisfies conditions I, II and III, we say that λ satisfies *Condition* A.

2. Let $f(t)$ be integrable in the Lebesgue sense in $(-\pi, \pi)$ and periodic with period 2π. Let the Fourier series of $f(t)$ be

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt), \tag{2.1}$$

and let

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\}.$$

In the proofs of the theorems which follow we make the usual simplifications and suppose that $a_0 = 0$, $x = 0$, $s = 0$, $f(t) = \phi(t)$.

THEOREM 1. *If*

$$\int_0^t |\phi(u)|\, du = o(t) \tag{2.2}$$

as $t \to 0$, *then a necessary and sufficient condition for the Fourier series* (2.1) *to be summable* (λ, k) *at* $t = x$ *to sum* s, *where* λ *satisfies Condition* A *and* $\kappa \geqslant 1$, *is that*

$$\int_0^{\mu'(\omega)} \phi(t) \frac{\sin \omega t}{t}\, dt = o(1)$$

as $\omega \to \infty$.

Proof. We have, by well-known arguments,

$$c_0(\omega) = \sum_{n \leqslant \omega} a_n = s_{[\omega]} = \frac{2}{\pi} \int_0^\pi \phi(t) \frac{\sin([\omega]+\frac{1}{2})t}{2 \sin \frac{1}{2}t} \, dt$$

$$= \frac{2}{\pi} \int_0^\pi \phi(t) \frac{\sin [\omega] t}{t} \, dt + o(1)$$

$$= \frac{2}{\pi} \int_0^\pi \phi(t) \frac{\sin \omega t}{t} \, dt + o(1)$$

as $\omega \to \infty$. Hence, for $\kappa > 0$, $\omega > 0$,

$$c_k(\omega) = \sum_{n \leqslant \omega} \{\lambda(\omega) - \lambda(n)\}^\kappa a_n$$

$$= \kappa \int_0^\omega \{\lambda(\omega) - \lambda(x)\}^{\kappa-1} \lambda'(x) c_0(x) \, dx$$

$$= \frac{2\kappa}{\pi} \int_0^\omega \{\lambda(\omega) - \lambda(x)\}^{\kappa-1} \lambda'(x) \, dx \int_0^\pi \phi(t) \frac{\sin xt}{t} \, dt + o\Big(\{\lambda(\omega)\}^\kappa\Big)$$

$$= \frac{2}{\pi} \int_0^\pi \phi(t) E_k(\omega, t) \, dt + o\Big(\{\lambda(\omega)\}^\kappa\Big)$$

as $\omega \to \infty$, where

$$E_\kappa(\omega, t) = \frac{\kappa}{t} \int_0^\omega \{\lambda(\omega) - \lambda(x)\}^{\kappa-1} \lambda'(x) \sin xt \, dx.$$

Now write

$$\frac{t}{\kappa} E_\kappa(\omega, t) = \int_0^A + \int_A^\omega = I_1 + I_2.$$

Then, for $\kappa > 0$, $\omega > A$,

$$|I_1| \leqslant \int_0^A \{\lambda(\omega) - \lambda(x)\}^{\kappa-1} \lambda'(x) \, dx = O\Big(\{\lambda(\omega)\}^{\kappa-1}\Big)$$

and, for $\kappa \geqslant 1$, since $\lambda(x)$ and $\lambda'(x)$ are monotone increasing,

$$I_2 = \{\lambda(\omega) - \lambda(A)\}^{\kappa-1} \lambda'(\omega) \int_{\omega'}^{\omega''} \sin xt \, dx \quad (0 < \omega' < \omega'' < \omega)$$

$$= O\{\lambda(\omega)\}^\kappa \mu'(\omega) O(t^{-1}).$$

by the Second Mean Value Theorem. Thus, for $t > 0$, $\omega > A$,

$$E_\kappa(\omega, t) = O\Big(\{\lambda(\omega)\}^{\kappa-1} t^{-1}\Big) + O\Big(\{\lambda(\omega)\}^\kappa \mu'(\omega) t^{-2}\Big).$$

On the other hand, for $\kappa > 0$,

$$E_\kappa(\omega, t) = \int_0^\omega \{\lambda(\omega)-\lambda(x)\}^\kappa \cos xt\, dx$$

$$= \int_0^\omega \{\lambda(\omega)\}^\kappa \left[1+\sum_{q=1}^{\infty}(-1)^q\binom{\kappa}{q}\left\{\frac{\lambda(x)}{\lambda(\omega)}\right\}^q\right]\cos xt\, dx$$

$$= \{\lambda(\omega)\}^\kappa \frac{\sin\omega t}{t} + O\left[\sum_{q=1}^{\infty}\left|\binom{\kappa}{q}\right|\{\lambda(\omega)\}^{\kappa-q}\int_0^\omega \{\lambda(x)\}^q\, dx\right].$$

Now, for $\omega > A$,

$$\int_0^\omega \{\lambda(x)\}^q\, dx = O\left\{\frac{1}{\mu'(\omega)}\right\}\int_0^\omega e^{q\mu(x)}\mu'(x)\, dx$$

$$= O\left\{\frac{e^{q\mu(\omega)}}{q\mu'(\omega)}\right\} = O\left(\frac{\{\lambda(\omega)\}^q}{q\mu'(\omega)}\right).$$

Hence

$$E_\kappa(\omega, t)-\{\lambda(\omega)\}^\kappa\frac{\sin\omega t}{t} = O\left\{\sum_{q=1}^{\infty}\left|\binom{\kappa}{q}\right|\{\lambda(\omega)^{\kappa-q}\frac{\{\lambda(\omega)\}^q}{q\mu'(\omega)}\right\}$$

$$= \frac{\{\lambda(\omega)\}^\kappa}{\mu'(\omega)}O\left\{\sum_{q=1}^{\infty}\frac{1}{q}\left|\binom{\kappa}{q}\right|\right\} = O\left(\frac{\{\lambda(\omega)\}^\kappa}{\mu'(\omega)}\right).$$

It follows that, if (2.2) is satisfied and $\kappa \geqslant 1$, then

$$c_\kappa(\omega)-\frac{2}{\pi}\int_0^{\mu'(\omega)}\phi(t)\frac{\sin\omega t}{t}\, dt$$

$$= O\left(\frac{\{\lambda(\omega)\}^\kappa}{\mu'(\omega)}\right)\int_0^{\mu'(\omega)}|\phi(t)|\, dt + O\{\lambda(\omega)\}^{\kappa-1}\int_{\mu'(\omega)}^{\pi}\frac{|\phi(t)|}{t}\, dt$$

$$+ O\left(\{\lambda(\omega)\}^\kappa\mu'(\omega)\right)\int_{\mu'(\omega)}^{\pi}\frac{|\phi(t)|}{t^2}\, dt$$

$$= J_1+J_2+J_3,$$

where, as $\omega\to\infty$,

$$J_1 = O\left(\frac{\{\lambda(\omega)\}^\kappa}{\mu'(\omega)}\right)o\{\mu'(\omega)\} = o\left(\{\lambda(\omega)\}^\kappa\right),$$

$$J_2 = O\left(\{\lambda(\omega)\}^{\kappa-1}\right)o\left\{\log\frac{1}{\mu'(\omega)}\right\} = O\{\lambda(\omega)\}^{\kappa-1}o\left\{\frac{1}{\mu'(\omega)}\right\}$$

$$= O\{\lambda(\omega)\}^{\kappa-1}o\left\{\frac{\lambda'(\omega)}{\mu'(\omega)}\right\} = o\left(\{\lambda(\omega)\}^\kappa\right),$$

$$J_3 = O\left(\{\lambda(\omega)\}^\kappa\mu'(\omega)\right)o\left\{\frac{1}{\mu'(\omega)}\right\} = o\left(\{\lambda(\omega)\}^\kappa\right),$$

which completes the proof.

As a corollary of Theorem 1 we have the following result:

THEOREM 2. *If* (2.2) *is satisfied, and if the Fourier series* (2.1) *is summable* (λ, κ) *at* $t = x$ *to sum* s, *where* λ *satisfies Condition* A *and* $\kappa > 1$, *then it is summable* $(\lambda, 1)$ *at* $t = x$ *to the same sum.*

If we make further restrictions on $\lambda(x)$, we can prove more precise results. In the rest of this section we consider only the special type $\lambda = e^{\omega^\alpha}$ $(0 < \alpha < 1)$ and prove the following theorems.

THEOREM 3. *If* (2.2) *is satisfied, then a necessary and sufficient condition for the Fourier series* (2.1) *to be summable* (e^{ω^α}, r) *at* $t = x$ *to sum* s, *where* $r > 0$, *is that*

$$\int_0^{\omega^{\alpha-1}} \phi(t) \frac{\sin \omega t}{t}\, dt = o(1)$$

as $\omega \to \infty$.

THEOREM 4. *If* (2.2) *is satisfied, and the Fourier series of* (2.1) *is summable* $(e^{\omega^\alpha}, \kappa)$ *at* $t = x$ *to sum* s, *for some* $\kappa > 0$, *then* (2.1) *is summable* (e^{ω^α}, r) *to* s *at* $t = x$, *for every* $r > 0$.

Theorem 4 is an immediate corollary of Theorem 3.

Proof of Theorem 3. By Theorem 1 we need consider only the case $0 < r < 1$. Then we have, as in Theorem 1,

$$c_r(\omega) = \alpha r \int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{r-1} e^{x^\alpha} x^{\alpha-1} c_0(x)\, dx$$

$$= \frac{2}{\pi} \int_0^\pi \phi(t)\, E_r(\omega, t)\, dt + o(e^{r\omega^\alpha}),$$

where $$E_r(\omega, t) = \frac{\alpha r}{t} \int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{r-1} e^{x^\alpha} x^{\alpha-1} \sin xt\, dx.$$

Now, for all sufficiently large ω, there is a unique number ω_1 such that

$$e^{\omega^\alpha} - e^{\omega_1{}^\alpha} = e^{\omega^\alpha} (\log \omega)^{-1/r}.$$

Write $$\frac{t}{\alpha r} E_r(\omega, t) = \int_0^A + \int_A^{\omega_1} + \int_{\omega_1}^{\omega} = I_1 + I_2 + I_3,$$

where A is chosen so that $e^{x^\alpha} x^{\alpha-1}$ is steadily increasing* for $x > A$. Then

* *E.g.* we may take $$A = \left(\frac{1-\alpha}{\alpha}\right)^{1/\alpha}.$$

for large ω,

$$I_1 = O\{e^{(r-1)\omega^\alpha}\}$$

$$I_2 = (e^{\omega^\alpha} - e^{\omega_1{}^\alpha})^{r-1} e^{\omega_1{}^\alpha} \omega_1^{\alpha-1} \int_{\omega'}^{\omega''} \sin xt\, dt \quad (A < \omega' < \omega'' < \omega)$$

$$= O\{e^{r\omega^\alpha} \omega^{\alpha-1} (\log \omega)^{(1-r)/r} t^{-1}\},$$

by the Second Mean Value Theorem, and

$$|I_3| \leqslant \int_{\omega_1}^{\omega} (e^{\omega^\alpha} - e^{x^\alpha})^{r-1} e^{x^\alpha} x^{\alpha-1}\, dx$$

$$= \frac{1}{\alpha r} (e^{\omega} - e^{\omega_1{}^\alpha})^r = O\{e^{r\omega} (\log \omega)^{-1}\}.$$

Hence, if (2.2) is satisfied, writing $\xi = \omega^{\alpha-1} (\log \omega)^{(1-r)/r}$, we easily see that

$$\int_\xi^\pi \phi(t)\, E_r(\omega, t)\, dt$$

$$= O\{e^{(r-1)\omega^\alpha}\} \int_\xi^\pi \frac{|\phi(t)|}{t}\, dt + O\{e^{r\omega^\alpha} \omega^{\alpha-1} (\log \omega)^{(1-r)/r}\} \int_\xi^\pi \frac{|\phi(t)|}{t^2}\, dt$$

$$+ O\{e^{r\omega^\alpha} (\log \omega)^{-1}\} \int_\xi^\alpha \frac{|\phi(t)|}{t}\, dt$$

$$= o\{e^{(r-1)\omega^\alpha} \log \omega\} + o(e^{r\omega^\alpha}) + o(e^{r\omega^\alpha}) = o(e^{r\omega^\alpha})$$

as $\omega \to \infty$.

Next let ω_2 be defined so that $\omega_2{}^\alpha = \omega^\alpha - 1$. Then writing

$$\frac{t}{\alpha r} E_r(\omega, t) = \int_0^A + \int_A^{\omega_2} + \int_{\omega_2}^{\omega} = I_4 + I_5 + I_6,$$

we have

$$I_4 = O\{e^{(r-1)\omega^\alpha}\},$$

$$I_5 = (e^{\omega^\alpha} - e^{\omega^\alpha - 1})^{r-1} e^{\omega^\alpha} \omega^{\alpha-1} O(t^{-1})$$

$$= O(e^{r\omega^\alpha} \omega^{\alpha-1} t^{-1}),$$

and

$$I_6 = \frac{e^{r\omega^\alpha}}{\alpha} \int_{\omega^\alpha - 1}^{\omega^\alpha} (1 - e^{u-\omega^\alpha})^{r-1} e^{u-\omega^\alpha} \sin u^{1/\alpha} t\, du$$

$$= \frac{e^{r\omega^\alpha}}{\alpha} \int_0^1 (1 - e^{-x})^{r-1} e^{-x} \sin (\omega^\alpha - x)^{1/\alpha} t\, dx.$$

Now, for $0<x<1$ and ω large,

$$(\omega^\alpha-x)^{1/\alpha}=\omega-\frac{1}{\alpha}\,\omega^{1-\alpha}x+O(\omega^{1-2\alpha}x^2),$$

and hence

$$\sin(\omega^\alpha-x)^{1/\alpha}t=\sin(\omega-\omega^{1-\alpha}\alpha^{-1}x)\,t+O(\omega^{(1-2\alpha}x^2t)$$

as $\omega\to\infty$. Thus, for ω large,

$$I_6=\frac{e^{r\omega^\alpha}}{\alpha}\int_0^1(1-e^{-x})^{r-1}e^{-x}\sin\left(\omega-\frac{1}{\alpha}\,\omega^{1-\alpha}x\right)t\,dx+O(e^{r\omega^\alpha}\omega^{1-2\alpha}t).$$

Further $$(1-e^{-x})^{r-1}e^{-x}=x^{r-1}+\rho(x),$$

when $\rho(x)$ is of bounded variation in (0, 1). Hence, if $S(x)=\sin x$ or $\cos x$,

$$\begin{aligned}\int_0^1(1-e^{-x})^{r-1}e^{-x}S(Tx)\,dx&=\int_0^1x^{r-1}S(Tx)+\int_0^1\rho(x)\,S(Tx)\,dx\\&=O(T^{-r})+O(T^{-1})\end{aligned}$$

for $T>0$, and so, writing

$$\sin(\omega-\frac{1}{\alpha}\,\omega^{1-\alpha}x)\,t=\sin\omega t\cos Tx-\cos\omega t\sin Tx,$$

we obtain, for ω large,

$$I_6=O(e^{r\omega^\alpha}\omega^{r(\alpha-1)}t^{-r})+O(e^{r\omega^\alpha}\omega^{\alpha-1}t^{-1})+O(e^{r\omega}\ \omega^{1-2\alpha}t).$$

It easily follows, when we write $\eta=\omega^{\alpha-1}$, that, if (2.2) is satisfied, then

$$\begin{aligned}&\int_\eta^\xi\phi(t)\,E_r(\omega,t)\\&\quad=O(e^{(r-1)\omega^\alpha})\int_\eta^\xi\frac{|\phi(t)|}{t}\,dt+O(e^{r\omega}\ \omega^{\alpha-1})\int_\eta^\xi\frac{|\phi(t)|}{t^2}\,dt+O(e^{r\omega}\ \omega^{r(\alpha-1)})\int_\eta^\xi\frac{\phi(t)}{t^{1+r}}\,dt\\&\qquad+O(e^{r\omega}\ \omega^{\alpha-1})\int_\eta^\xi\frac{|\phi(t)|}{t^2}\,dt+O(e^{r\omega^\alpha}\omega^{1-2\alpha})\int_\eta^\xi|\phi(t)|\,dt\\&\quad=o(e^{(r-1)\omega^\alpha})\log\omega+o(e^{r\omega^\alpha})+o(e^{r\omega^\alpha})+o(e^{r\omega^\alpha})+o\{e^{r\omega}\ \omega^{-\alpha}(\log\omega)^{(1-r)/r}\}\\&\quad=o(e^{r\omega^\alpha})\end{aligned}$$

as $\omega\to\infty$. Finally, as in the proof* of Theorem 1,

$$\int_0^\eta\phi(t)\,E_r(\omega,t)\,dt=e^{r\omega^\alpha}\int_0^\eta\phi(t)\,\frac{\sin\omega t}{t}\,dt+o(e^{r\omega^\alpha}),$$

which completes the proof.

* If $\mu(\omega)=\omega^\alpha$, then $\eta=\mu'(\omega)/\alpha$. The relevant part of Theorem 1 was proved for $\kappa>0$.

The following theorem has been proved elsewhere* in the case $r = 1$.

THEOREM 5. *If*

$$\int_0^t |\phi(u)|\,du = o\left(\frac{t}{\log 1/t}\right) \tag{2.3}$$

as $t \to +0$, *then the Fourier series* (2.1) *is summable* (e^{ω^α}, r) *to sum* s *at* $t = x$, *where* $r > 0$.

Theorem 5 is a corollary of the known case $r = 1$ and Theorem 4.

Alternatively, Theorem 5 may be deduced from Theorem 3 by writing

$$\int_0^{\omega^{\alpha-1}} \phi(t)\,\frac{\sin \omega t}{t}\,dt = \int_0^{\omega^{-1}} + \int_{\omega^{-1}}^{\omega^{\alpha-1}} = J_1 + J_2.$$

Then
$$J_1 = O(\omega)\int_0^{\omega^{-1}} |\phi(t)|\,dt = o(1),$$

while, on integration by parts,

$$\begin{aligned} |J_2| &\leqslant \int_{\omega^{-1}}^{\omega^{\alpha-1}} \frac{|\phi(t)|}{t}\,dt \\ &= o\{(\log \omega)^{-1}\} + o\left\{\int_{\omega^{-1}}^{\omega^{\alpha-1}} \frac{dt}{t \log 1/t}\right\} \\ &= o\{(\log \omega)^{-1}\} + o(1) = o(1). \end{aligned}$$

3. The appropriate Tauberian theorem for summability (λ, r) is as follows†.

THEOREM A. *If* Σa_n *is summable* (λ, r) *and*

$$a_n > -K\,\frac{\lambda(n) - \lambda(n-1)}{\lambda(n)}, \tag{3.1}$$

then Σa_n *is convergent.*

It has been observed elsewhere‡ that, as a corollary§ of Theorem 5 and Theorem A, we have the following theorem of Hardy and Littlewood‖.

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* Wang, **4**.
† Hardy, **1**.
‡ Wang, **4**.
§ In the case $r = 1$.
‖ Hardy and Littlewood, **2**, **3**.

THEOREM B. *If* (2.3) *is satisfied and*

$$a_n > -K n^{-\delta}, \tag{3.2}$$

where $\delta > 0$, *then the Fourier series* (2.1) *is convergent to sum* s *at* $t = x$.

Hardy and Littlewood* have raised the question whether condition (2.3) in Theorem B may be replaced by the weaker condition

$$\int_0^t \phi(u)\,du = o\left(\frac{t}{\log 1/t}\right) \tag{3.3}$$

as $t \to +0$. In this section we give an example to show that this is not possible.

THEOREM 6. *There is an even function* $\phi(t)$ *satisfying condition* (3.3) *for* $x = 0$, *whose Fourier series* (2.1) *is divergent at* $t = 0$, *and such that* $a_n = O(n^{\alpha-1})$ *for every* $\alpha > 0$.

Proof. We choose sequences $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{c_n\}$, such that

$$\pi = \alpha_0 > \alpha_1 > \alpha_2 > \ldots > \alpha_n \to 0, \quad 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$$

and $c_n \to 0$ as $n \to \infty$, and define an even function $\phi(t)$ such that

$$\phi(t) = c_i \sin \lambda_i t \quad \text{for} \quad \alpha_i < t \leqslant \alpha_{i-1} \quad (i = 1, 2, \ldots).$$

Then $\phi(t)$, being bounded, is integrable in the Lebesgue sense in $(0, \pi)$.

We take

$$\lambda_n = (n+1)!, \quad \alpha_n = \frac{\log \lambda_n}{\lambda_n} = \frac{\log\{(n+1)!\}}{(n+1)!} \quad (n \geqslant 1)$$

and

$$c_n = \{\log(n+1)\}^{-\frac{1}{2}}.$$

Observing that

$$\int_{\alpha_i}^{\alpha_{i-1}} \phi(t)dt = O\left(\frac{c_i}{\lambda_i}\right),$$

we see that, if $\alpha_n < t \leqslant \alpha_{n-1}$,

$$\int_0^t \phi(u)\,du = o\left(\sum_{i=n}^{\infty} \frac{1}{\lambda_i}\right) = o\left\{\frac{1}{(n+1)!}\right\},$$

while

$$\frac{1}{t} \log \frac{1}{t} < \frac{1}{\alpha_n} \log \frac{1}{\alpha_n} = \frac{\lambda_n}{\log \lambda_n} (\log \lambda_n - \log \log \lambda_n).$$

$$\leqslant \lambda_n = (n+1)!,$$

* Hardy and Littlewood, 3.

and hence

$$\frac{1}{t}\log\frac{1}{t}\int_0^t \phi(u)\,du = O\{(n+1)!\}\,o\left\{\frac{1}{(n+1)!}\right\} = o(1)$$

as $t \to +0$. Thus (3.3) is satisfied.

We now write

$$\begin{aligned} s_{\lambda_n} &= \frac{2}{\pi}\int_0^\pi \phi(t)\,\frac{\sin\lambda_n t}{t}\,dt + o(1) \\ &= \frac{2}{\pi}\sum_{i=1}^{\infty} c_i \int_{a_i}^{a_{i-1}} \frac{\sin\lambda_i t \sin\lambda_n t}{t}\,dt + o(1) \\ &= \sum_{i=1}^{\infty} I_i + o(1) \\ &= \sum_{i<n} I_i + I_n + \sum_{i>n} I_i + o(1). \end{aligned}$$

Then

$$\begin{aligned} I_n &= \frac{2}{\pi}\,c_n \int_{a_n}^{a_{n-1}} \frac{\sin^2\lambda_n t}{t}\,dt = \frac{c_n}{\pi}\int_{a_n\lambda_n}^{a_{n-1}\lambda_n} \frac{1-\cos 2u}{u}\,du \\ &\sim \frac{c_n}{\pi}\log\frac{a_{n-1}}{a_n} = \frac{1}{\pi}\,\frac{1}{\sqrt{\log(n+1)}}\log\left((n+1)\,\frac{\log(n!)}{\log\{(n+1)!\}}\right) \\ &\sim \frac{\sqrt{\log(n+1)}}{\pi} \end{aligned}$$

as $n \to \infty$.

Now

$$\begin{aligned} I_i &= \frac{c_i}{\pi}\int_a^{a_{i-1}} \frac{\cos(\lambda_n-\lambda_i)t - \cos(\lambda_n+\lambda_i)t}{t}\,dt \\ &= O\left\{\frac{c_i}{a_i|\lambda_n-\lambda_i|}\right\} \end{aligned}$$

uniformly in i and n $(i \neq n)$. Hence

$$\begin{aligned} \sum_{i<n} I_i &= \sum_{i=1}^{n-1} o\left\{\frac{(i+1)!}{(n+1)!-(i+1)!}\right\} \\ &= \frac{1}{(n+1)!-n!}\,o\left\{\sum_{i=1}^{n-1}(i+1)!\right\} \\ &= o\left\{\frac{n!}{(n+1)!-n!}\right\} \\ &= o(1). \end{aligned}$$

as $n \to \infty$.

Again, since the total variation of $\sin\lambda_n t$ in (a_i, a_{i-1}) is $O(\lambda_n a_{i-1})$, we have, by the Second Mean Value Theorem,

$$\begin{aligned} I_i &= c_i \int_{a_i}^{a_{i-1}} \frac{\sin\lambda_i t \sin\lambda_n t}{t}\, dt \\ &= O(c_i \lambda_n a_{i-1}) \max_{a_i < a' < a'' < a_{i-1}} \left| \int_{a'}^{a''} \frac{\sin\lambda_i t}{t}\, dt \right| \\ &= O\left(\frac{c_i \lambda_n a_{i-1}}{\lambda_i a_i}\right) \end{aligned}$$

uniformly in i and n. Hence

$$\begin{aligned} \sum_{i>n} I_i &= (n+1)! \sum_{i=n+1}^{\infty} o\left(\frac{1}{i!} \frac{\log(i!)}{\log\{(i+1)!\}}\right) \\ &= (n+1)!\, o\left\{\sum_{i=n+1}^{\infty} \frac{1}{i!}\right\} = o(1) \end{aligned}$$

as $n \to \infty$. Thus $s_{\lambda_n} \to \infty$ as $n \to \infty$, *i.e.* (2.1) diverges for $t = x$.

Now let $\{\rho_n\}$ be a non-decreasing sequence of positive integers, such that $\lambda_{\rho_n} > n$ for all sufficiently large n, and write

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt\, dt \\ &= \frac{2}{\pi} \sum_{i=1}^{\infty} c_i \int_{a_i}^{a_{i-1}} \sin\lambda_i t \cos nt\, dt \\ &= \sum_{i=1}^{\infty} J_i \\ &= \sum_{i \le \rho_n} J_i + \sum_{i > \rho_n} J_i. \end{aligned}$$

Then, since the total variation of $\sin\lambda_i t$ in (a_i, a_{i-1}) is $O(\lambda_i a_{i-1})$ we have, by the Second Mean Value Theorem,

$$\begin{aligned} \sum_{i \le \rho_n} J_i &= \sum_{i \le \rho_n} O\left(\frac{c_i \lambda_i a_{i-1}}{n}\right) \\ &= \sum_{i \le \rho_n} o\left\{\frac{(i+1)\log(i!)}{n}\right\} = \frac{1}{n}\, o\left\{\sum_{i \le \rho_n} (i+1)\log(i!)\right\} \\ &= \frac{1}{n}\, o\left\{\sum_{i \le \rho_n} i^2 \log i\right\} = o\left(\frac{\rho_n{}^3 \log \rho_n}{n}\right) \end{aligned}$$

as $n \to \infty$. On the other hand $J_i = O\{c_i/|\lambda_i - n|\}$ uniformly in n and

i $(n \neq \lambda_i)$, and hence

$$\sum_{i>\rho_n} J_i = \sum_{i>\rho_n} o\left\{\frac{1}{(i+1)!-n}\right\}$$

$$= o\left\{\sum_{i>\rho_n} \frac{1}{(i+1)!-i!}\right\},$$

since $i! = \lambda_{i-1} \geqslant \lambda_{\rho_n} > n$. But this is

$$o\left\{\frac{1}{(\rho_n)!}\right\} = o\{\rho_n/e)^{-\rho_n}\}$$

as $n \to \infty$.

Hence, if we choose $\rho_n = [n^{\frac{1}{2}\alpha}]$, we easily see* that $a_n = O(n^{\alpha-1})$.

4. In this section we begin by supposing that $\phi(t)$ satisfies a condition similar to (3.3), but stronger.

THEOREM 7. *If* $\kappa > 1$, $a = (\kappa-1)/\kappa$ *and*

$$\phi_1(t) = \int_0^t \phi(u)\,du = o(t^\kappa) \tag{4.1}$$

as $t \to +0$, *then the Fourier series* (2.1) *is summable* (e^{ω^α}, r) *at* $t = x$ *to sum* s, *for integral* r *greater than* κ.

Proof†. Defining $E_r(\omega, t)$ as in Theorem 3, we have, since

$$\phi_1(t)\,E_r(\omega, t) = o(1) \quad \text{as} \quad t \to +0\ddagger,$$

$$\int_0^\pi \phi(t)\,E_r(\omega, t)\,dt = \phi_1(\pi)\,E_r(\omega, \pi) - \left\{\int_0^{\omega^{\alpha-1}} + \int_{\omega^{\alpha-1}}^\pi\right\} \phi_1(t)\,\frac{\partial}{\partial t}\,E_r(\omega, t)\,dt.$$

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* In fact we may show, for example by taking $\rho_n = [\log n\,(\log\log n)^{-\frac{1}{2}}]$ for $n \geqslant 3$, that $a_n = o\{n^{-1}(\log n)^3\}$.

† The preliminary simplification is easily effected. For example, if $x = 0$, we may replace $f(t)$ by

$$f(t) - \tfrac{1}{2}a_0 - (\tfrac{1}{2}a_0 - s)\sum_{n=1}^{[\kappa]} p_n \cos nt,$$

where

$$1 + \sum_{n=1}^{[\kappa]} p_n \cos nt = (1-\cos t)^{[\kappa]},$$

so that $\phi(t)$ is replaced by

$$\phi(t) - (\tfrac{1}{2}a_0 - s)(1-\cos t)^{[\kappa]}$$

and condition (4.1) is unchanged.

‡ *E.g.* because

$$|E_r(\omega t)| = \left|\int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^r \cos xt\,dt\right| < \omega e^{r\omega^\alpha}$$

uniformly in t.

Now arguing as in Theorem 1 we obtain, for $r \geqslant 1$,

$$E_r(\omega, t) = O(e^{(r-1)\omega^\alpha} t^{-1}) + O(e^{r\omega^\alpha} \omega^{\alpha-1} t^{-2}),$$

and hence, in particular, $E_r(\omega, \pi) = o(e^{r\omega^\alpha})$.

Next, for $r \geqslant 1$,

$$\frac{\partial}{\partial t}\{tE_r(\omega, t)\} = \alpha r \int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{r-1} e^{x^\alpha} x^\alpha \cos xt\, dx$$

$$= \alpha r (e^{\omega^\alpha} - 1)^{r-1} e^{\omega^\alpha} \int_{\omega'}^{\omega''} \cos xt \quad (0 < \omega' < \omega'' < \omega),$$

i.e.

$$t \frac{\partial}{\partial t} E_r(\omega, t) + E_r(\omega, t) = O(e^{r\omega^\alpha} \omega^\alpha t^{-1}),$$

while

$$\frac{t}{\alpha r} E_r(\omega, t) = \int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{r-1} e^{x^\alpha} x^\alpha \frac{\sin xt}{x}\, dx$$

$$= (e^{\omega^\alpha} - 1)^{r-1} e^{\omega^\alpha} \omega^\alpha \int_{\omega'}^{\omega''} \frac{\sin xt}{x}\, dx \quad (0 < \omega' < \omega'' < \omega)$$

$$= O(e^{r\omega^\alpha} \omega^\alpha).$$

Hence

$$\frac{\partial}{\partial t} E_r(\omega, t) = O(e^{r\omega^\alpha} \omega^\alpha t^{-2}),$$

and so, for $r \geqslant 1$,

$$\int_0^{\omega^{\alpha-1}} \phi_1(t) \frac{\partial}{\partial t} E_r(\omega, t)\, dt = O(e^{r\omega^\alpha} \omega^\alpha) \int_0^{\omega^{\alpha-1}} o(t^{-2})\, dt$$

$$= O(e^{r\omega^\alpha} \omega^\alpha)\, o\{\omega^{(\alpha-1)(\kappa-1)}\}$$

$$= o(e^{r\omega^\alpha})$$

as $\omega \to \infty$, since $\alpha + (\alpha-1)(\kappa-1) = 0$.

Again, writing

$$\frac{1}{\alpha r} \frac{\partial}{\partial t}\{tE_r(\omega, t)\} = \left\{\int_0^A + \int_A^\omega\right\} (e^{\omega^\alpha} - e^{x^\alpha})^{r-1} e^{x^\alpha} x^\alpha \cos xt\, dx$$

and integrating the second integral by parts $r-1$ times, r being a positive integer, we easily obtain, for $\omega > A > 0$,

$$\frac{1}{\alpha r} \frac{\partial}{\partial t}\{tE_r(\omega, t)\}$$

$$= O(e^{(r-1)\omega^\alpha} t^{-r+1}) \pm \frac{1}{t^{r-1}} \int_A^\omega \left(\frac{d}{dx}\right)^{r-1} \{(e^{\omega^\alpha} - e^{x^\alpha})^{r-1} e^{x^\alpha} x^\alpha\}\, S(xt)\, dx,$$

where $S(x) = \sin x$ or $\cos x$.

Now, by the binomial theorem,

$$\left(\frac{d}{dx}\right)^{r-1}\{(e^{\omega^\alpha}-e^{x^\alpha})^{r-1}e^{x^\alpha}x^\alpha\} = \sum_{i=1}^{r} a_i e^{(r-i)\omega^\alpha}\left(\frac{d}{dx}\right)^{r-1}(e^{ix^\alpha}x^\alpha)$$

$$= \sum_{i=1}^{r} a_i e^{(r-i)\omega^\alpha} e^{ix^\alpha} \sum_{j=1}^{r} \beta_j x^{j\alpha-r+1},$$

and, if A is chosen large enough,

$$\int_A^\omega e^{ix^\alpha} x^{j\alpha-r+1} S(xt)\,dx = e^{i\omega^\alpha}\omega^{j\alpha-r+1}\int_{\omega'}^{\omega} S(xt)\,dx \quad (A<\omega'<\omega)$$

$$= O(e^{i\omega^\alpha}\omega^{j\alpha-r+1}t^{-1}).$$

Hence

$$\frac{1}{t^{r-1}}\int_A^\omega \left(\frac{d}{dx}\right)^{r-1}\{(e^{w^\alpha}-e^{x^\alpha})^{r-1}e^{x^\alpha}x^\alpha\}S(xt)\,dx = \sum_{i=1}^{r}\sum_{j=1}^{r} O(e^{(r-i)\omega^\alpha}e^{i\omega^\alpha}\omega^{j\alpha-r+1}t^{-r})$$

$$= O(e^{r\omega^\alpha}\omega^{r\alpha-r+1}t^{-r}),$$

and thus, since $E_r(\omega, t) = O(e^{(r-1)\omega^\alpha}t^{-1}) + O(e^{r\omega^\alpha}\omega^{\alpha-1}t^{-2})$, we obtain

$$\frac{\partial}{\partial t}E_r(\omega, t) = O(e^{(r-1)\omega^\alpha}t^{-r}) + O(e^{r\omega^\alpha}\omega^{r\alpha-r+1}t^{-r-1}) + O(e^{r\omega^\alpha}\omega^{\alpha-1}t^{-3}),$$

if r is an integer greater than 1

It follows that, if r is an integer and $r>\kappa>1$,

$$\int_{\omega^{\alpha-1}}^{\pi}\phi_1(t)\frac{\partial}{\partial t}E_r(\omega, t)\,dt$$

$$= O(e^{(r-1)\omega^\alpha})\int_{\omega^{\alpha-1}}^{\pi} o(t^{\kappa-r})\,dt + O(e^{r\omega^\alpha}\omega^{r\alpha-r+1})\int_{\omega^{\alpha-1}}^{\pi} o(t^{\kappa-r-1})\,dt$$

$$+ O(e^{r\omega^\alpha}\omega^{\alpha-1})\int_{\omega^{\alpha-1}}^{\pi} o(t^{\kappa-3})\,dt$$

$$= O(e^{(r-1)\omega^\alpha})\{C+\omega^{(\alpha-1)(\kappa-r+1)}\} + O(e^{r\omega^\alpha}\omega^{r\alpha-r+1})\,o\{\omega^{(\alpha-1)(\kappa-r)}\}$$

$$+ O(e^{r\omega^\alpha}\omega^{\alpha-1})\,o(\omega^{-(\alpha-1)})$$

$$= o(e^{r\omega^\alpha}) + o(e^{r\omega^\alpha}) + o(e^{r\omega^\alpha}) = o(e^{r\omega^\alpha})$$

as $\omega\to\infty$, which completes the proof.

As a corollary of Theorem 7 and Theorem A we have

THEOREM 8. *If $\kappa>1$, and $\phi(t)$ satisfies* (4.1), *i.e.*

$$\int_0^t \phi(u)\,du = o(t^\kappa), \tag{4.2}$$

and if $$a_n > -Kn^{-1/\kappa}, \tag{4.3}$$

then the Fourier series (2.1) *is convergent at* $t = x$ *to sum* s.

It follows from the next theorem that the indices in Theorem 8 cannot be improved.

THEOREM 9. *If* $1 < \kappa' < \kappa$, *there is an even function* $\phi(t)$ *such that*

$$\int_0^t \phi(u)\,du = o(t^{\kappa'}) \tag{4.4}$$

and $a_n = O(n^{-1/\kappa})$, *while the Fourier series of* $\phi(t)$ *diverges at* $t = 0$.

Proof. We define $\phi(t)$ is in Theorem 6, but this time take

$$a_n = \lambda_n^{-1/\kappa'} = \{(n+1)!\}^{-1/\kappa'} \quad (n \geqslant 1).$$

Then, as in Theorem 6, if $a_n < t \leqslant a_{n-1}$,

$$\int_0^t \phi(u)\,du = o\left\{\frac{1}{(n+1)!}\right\},$$

while $$t^{-\kappa'} < a_n^{-\kappa'} = (n+1)!,$$

and hence (4.4) is satisfied.

Now write, as in Theorem 6,

$$s_{\lambda_n} = \frac{2}{\pi} \sum_{i=1}^{\infty} c_i \int_{a_i}^{a_{i-1}} \frac{\sin \lambda_i t \sin \lambda_n t}{t}\,dt + o(1)$$

$$= \sum_{i=1}^{\infty} I_i + o(1) = \sum_{i<n} I_i + I_n + \sum_{i>n} I_i + o(1).$$

Then $$I_n \sim \frac{c_n}{\pi} \log \frac{a_{n-1}}{a_n} = \frac{1}{\pi\kappa'} \sqrt{\log (n+1)}.$$

On the other hand

$$\sum_{i<n} I_i = \sum_{i=1}^{n-1} O\left\{\frac{c_i}{a_i(\lambda_n - \lambda_i)}\right\} = o\left\{\sum_{i-1}^{n-1} \frac{\{(i+1)!\}^{1/\kappa'}}{(n+1)! - (i+1)!}\right\} = o(1),$$

while $$\sum_{i>n} I_i = \sum_{i=n+1}^{\infty} O\left(\frac{c_i \lambda_n a_{i-1}}{\lambda_i a_i}\right)$$

$$= (n+1)!\, o\left\{\sum_{i=n+1}^{\infty} \frac{(i+1)^{1/\kappa'}}{(i+1)!}\right\} = o(1).$$

Thus $s_{\lambda_n} \to \infty$ as $n \to \infty$.

Finally, as in Theorem 6, if $\rho_n = [\log n / \log \log n]$,

$$a_n = \frac{2}{\pi} \sum_{i=1}^{\infty} c_i \int_{a_i}^{a_{i-1}} \sin \lambda_i t \sin nt\,dt = \sum_{i=1}^{\infty} J,$$

where
$$\sum_{i\leqslant\rho_n} J_i = \sum_{i\leqslant\rho_n} O\left(\frac{c_i\lambda_i a_{i-1}}{n}\right) = \frac{1}{n}\sum_{i\leqslant\rho_n} o\{(i+1)(i!)^{1-1/\kappa}\}$$
$$= o\left\{\frac{1}{n}\sum_{i\leqslant\rho_n} i^{i(1-1/\kappa)}\right\} = o\left\{\frac{1}{n}\rho_n^{\rho_n(1-1/\kappa)}\right\}$$
$$= o\left\{\frac{1}{n}e^{(1-1/\kappa)\log n}\right\} = o(n^{-1/\kappa})$$

and
$$\sum_{i>\rho_n} J_i = \sum_{i>\rho_n} O\left(\frac{c_i}{\lambda_i - n}\right) = o\{(\rho_n/e)^{-\rho_n}\}$$
$$= o\{e^{(-1+o(1))\log n}\}$$
$$= o(n^{-1/\kappa}).$$

Finally, we return to conditions (3.3) and consider the type λ, where $\lambda(\omega) = e^{(\log\omega)^2}$ $(\omega > 1)$, $\lambda(\omega) = 1$ $(0 \leqslant \omega \leqslant 1)$.

Theorem 10. *If* (3.3) *holds, i.e.*

$$\phi_1(t) = \int_0^t \phi(u)\,du = o\left(\frac{t}{\log 1/t}\right) \tag{4.5}$$

as $t \to +0$, *then the Fourier series* (2.1) *is summable** $(e^{(\log\omega)^2}, 2)$ *at* $t = x$ *to sum* s.

Proof. Writing, for $\omega > 1$,

$$E(\omega, t) = \frac{4}{t}\int_1^\omega \{e^{(\log\omega)^2} - e^{(\log x)^2}\} e^{(\log x)^2}\frac{\log x}{x}\sin xt\,dx$$
$$= (e^{(\log\omega)^2} - 1)^2\int_0^1 \cos xt\,dx + \int_1^\omega \{e^{(\log\omega)^2} - e^{(\log x^2)}\}^2 \cos xt\,dx,$$

we have, since $E(\omega, t) = O\{\omega e^{2(\log\omega)^2}\}$ uniformly in t, for $\omega > 1$,

$$\tfrac{1}{2}\pi c_2(\omega) = \phi_1(\pi)E(\omega, \pi) - \left\{\int_0^{\omega^{-1}\log\omega} + \int_{\omega^{-1}\log\omega}^{\pi}\right\}\phi_1(t)\frac{\partial}{\partial t}E(\omega, t)\,dt + o(e^{2(\log\omega)^2}.$$

Now, for $\omega < 1$, by the second mean value theorem,

$$\frac{\partial}{\partial t}E(\omega, t) = -(e^{(\log\omega)^2} - 1)^2\int_0^1 x\sin xt\,dx - \int_1^\omega \{e^{(\log\omega)^2} - e^{(\log x)^2}\}^2 x\sin xt\,dx$$
$$= O(e^{2(\log\omega)^2}) + O(e^{2(\log\omega)^2}\omega t^{-1}),$$

* *I.e.* $(\lambda, 2)$, where $\lambda(\omega)$ is as just defined.

and hence

$$\int_0^{\omega^{-1}\log\omega} \phi_1(t)\frac{\partial}{\partial t}E(\omega,t)\,dt$$

$$= O(e^{2(\log\omega)^2})\int^{\omega^{-1}\log\omega} o\left(\frac{t}{\log 1/t}\right)dt + O(e^{2(\log\omega)^2}\omega)\int_0^{\omega^{-1}\log\omega} o\left(\frac{1}{\log 1/t}\right)dt$$

$$= O(e^{2(\log\omega)^2}o(\omega^{-2}\log\omega) + O(e^{2(\log\omega)^2}\omega)\,o(\omega^{-1})$$

$$= o(e^{2(\log\omega)^2}$$

as $\omega\to\infty$.

Again*, as in Theorem 1, for $\omega > 1$,

$$E(\omega,t) = O(e^{(\log\omega)^2}t^{-1}) + O(e^{2(\log\omega)^2}\log\omega\,\omega^{-1}t^{-2}),$$

and hence, in particular, $E(\omega,\pi) = o(e^{2(\log\omega)^2})$.

Also, as in Theorem 7, for $\omega > 1$,

$$\frac{\partial}{\partial t}\{tE(\omega,t)\} = 4\int_1^{\omega}(e^{(\log\omega)^2} - e^{(\log x)^2})\,e^{(\log x)^2}\log x\cos xt\,dx$$

$$= -\frac{4}{t}\int_1^{\omega}\frac{d}{dx}\{(e^{(\log\omega)^2} - e^{(\log x)^2})\,e^{(\log x)^2}\log x\}\sin xt\,dx$$

$$= O(e^{(\log\omega)^2}t^{-1}) + O\{e^{2(\log\omega)^2}(\log\omega)^2\omega^{-1}t^{-2}\},$$

and so

$$\frac{\partial}{\partial t}E(\omega,t) = O(e^{(\log\omega)^2}t^{-2}) + O\{e^{2(\log\omega)^2}(\log\omega)^2\omega^{-1}t^{-3}\}.$$

Hence

$$\int_{\omega^{-1}\log\omega}^{\pi}\phi_1(t)\frac{\partial}{\partial t}E(\omega,t)\,dt$$

$$= O(e^{(\log\omega)^2})\int_{\omega^{-1}\log\omega}^{\pi} o\left(\frac{1}{t\log 1/t}\right)dt$$

$$+ O\{e^{2(\log\omega)^2}(\log\omega)^2\omega^{-1}\}\int_{\omega^{-1}\log\omega}^{\pi} o\left(\frac{1}{t^2\log 1/t}\right)dt$$

$$= O(e^{(\log\omega)^2})\,o(\log\log\omega) + O\{e^{2(\log\omega)^2}(\log\omega^2)\,\omega^{-1}\}\,o\{\omega(\log\omega)^{-2}\}$$

$$= o(e^{2(\log\omega)^2}) + o(e^{2(\log\omega)^2})$$

$$= o(e^{2(\log\omega)^2})^2$$

as $\omega\to\infty$, which completes the proof.

As a corollary of Theorem 10 and Theorem A we now have

* $\lambda(x)$ is differentiable even at $x = 1$.

THEOREM 11. *If*

$$\int_0^t \phi(u)\,du = o\left(\frac{t}{\log 1/t}\right) \tag{4.6}$$

as $t \to +0$, *and*

$$a_n > -K\frac{\log n}{n}, \tag{4.7}$$

then the Fourier series (2.1) *is convergent to sum* s *at* $t = x$.

If in the example to Theorem 6 we choose c_n so that $c_{n_i} = \{\log(n_i+1)\}^{-\frac{1}{2}}$ and $c_n = 0$ for $n \neq n_i$, where $\{n_i\}$ is a rapidly increasing sequence of integers, we can make

$$\sum_{i \leqslant \rho_n} J_i = o(n^{-1}\rho_n{}^2 \log \rho_n),$$

and thus, taking $\rho_n = [(\log n)(\log\log n)^{-\frac{1}{2}}]$ for $n \geqslant 3$, make

$$a_n = o\{n^{-1}(\log n)^2\},$$

while $s_{\lambda_{n_i}} \to \infty$ as $i \to \infty$. Thus *there is an even function* $\phi(t)$ *satisfying* (4.6) *and such that its Fourier series* (2.1) *diverges at* $t = 0$, *while*

$$a_n = o\{n^{-1}(\log n)^2\}.$$

The problems of bridging the gap between this example and Theorem 11 still remains.

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ON THE CONVERGENCE AND SUMMABILITY OF POWER SERIES ON THE CIRCLE OF CONVERGENCE (II)

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This paper is a continuation of another paper which has appeared elsewhere†. The connexion between the two parts consists in the fact that both of them treat the problem of the behaviour of power series of class H^λ on the circle of convergence. The first part was concerned with the case $\lambda \geqslant 1$. The present part deals with the case $0 < \lambda < 1$, first discussed by Hardy and Littlewood‡. Different methods are used in the two parts, and a knowledge of the first part of the paper is not indispensable here.

1.1. Let

$$F(z) = c_0 + c_1 z + c_2 z^2 + \dots \tag{1.1.1}$$

be a function analytic for $|z| < 1$. If, for some $\lambda > 0$, the integral

$$\int_{-\pi}^{\pi} |F(re^{i\theta})|^\lambda \, d\theta$$

remains bounded when $r \to 1$, the function $F(z)$ and the series $c_0 + c_1 z + \dots$ are said to belong to the class H^λ. We then write $F(z) \in H^\lambda$. Instead of H^1 we write H.

If $\lambda \geqslant 1$, a necessary and sufficient condition for the function $F(z)$ to belong to the class H^λ is that the series

$$c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \dots \tag{1.1.2}$$

is the Fourier series of a function of the class L^λ. Hence, in virtue of a well-known result, if $\lambda \geqslant 1$, the power series of the function $F(z) \in H^\lambda$ is

† See Zygmund [2].

‡ See Hardy and Littlewood [1].

summable (C, ϵ), $\epsilon > 0$, at almost every point of the circumference $|z| = 1$. The problem whether in this result we may replace summability (C, ϵ) by ordinary convergence remains open, but the answer is probably negative†.

It is well known that, if $F(z) \epsilon H^\lambda$, then, for almost every θ, the limit

$$\lim_{z \to e^{i\theta}} F(z) = F(e^{i\theta}) \tag{1.1.3}$$

exists and is finite, provided that z tends to $e^{i\theta}$ along non-tangential paths. Moreover, it has recently been shown that, if $0 < \lambda < 1$, the series (1.1.2) is summable $(C, \alpha+\epsilon)$ at almost every point, where

$$\alpha = \frac{1}{\lambda} - 1 \tag{1.1.4}$$

and ϵ is any positive number‡.

The object of this section is to show that the latter result holds with $\epsilon = 0$. In other words we have

THEOREM 1. *If $0 < \lambda < 1$, if the number α is given by* (1.1.4), *and if* (1.1.1) *is a power series of the class H^λ, then the series* (1.1.2) *is summable* (C, α) *for almost every θ.*

1.2. We recall the definition and certain properties of summability (C, α).

Given a series

$$\sum_{\nu=0}^{\infty} u_\nu, \tag{1.2.1}$$

we consider the numbers S_ν^α and A_ν^α defined by the equations

$$\sum_{\nu=0}^{\infty} S_\nu^\alpha z^\nu = \frac{1}{(1-z)^{\alpha+1}} \sum_{\nu=0}^{\infty} u_\nu z^\nu, \tag{1.2.2}$$

$$\sum_{\nu=0}^{\infty} A_\nu^\alpha z^\nu = \frac{1}{(1-z)^{\alpha+1}}, \tag{1.2.3}$$

† In the well known examples, given by Kolmogoroff [1], [2], of functions $f(\theta) \epsilon L$ whose Fourier series diverge almost everywhere (or even everywhere) the conjugate series is not a Fourier series, so that the corresponding power series does not belong to H.

‡ Hardy and Littlewood [1]. That the result fails for $\epsilon < 0$, may be seen from the function $F(z) = (1-z)^{\eta-1/\lambda}$, which belongs to H^λ if $\eta > 0$. Since the coefficients c_n of the Maclaurin development of $F(z)$ are exactly of the order $n^{-\eta-1+1/\lambda} = n^{\alpha-\eta}$, the series (1.1.2) is in this case summable $(C, \alpha-\eta)$ for no θ,

and we say that the series (1.2.1) is summable (C, α) to sum s, if the ratio

$$\sigma_n^{\alpha} = \frac{S_n^{\alpha}}{A_n^{\alpha}}$$

tends to s as n tends to $+\infty$. The numbers S_ν^{α} and σ_ν^{α} are called respectively the α-th Cesàro sums and the α-th Cesàro means of the series (1.2.1). It is easy to see that

$$\lim_{n\to\infty} \frac{A_n^{\alpha}}{n^{\alpha}} = \frac{1}{\Gamma(\alpha+1)}. \tag{1.2.4}$$

If the series (1.2.1) is summable (C, α) to s, it is also summable (C, α') to s, provided that $\alpha' > \alpha > -1$.

Let U_n^{α}, V_n^{α}, W_n^{α} denote respectively the Cesàro sums of the series

$$\sum_{\nu=0}^{\infty} u_\nu, \quad \sum_{\nu=0}^{\infty} v_\nu, \quad \sum_{\nu=0}^{\infty} w_\nu.$$

If the last series is the Cauchy product of the first two, then

$$W_n^{\beta+\gamma+1} = \sum_{\nu=0}^{n} U_\nu^{\beta} V_{n-\nu}^{\gamma}. \tag{1.2.5}$$

This formula easily gives the well-known result that *the product of two series summable respectively* (C, β) *to* s *and* (C, γ) *to* t *(where* $\beta > -1, \gamma > -1$*) is summable* $(C, \beta+\gamma+1)$ *to* st.

1.3. In this paper we always use C to mean an absolute constant, not necessarily the same at each occurrence.

The proof of Theorem 1 is based on a number of lemmas.

LEMMA 1. *Let* $f(\theta)$ *be a function of the class* L^k, $k \geqslant 1$, *over the interval* $-\pi \leqslant \theta \leqslant \pi$, *and let* $u(r, \theta)$ *be the Poisson integral of* $f(\theta)$. *If*

$$\left| \int_0^{\theta} |f(t)|^k dt \right| \leqslant \mu |\theta| \quad \textit{for} \quad |\theta| \leqslant \pi,$$

then

$$|u(r, \theta)| \leqslant C \left\{ \mu \left(1 + \frac{|\theta|}{\delta} \right) \right\}^{1/k}, \tag{1.3.1}$$

$$\left| \int_0^{\theta} |u(n, t)|^k dt \right| \leqslant C\mu |\theta|, \tag{1.3.2}$$

where $\delta = 1-r$.

In the case $k = 1$ this lemma was proved by Hardy and Littlewood†. The general result follows from this special case by a simple application of Jensen's inequality.

For let

$$P_r(t) = \frac{1-r^2}{1-2r\cos t+r^2}$$

be the Poisson kernel. Since the function x^k, with $k \geqslant 1$, is convex in the interval $(0, +\infty)$, and the integral of $P_r(t)$ over the interval $(-\pi, +\pi)$ is equal to 2π,

$$|u(r,\theta)|^k = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta+t)\,P_r(t)\,dt\right|^k \leqslant \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta+t)|\,P_r(t)\,dt\right|^k$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta+t)|^k\,P_r(t)\,dt.$$

Hence $|u(r,\theta)|^k$ is majorized by the Poisson integral of the function $|f|^k$. From this and the special inequality of Hardy and Littlewood just mentioned we obtain (1.3.1) and (1.3.2) (we may assume that $C \geqslant 1$, so that $C^{1/k} \leqslant C$)‡.

LEMMA 2. *Let $\{f_n(\theta)\}$ be a sequence of non-negative measurable functions of period 2π, tending to 0 almost everywhere, and satisfying for $n = 1, 2, \ldots$ an inequality*

$$f_n(\theta) \leqslant f(\theta), \quad \textit{where} \quad f(\theta) \in L.$$

Let $\mu_n(\theta)$ denote the smallest number such that

$$\left|\frac{1}{h}\int_{\theta}^{\theta+h} f_n(u)\,du\right| \leqslant \mu_n(\theta) \quad \textit{for} \quad |h| \leqslant \pi.$$

Under the above conditions, the sequence $\{\mu_n(\theta)\}$ tends to 0 almost everywhere.

Proof. From the conditions of the lemma it follows that we can find a set $E \subset (-\pi, \pi)$ such that on E the functions $f_n(\theta)$ tend to 0 uniformly. In particular

$$f_n(\theta) \leqslant \epsilon \quad \text{for} \quad \theta \in E, \quad n > n_0(\epsilon). \tag{1.3.3}$$

Let CE denote the set complementary to E, and let $\chi(\theta)$ be the characteristic function of the set CE. The function $f(\theta)\chi(\theta)$ is integrable

† See Hardy and Littlewood [2], 178 and 182.

‡ If *e.g.* $|\theta| \geqslant \delta$, the term 1 on the right-hand side of (1.3.1) may obviously be dropped.

and vanishes in E, so that

$$(1.3.4) \qquad \left|\int_\theta^{\theta+h} f_n(t)\chi(t)\,dt\right| \leqslant \left|\int_\theta^{\theta+h} f(t)\chi(t)\,dt\right| = o(|h|)$$

at almost every point $\theta \in E$.

Let E_1 be a sub-set of E, such that (1.3.4) holds uniformly on E_1. In particular,

$$(1.3.5) \quad \left|\int_\theta^{\theta+h} f_n(t)\chi(t)\,dt\right| \leqslant \left|\int_\theta^{\theta+h} f(t)\chi(t)\,dt\right| \leqslant \epsilon|h|$$

$$\text{for} \quad \theta \in E_1, \quad |h| \leqslant h_0(\epsilon).$$

From (1.3.3) we see that

$$\left|\int_\theta^{\theta+h} f_n(t)\{1-\chi(t)\}\,dt\right| \leqslant \epsilon|h| \quad \text{for} \quad \theta \in E, \quad n > n_0(\epsilon).$$

From this and (1.3.5) it follows that

$$(1.3.6) \quad \left|\frac{1}{h}\int_\theta^{\theta+h} f_n(t)\,dt\right| \leqslant 2\epsilon \quad \text{for} \quad \theta \in E_1, \quad n > n_0, \quad |h| \leqslant h_0.$$

On the other hand,

$$(1.3.7) \quad \left|\frac{1}{h}\int_\theta^{\theta+h} f_n(t)\,dt\right| \leqslant \frac{1}{h_0}\int_{-\pi}^{\pi} f_n(t)\,dt \to 0 \quad \text{for} \quad h_0 \leqslant |h| \leqslant \pi.$$

From (1.3.6) and (1.3.7) it follows that $\mu_n(\theta) \leqslant 2\epsilon$ for $n > n_1$ and $\theta \in E_1$. Since the measure of the difference $(-\pi, \pi) - E_1$ may be as small as we please, the lemma follows.

LEMMA 3. *Let $\mathfrak{E}$ be an arbitrary perfect set of positive measure and of period 2π, and let $\phi(\theta)$ denote the function equal to 0 in $\mathfrak{E}$ and equal to d if θ belongs to an interval contiguous to $\mathfrak{E}$ and of length d. If α is any positive number, then for almost every point $\theta \in \mathfrak{E}$ the integral*

$$(1.3.8) \qquad \int_{-\pi}^{\pi} \frac{\phi^\alpha(\theta+u)}{|u|^{\alpha+1}}\,du$$

is finite.

This lemma is known†.

† See Marcinkiewicz [1], 55. The Lemma is stated there for the case when α is a positive integer, but the proof is perfectly general.

1.4. The following lemma is fundamental for the whole proof of Theorem 1.

LEMMA 4. *Let $F(z)$ be a function analytic for $|z|<1$ and of the class H^λ, where $\frac{1}{2}\leqslant\lambda<1$. Let α be given by the formula* (1.1.4). *If for every point θ of a perfect set $\mathfrak{E}\subset(-\pi,\pi)$ we have the inequality*

$$\left|\int_\theta^{\theta+h}|F(e^{iu})|^\lambda\,du\right|\leqslant\mu|h| \quad \text{for} \quad |h|\leqslant\pi \tag{1.4.1}$$

(with μ independent of θ), then the Cesàro means $\sigma_n^\alpha(\theta, F)$ of the series (1.1.1) *at the point $z=e^{i\theta}$ satisfy, for every $\theta\in\mathfrak{E}$, the inequality*

$$|\sigma_n^\alpha(\theta, F)|\leqslant\frac{C}{\alpha}\,\mu^{1/\lambda}\left\{1+\frac{1}{\mu}\int_{-\pi}^{\pi}\frac{\phi^\alpha(v+\theta)\,|F(e^{i(v+\theta)})|^\lambda}{|v|^{\alpha+1}}\,dv\right\}, \tag{1.4.2}$$

where $\phi(\theta)$ has the same meaning as in Lemma 3.

Moreover, the integral on the right-hand side of the inequality (1.4.2) *is finite for almost every point $\theta\in\mathfrak{E}$.*

Proof. Let us assume first that $F(z)\neq0$ for $|z|<1$. Then

$$F(z)=\Phi^2(z), \tag{1.4.3}$$

where the function $\Phi(z)$, any branch of $F^{\frac{1}{2}}(z)$, is analytic for $|z|<1$ and of the class $H^{2\lambda}$. Since $\lambda\geqslant\frac{1}{2}$, $\Phi(z)$ belongs *a fortiori* to H.

Besides (1.4.3) we also have to consider the decomposition

$$F(z)=\Psi_1(z)\,\Psi_2(z), \tag{1.4.4}$$

where

$$\Psi_1(z)=F^\lambda(z)\in H, \quad \Psi_2(z)=F^{1-\lambda}(z)\in H^{\lambda/(1-\lambda)}. \tag{1.4.5}$$

Let $S_n^\alpha(\theta, F)$ denote the α-th Cesàro sums of the series (1.1.1) at the point $z=e^{i\theta}$. By reason of (1.2.5) and (1.4.3), we have

$$S_n^\alpha(\theta, F)=\sum_{\nu=0}^{n}S_\nu^{\frac{1}{2}(\alpha-1)}(\theta, \Phi)\,S_{n-\nu}^{\frac{1}{2}(\alpha-1)}(\theta, \Phi),$$

and so, by Schwarz's inequality,

$$|S_n^\alpha(\theta, F)|\leqslant\sum_{\nu=0}^{n}|S_\nu^{\frac{1}{2}(\alpha-1)}(\theta, \Phi)|^2. \tag{1.4.6}$$

If

$$r=1-\frac{1}{n+1}, \quad \delta=1-r=\frac{1}{n+1},$$

the right-hand side of (1.4.6) does not exceed

$$e^2 \sum_{\nu=0}^{n} |S_\nu^{\frac{1}{2}(\alpha-1)}(\theta, \Phi)|^2 r^{2\nu} \leqslant e^2 \sum_{\nu=0}^{\infty} |S_\nu^{\frac{1}{2}(\alpha-1)}(\theta, \Phi)|^2 r^{2\nu}$$

$$= \frac{e^2}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{\nu=0}^{\infty} S_\nu^{\frac{1}{2}(\alpha-1)}(\theta, \Phi) r^\nu e^{i\nu\psi} \right|^2 d\psi$$

$$= \frac{e^2}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\Phi(re^{i(\theta+\psi)})}{(1-re^{i\psi})^{\frac{1}{2}(\alpha+1)}} \right|^2 d\psi \dagger$$

$$= \frac{e^2}{2\pi} \int_{-\pi}^{\pi} \frac{|F(re^{i(\theta+\psi)}|}{\Delta_r^{\alpha+1}(\psi)} d\psi,$$

where $\Delta_r(\psi) = (1-2r\cos\psi+r^2)^{\frac{1}{2}} = (\delta^2+4r\sin^2\frac{1}{2}\psi)^{\frac{1}{2}}$.

In order to simplify the notation we confine our attention to the point $\theta = 0$, which we suppose to belong to $\mathfrak{E}$.

The function $\Psi_1(z)$ [*cf.* (1.4.5)] is represented by the Poisson integral, *i.e.*,

$$\Psi_1(re^{i\psi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi_1(e^{iv})\, P_r(\psi-v)\, dv.$$

Hence, replacing the function F by $\Psi_1\Psi_2$ in the last integral but one (with $\theta = 0$), we obtain

$$(1.4.7) \quad |S_n^\alpha(0,\ F)| \leqslant \frac{e^2}{2\pi} \int_{-\pi}^{\pi} \frac{|\Psi_2(re^{i\psi})|}{\Delta_r^{\alpha+1}(\psi)} d\psi \, \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Psi_1(e^{iv})|\, P_r(\psi-v)\, dv$$

$$\leqslant \int_{-\pi}^{\pi} |\Psi_1(e^{iv})|\, dv \int_{-\pi}^{\pi} \frac{|\Psi_2(re^{i\psi})|\, P_r(\psi-v)}{\Delta_r^{\alpha+1}(\psi)} d\psi.$$

The last double integral may be decomposed into four integrals extended respectively over the four squares

$$0 \leqslant \pm v \leqslant \pi, \quad 0 \leqslant \pm\psi \leqslant \pi.$$

All the four integrals have a similar structure and it is sufficient to investigate one of them, *e.g.* the integral‡

$$I(r) = \int_0^\pi |\Psi_1(e^{iv})|\, dv \int_0^\pi \frac{|\Psi_2(re^{i\psi})|\, P_r(v-\psi)}{\Delta_r^{\alpha+1}(\psi)} d\psi.$$

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† *Cf.* (1.2.2).

‡ Observe that $P_r(v+\psi) \leqslant P_r(v-\psi)$ for $0 \leqslant v \leqslant \pi$, $0 \leqslant \psi \leqslant \pi$.

The integral $I(r)$ we decompose, in turn, into three integrals

$$(1.4.8)\quad I(r)=\int_0^{\delta}|\Psi_1(e^{iv})|\,dv\int_0^{\pi}\frac{|\Psi_2(re^{i\psi})|\,P_r(v-\psi)}{\Delta_r^{\alpha+1}(\psi)}\,d\psi$$

$$+\int_{\delta}^{\pi}|\Psi_1(e^{iv})|\,dv\int_0^{\frac{1}{2}v}\frac{|\Psi_2(re^{i\psi})|\,P_r(v-\psi)\,d\psi}{\Delta_r^{\alpha+1}(\psi)}$$

$$+\int_{\delta}^{\pi}|\Psi_1(e^{iv})|\,dv\int_{\frac{1}{2}v}^{\pi}\frac{|\Psi_2(re^{i\psi})|\,P_r(v-\psi)\,d\psi}{\Delta_r^{\alpha+1}(\psi)}$$

$$=I_1(r)+I_2(r)+I_3(r),$$

say, and we begin by showing that

$$(1.4.9)\qquad I_k(r)\leqslant\frac{C}{\alpha}\,\mu^{1/\lambda}\,\delta^{-\alpha}\quad\text{for}\quad k=1,\,2.$$

1.5. We note that

$$(1.5.1)\qquad \left|\int_0^h|\Psi|_1(e^{it})|\,dt\right|\leqslant\mu|h|,\qquad \left|\int_0^h|\Psi_2(e^{it})|\,dt\right|\leqslant\mu^{\alpha}|h|,$$

for $|h|\leqslant\pi$. The first of these inequalities is identical with (1.4.1) (for $\theta=0$). In order to obtain the second inequality, it is sufficient to replace $|\Psi_2|$ by $|\Psi_1|^{(1-\lambda)/\lambda}=|\Psi_1|^{\alpha}$ and then to apply Hölder's inequality.

In order to estimate $I_1(r)$, we break up the inner integral of $I_1(r)$ into two integrals, thus

$$(1.5.2)\qquad \int_0^{2\delta}\frac{|\Psi_2(re^{i\psi})|\,P_r(v-\psi)}{\Delta_r^{\alpha+1}(\psi)}\,dx+\int_{2\delta}^{\pi}\frac{|\Psi_2(re^{i\psi})|\,P_r(v-\psi)}{\Delta_r^{\alpha+1}(\psi)}\,d\psi$$

$$=A(r,v)+B(r,v),$$

say, and observe that $\Delta_r(\psi)$ satisfies the two inequalities

$$\Delta_r(\psi)\geqslant\delta,$$

$$\Delta_r(\psi)\geqslant C\psi\quad\text{for}\quad 0\leqslant\psi\leqslant\pi,$$

By reason of Lemma 1 (cf. (1.3.1) with $k=1$) and of the second inequality (1.5.1) we have

$$(1.5.3)\qquad A(r,v)\leqslant C\delta^{-\alpha-1}.\,C\mu^{\alpha}.\int_0^{2\delta}P_r(v-\psi)\,d\psi\leqslant C\mu^{\alpha}\delta^{-\alpha-1}.$$

If $0\leqslant v\leqslant\delta$, $2\delta\leqslant\psi\leqslant\pi$, then

$$P_r(\psi-v)\leqslant P_r(\tfrac{1}{2}\psi)\leqslant C\delta\psi^{-2},$$

and so

$$B(r,v)\leqslant\int_{2\delta}^{\pi}|\Psi_2(re^{i\psi})|.\,C\delta\psi^{-2}.\,C\psi^{-\alpha-1}\,d\psi\leqslant C\delta\int_{2\delta}^{\pi}\frac{|\Psi_2(re^{i\psi})|}{\psi^{\alpha+3}}\,d\psi.$$

An application of Lemma 1 (the inequality (1.3.1) with $k=1$) shows that $B(r, v)$ does not exceed $C\mu^\alpha \delta^{-\alpha-1}$. From this and (1.5.3) we see that the sum (1.5.2), which represents the inner integral of $I_1(r)$, does not exceed $C\mu^\alpha \delta^{-\alpha-1}$. Taking account of the first inequality (1.5.1), we obtain at once

$$I_1(r) \leqslant C\mu^{1+\alpha}\delta^{-\alpha} = C\mu^{1/\lambda}\delta^{-\alpha}, \tag{1.5.4}$$

which proves (1.4.9) for $k=1$.

Passing on to the integral $I_2(r)$, we observe that

$$\int_0^{\frac{1}{2}v} \frac{|\Psi_2(re^{i\psi})|\, P_r(v-\psi)}{\Delta_r^{\alpha+1}(\psi)}\, d\psi \leqslant P_r(\tfrac{1}{2}v) \int_0^{\frac{1}{2}v} \frac{|\Psi_2(re^{i\psi})|}{\Delta_r^{\alpha+1}(\psi)}\, d\psi. \tag{1.5.5}$$

Now
$$\int_0^{\frac{1}{2}\delta} \frac{|\Psi_2(re^{i\psi})|}{\Delta_r^{\alpha+1}(\psi)}\, d\psi \leqslant C\delta^{-\alpha-1} \int_0^{\frac{1}{2}\delta} |\Psi_2(re^{i\psi})|\, d\psi \leqslant C\mu^\alpha \delta^{-\alpha}$$

[*cf.* (1.5.1)], and, if $v \geqslant \delta$,

$$\int_{\frac{1}{2}\delta}^{\frac{1}{2}v} \frac{|\Psi_2(re^{i\psi})|}{\Delta_r^{\alpha+1}(\psi)}\, d\psi \leqslant C \int_{\frac{1}{2}\delta}^{\frac{1}{2}v} \frac{|\Psi_2(re^{i\psi})|}{\psi^{\alpha+1}}\, d\psi.$$

If we integrate this by parts and use the second inequality (1.5.1), then Lemma 1 (cf. (1.3.2)) shows that the last expression does not exceed $C\alpha^{-1}\mu^\alpha\delta^{-\alpha}$. Hence, by reason of (1.5.5),

$$I_2(r) \leqslant \frac{C}{\alpha}\, \mu^\alpha \delta^{-\alpha} \cdot \delta \int_\delta^\pi \frac{|\Psi_1(e^{iv})|}{v^2}\, dv.$$

Integrating this by parts and applying (1.5.1), we obtain (1.4.9) for $k=2$.

It remains to estimate $I_3(r)$. It is easy to see that

$$I_3(r) \leqslant C \int_\delta^\pi \frac{|\Psi_1(e^{iv})|}{v^{\alpha+1}}\, dv \int_{\frac{1}{2}v}^\pi |\Psi_2(re^{i\psi})|\, P_r(v-\psi)\, d\psi.$$

Here the inner integral divided by 2π does not exceed

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Psi_2(re^{i\psi})|\, P_r(v-\psi)\, d\psi &\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(v-\psi)\, d\psi\, \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Psi_2(e^{it})|\, P_r(\psi-t)\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Psi_2(e^{it})|\, dt\, \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(v-\psi)\, P_r(\psi-t)\, d\psi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Psi_2(e^{it})|\, P_{r^2}(v-t)\, dt,\end{aligned}$$

and the last integral is the Poisson integral $U(r^2, v)$ of the function

$$|\Psi_2(e^{it})| \in L^{\lambda/(1-\lambda)} \subset L.$$

Hence

$$(1.5.6) \qquad I_3(r) \leqslant C \int_\delta^\pi \frac{|\Psi_1(e^{iv})|\, U(r^2, v)}{v^{\alpha+1}}\, dv.$$

Let $\mathfrak{E}(\delta)$ denote the portion of $\mathfrak{E}$ contained in the interval (δ, π), and let $S(\delta) = (\delta, \pi) - \mathfrak{E}(\delta)$. Then

$$(1.5.7) \qquad \int_\delta^\pi \frac{|\Psi_1(e^{iv})|\, U(r^2, v)}{v^{\alpha+1}}\, dv = \int_{\mathfrak{E}(\delta)} + \int_{S(\delta)} = J(r) + K(r),$$

say. An application of Lemma 1 shows that

$$U(r^2, v) \leqslant C\mu^\alpha \quad \text{for } v \in \mathfrak{E}(\delta),$$

and so

$$(1.5.8) \qquad J(r) \leqslant C\mu^\alpha \int_{\mathfrak{E}(\delta)} \frac{|\Psi_1(e^{iv})|}{v^{\alpha+1}}\, dv \leqslant C\mu^\alpha \int_\delta^\pi \frac{|\Psi_1(e^{iv})|}{v^{\alpha+1}}\, dv$$

$$\leqslant \frac{C}{\alpha}\, \mu^{1/\lambda}\, \delta^{-\alpha}$$

[*cf.* (1.5.1)].

The set $S(\delta)$ consists of an enumerable sequence of intervals $d_i = (a_i, b_i)$ (the length of d_i will also be denoted by d_i). If we apply Lemma 1 [the inequality (1.3.1) with $k = \lambda/(1-\lambda)$] to the function $U(r^2, v)$ and to each of the intervals d_i, and observe that $(\delta + d_i)^\alpha \leqslant \delta^\alpha + d_i^\alpha$, we obtain†

$$K(r) \leqslant C\mu^\alpha \delta^{-\alpha} \sum_i \int_{a_i}^{b_i} (\delta + d_i)^\alpha \frac{|\Psi_1(e^{iv})|}{v^{\alpha+1}}\, dv$$

$$\leqslant C\mu^\alpha \int_\delta^\pi \frac{|\Psi_1(e^{iv})|}{v^{\alpha+1}}\, dv + C\mu^\alpha \delta^{-\alpha} \int_\delta^\pi \frac{\phi^\alpha(v)\, |\Psi_1(e^{iv})|}{v^{\alpha+1}}\, dv$$

$$= K_1(r) + K_2(r),$$

say, where the function $\phi(v)$ has the same meaning‡ as in Lemma 3. An

† The first inequality (1.5.1) may be written

$$\left| \int_0^h |\Psi_2|^{1/\alpha}\, dt \right| \leqslant \mu |h|.$$

‡ The two extreme intervals of $S(\delta)$ (if such intervals exist) may be only parts of intervals contiguous to $\mathfrak{E}$, but this is not relevant for our purpose.

integration by parts gives

$$K_1(r) \leqslant \frac{C}{\alpha}\,\mu^{1+\alpha}\,\delta^{-\alpha} = \frac{C}{\alpha}\,\mu^{1/\lambda}\,\delta^{-\alpha}.$$

Hence, taking into account (1.4.8), (1.4.9), (1.5.6), (1.5.7), (1.5.8), and the inequalities for $K(r)$ and $K_1(r)$, we obtain

$$I(r) \leqslant \frac{C}{\alpha}\,\mu^{1/\lambda}\,\delta^{-\alpha}\Big(1+\frac{1}{\mu}\int_{-\pi}^{\pi}\frac{\phi^{\alpha}(v)\,|F(e^{iv})|^{\lambda}}{|v|^{\alpha+1}}\,dv\Big).$$

We now recall that the right-hand side of (1.4.7) consists of four integrals one of which is $I(r)$, and that the remaining three are analogous to $I(r)$. Hence, observing that $\delta^{-\alpha} = (n+1)^{\alpha}$, and making use of (1.2.4), we deduce from the last inequality for $I(r)$ the inequality (1.4.2) in the case $\theta = 0$. The passage to general $\theta \in \mathfrak{E}$ is immediate.

We now show that the integral in (1.4.2) is finite at every point of density of $\mathfrak{E}$ at which the integral (1.3.8) is finite, and so, by Lemma 3, almost everywhere in $\mathfrak{E}$. Let us suppose for simplicity that $\theta = 0$ is such a point. Then

$$\int_{-\pi}^{\pi}\frac{\phi^{\alpha}(v)\,|F(e^{iv})|^{\lambda}}{|v|^{\alpha+1}}\,dv = \int_{-\epsilon}^{+\epsilon} + \int_{R},$$

where the set R consists of the two intervals $(-\pi, -\epsilon)$ and (ϵ, π). The integral over R is finite for every $\epsilon > 0$. Since $\theta = 0$ is a point of density of $\mathfrak{E}$, we may take ϵ so small that for every interval $d_i = (a_i, b_i)$ contiguous to $\mathfrak{E}$ and contained *e.g.* in $(0, \epsilon)$,

$$b_i \leqslant 2a_i.$$

For each such interval

$$(1.5.9)\qquad \int_{a_i}^{b_i}\frac{\phi^{\alpha}(v)\,|F(e^{iv})|^{\lambda}}{v^{\alpha+1}}\,dv \leqslant \frac{d_i^{\alpha}}{a_i^{\alpha+1}}\int_{a_i}^{b_i}|\Psi_1(e^{iv})|\,dv$$

$$\leqslant 2^{\alpha+1}\,\frac{d_i^{\alpha}}{b_i^{\alpha+1}}\,\mu d_i \leqslant 2^{\alpha+1}\,\mu\int_{a_i}^{b_i}\frac{\phi_{\alpha}(v)}{v^{\alpha+1}}\,dv,$$

so that

$$\int_0^{\epsilon}\frac{\phi^{\alpha}(v)\,|F(e^{iv})|^{\lambda}}{v^{\alpha+1}}\,dv \leqslant 2^{\alpha+1}\,\mu\int_0^{\pi}\frac{\phi^{\alpha}(v)}{v^{\alpha+1}}\,dv < \infty.$$

In the same way we prove the finiteness of the integral extended over the interval $(-\epsilon, 0)$, and so the last assertion of Lemma 4 is established.

It only remains to remove the hypothesis that the function $F(z)$ has no zeros for $|z| < 1$. It is well known that, if $F(z) \in H^{\lambda}$, then

$$F(z) = F_1(z) + F_2(z),$$

where the functions F_1 and F_2 belong to H^λ, and

$$F_k(z) \neq 0, \quad |F_k(e^{i\theta})| \leqslant 2|F(e^{i\theta})| \quad \text{for} \quad |z| < 1 \quad \text{and}\dagger \quad k = 1, 2.$$

Since Lemma 4 holds for the functions F_1 and F_2, it also holds for the function $F = F_1 + F_2$.

1.6. From Lemma 4 it follows that the sequence of the functions $\sigma_n^{\alpha}(\theta, F)$ is bounded at almost every point θ of the set $\mathfrak{E}$. In order to prove that for $\frac{1}{2} \leqslant \lambda < 1$ the expression $\sigma_n^{\alpha}(\theta, F)$ tends to F almost everywhere, we make for every $0 < \rho < 1$ the decomposition

$$(1.6.1) \quad F(z) = F_\rho(z) + G_\rho(z), \text{ where } F_\rho(z) = F(\rho z), \quad G_\rho(z) = F(z) - F_\rho(z).$$

When $\rho \to 1$, the function $|G_\rho(e^{i\theta})|^\lambda$ tends to 0 at almost every point θ and is majorized by an integrable function‡. In particular,

$$\int_{-\pi}^{\pi} |G_\rho(e^{i\theta})|^\lambda d\theta \to 0 \quad \text{as} \quad \rho \to 1.$$

From Lemma 2 it follows that the functions

$$\mu_\rho(\theta) = \max_{|h| \leqslant \pi} \frac{1}{h} \int_{\theta}^{\theta+h} |G_\rho(e^{iu})|^\lambda du$$

tend to 0 almost everywhere as $\rho \to 1$.

Let $\mathfrak{E} \subset (-\pi, \pi)$ be any perfect set on which $\mu_\rho(\theta)$ tends to 0 uniformly, and let μ_ρ denote the upper bound of $\mu_\rho(\theta)$ on $\mathfrak{E}$, so that $\mu_\rho \to 0$. Let $\phi(\theta)$ be the function defined in the statement of Lemma 3. Let $\mathfrak{E}_1$ be a perfect subset of $\mathfrak{E}$, such that the integral (1.3.8) is uniformly bounded on $\mathfrak{E}$, and such that§

$$\lim_{h \to 0} \frac{\text{meas } \mathfrak{E}(\theta - h, \theta + h)}{2h} = 1$$

uniformly for $\theta \in \mathfrak{E}_1$. We shall show that for $\theta \in \mathfrak{E}_1$ and $n = 1, 2, \ldots$,

$$(1.6.2) \qquad |\sigma_n^{\alpha}(\theta, G_\rho)| \leqslant \eta_\rho, \quad \textit{where} \quad \eta_\rho \to 0 \quad \text{as} \quad \rho \to 1.$$

---

† Hardy and Littlewood [3]. If $\omega(z)$ is the Blaschke product corresponding to the function $F(z)$, and so $F(z) = \omega(z) G(z)$, where $G(z) \neq 0$ for $|z| = 1$ (*cf.* F. Riesz [1]), then it is sufficient to take $F_1(z) = \big(\omega(z) - 1\big) G(z)$, $F_2(z) = G(z)$.

‡ Hardy and Littlewood [4].

§ Here $\mathfrak{E}(\theta - h, \theta + h)$ denotes the portion of the set $\mathfrak{E}$ contained in the interval $(\theta - h, \theta + h)$.

For the inequality (1.4.2) applied to the function G_ρ may be written

$$(1.6.3)\quad |\sigma_n^\alpha(\theta, G_\rho)| \leqslant \frac{C}{\alpha}\mu_\rho^{1/\lambda} + \frac{C}{\alpha}\mu_\rho^{-1+1/\lambda}\int_{-\epsilon}^{+\epsilon}\frac{\phi^\alpha(\theta+v)|G_\rho(e^{i(\theta+v)})|^\lambda}{|v|^{\alpha+1}}\,dv + \frac{C}{\alpha}\mu_\rho^{-1+1/\lambda}\int_R \frac{\phi^\alpha(\theta+v)|G_\rho(e^{i(\theta+v)})|^\lambda}{|v|^{\alpha+1}}\,dv,$$

where $R=(-\pi, -\epsilon)+(\epsilon, \pi)$. The first term on the right tends to 0 as $\rho\to 1$. The second term tends uniformly to 0, if ϵ is small enough and $\theta\,\epsilon\,\mathfrak{E}_1$. For the argument (1.5.9) shows that, if ϵ is small enough, the term in question does not exceed

$$\frac{C}{\alpha}\mu_\rho^{1/\lambda}2^{\alpha+1}\int_{-\epsilon}^{+\epsilon}\frac{\phi^\alpha(\theta+v)}{|v|^{\alpha+1}}\,dv \leqslant \frac{C}{\alpha}\mu_\rho^{1/\lambda}\int_{-\pi}^{\pi}\frac{\phi^\alpha(\theta+v)}{|v|^{\alpha+1}}\,dv,$$

and so it tends uniformly to 0 on the set $\mathfrak{E}_1$.

Finally, the third term on the right-hand side of (1.6.3) does not exceed

$$\frac{C}{\alpha}\mu_\rho^{-1+1/\lambda}\frac{1}{\epsilon^{\lambda+1}}\int_{-\pi}^{\pi}|G_\rho(e^{iu})|^\lambda\,du = o(1).$$

Hence (1.6.2) is established for $\theta\,\epsilon\,\mathfrak{E}_1$.

From (1.6.1) we obtain

$$\sigma_n^\alpha(\theta, F) = \sigma_n^\alpha(\theta, F_\rho) + \sigma_n^\alpha(\theta, G_\rho).$$

Here the second term on the right is uniformly small for $\theta\,\epsilon\,\mathfrak{E}_1$, provided that ρ is sufficiently near to 1 but fixed. On the other hand, for fixed $\rho<1$, the first term on the right tends uniformly to $F_\rho(e^{i\theta})=F(\rho e^{i\theta})$. Since $F(\rho e^{i\theta})$ tends to $F(e^{i\theta})$ for almost every θ, it follows that

$$\sigma_n^\alpha(\theta, F)\to F(e^{i\theta}) \textit{ almost everywhere in } \mathfrak{E}_1.$$

Observing that the measure of the set $\mathfrak{E}_1$ may be as near to 2π as we please, we deduce that

$$\sigma_n^\alpha(\theta, F)\to F(e^{i\theta}) \textit{ almost everywhere,}$$

and so Theorem 1 is established for $\frac{1}{2}\leqslant\lambda<1$.

To remove the last restriction, let us suppose *e.g.* that $F(z)\,\epsilon\,H^\lambda$ where $\frac{1}{4}\leqslant\lambda<\frac{1}{2}$. We may assume that $F(z)\neq 0$ for $|z|<1$, for otherwise $F(z)$ is a sum of two functions of the class H^λ satisfying this condition. We have then the formula (1.4.3), where $\Phi(z)\,\epsilon\,H^{2\lambda}$ and $\frac{1}{2}\leqslant 2\lambda<1$. Since, by the case already dealt with, the power series of the function $\Phi(z)$ is summable $\{C, (1/2\lambda)-1\}$ at almost every point $e^{i\theta}$, the power series of $F(z)$

is summable at those points by Cesàro's method of order

$$\left(\frac{1}{2\lambda}-1\right)+\left(\frac{1}{2\lambda}-1\right)+1=\frac{1}{\lambda}-1=\alpha.$$

Similarly we prove the theorem in the case $\frac{1}{8}\leqslant\lambda<\frac{1}{4}$, and so on. The proof of Theorem 1 is thus complete†.

2.1. It is well known that, if $S_n(\theta, f)$ denotes the n-th partial sum, at the point θ, of the Fourier series of a function $f\epsilon L^r$, where $r>1$, and if q is any positive number, then at almost every point θ

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu(\theta,f)-f(\theta)|^q=0. \tag{2.1.1}$$

More precisely, this equation holds at every point θ at which

$$\lim_{h\to0}\frac{1}{h}\int_\theta^{\theta+h}|f(\theta+u)-f(\theta)|^r\,du=0, \tag{2.1.2}$$

a relation which is known to be true almost everywhere‡.

Hardy and Littlewood have shown that, if $f\epsilon L$, then the relation (2.1.2), with $r=1$, may be true at individual points, without (2.1.1) being true, even with $q=1$, at those points§; and it was an open problem whether (2.1.1) holds almost everywhere for some $q>0$. Recently, however, Marcinkiewicz succeeded in proving that, if $f\epsilon L$ and $q=2$, then (2.1.1) holds at almost every point‖. Since the relation (2.1.1) is the stronger the larger is the exponent q, it may be of some interest to investigate whether Marcinkiewicz's result holds for $q>2$. To prove that this is really so is the object of this paragraph (see Theorem 2 below).

In proving his theorem, Marcinkiewicz made use of a formula of Hardy and Littlewood¶. It would be natural to look for an extension of that formula that would give the general relation (2.1.1). It seems, however, that such an argument, even if successful, would lead to rather complicated computations. I prefer, therefore, to apply another device

† From Theorem 1 we deduce that, if $F(z)$ is any function regular and schlicht for $z<1$, the Maclaurin expansion of $F(z)$ is summable (C, k), with $k>1$, at almost every point on the circumference $|z|=1$. This follows from Theorem 1 and from the fact that schlicht functions belong to every class H^λ with $\lambda<\frac{1}{2}$. *Cf.* Littlewood [1].

‡ See *e.g.* Zygmund [1], 238.

§ Hardy and Littlewood [2].

‖ Marcinkiewicz [2].

¶ Hardy and Littlewood [5]. The authors applied their formula to prove (2.1.1) in the case $r=q=2$. *Cf.* also Fejér [1].

of Hardy and Littlewood†, combined with a familiar argument based on the Hausdorff-Young inequality.

Henceforth $\bar{f}(\theta)$ will denote the function conjugate to $f(\theta)$ (*i.e.* the $(C, 1)$ sum of the series conjugate to the Fourier series of $f(\theta)$). The polynomial conjugate to $s_n(\theta, f)$ will be denoted by $\bar{s}_n(\theta, f)$. By C, as before, we shall mean an absolute constant, not necessarily the same at each occurrence.

2.2. THEOREM 2. *If $f(\theta)$ is an integrable function of period 2π, then, for every $q > 0$ and almost every θ,*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu(\theta, f) - f(\theta)|^q = 0, \tag{2.2.1}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |\bar{s}_\nu(\theta, f) - \bar{f}(\theta)|^q = 0. \tag{2.2.2}$$

It is known‡ that the relation (2.2.2) is at almost every point a consequence of (2.2.1), so that it would be sufficient to prove (2.2.1). The proof which follows will, however, give (2.2.2) simultaneously with (2.2.1). This proof has many features in common with that of Theorem 1, so that in places we may be concise.

We may, of course, assume that $f \geqslant 0$.

Let

$$f(\theta) \sim \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta), \tag{2.2.3}$$

so that $s_\nu(\theta, f)$ is the ν-th partial sum of the series (2.2.3). By $\sigma_n(\theta, f)$ and $\bar{\sigma}_n(\theta, f)$ we denote respectively the first arithmetic means of the sequences $\{s_\nu(\theta, f)\}$ and $\bar{s}_\nu\{(\theta, f)\}$, where $\nu = 0, 1, 2, \ldots$.

Let

$$t_n(\theta, f) = s_n(\theta, f) + i\bar{s}_n(\theta, f), \quad \tau_n(\theta, f) = \sigma_n(\theta, f) + i\bar{\sigma}_n(\theta, f).$$

Since $\sigma_n(\theta, f) \to f(\theta)$ and $\bar{\sigma}_n(\theta, f) \to \bar{f}(\theta)$ almost everywhere, it is sufficient to show that

$$\sum_{\nu=0}^{n} |t_\nu(\theta, f) - \tau_\nu(\theta, f)|^q = o(n) \quad \text{for almost every } \theta. \tag{2.2.4}$$

† *Cf.* Hardy and Littlewood [2], 184–185.

‡ Marcinkiewicz and Zygmund [1], 5. *Cf.* also Marcinkiewicz [2].

Let $\mathfrak{E}$ be a perfect set situated in the interval $(-\pi, \pi)$ and such that

$$(2.2.5) \qquad \left|\int_{\theta}^{\theta+h} f(u)\,du\right| \leqslant \mu|h| \quad \text{for} \quad \theta \in \mathfrak{E} \quad \text{and} \quad |h| \leqslant \pi,$$

where μ is independent of θ. At every point $\theta \in \mathfrak{E}$ we shall obtain an inequality analogous to (1.4.2). We assume for simplicity that the point $\theta = 0$ belongs to $\mathfrak{E}$, and we investigate the behaviour of the left-hand side of (2.2.4) for $\theta = 0$.

2.3. Let

$$D_n = t_n(0, f) - \tau_n(0, f) = \frac{1}{n+1} \sum_{\nu=1}^{n} \nu(a_\nu - ib_\nu).$$

Then $(n+1)\,D_n$ is the n-th partial sum of the power series

$$(2.3.1) \qquad \sum_{\nu=0}^{\infty} \nu(a_\nu - ib_\nu)\, z^\nu$$

at the point $z = 1$. If $U(r, \theta)$ and $V(r, \theta)$ denote respectively the Poisson integral of $f(\theta)$ and the harmonic function conjugate to $U(r, \theta)$, and if we write

$$F(z) = U(r, \theta) + iV(r, \theta) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu - ib_\nu)\, z^\nu, \quad \text{where} \quad z = re^{i\theta},$$

then the function (2.3.1) is equal to $zF'(z)$. It follows that

$$\sum_{\nu=0}^{\infty} (\nu+1)\, D_\nu z^\nu = \frac{zF'(z)}{1-z}.$$

We may assume that $q > 2$, so that, by the Hausdorff-Young theorem,

$$(2.3.2) \qquad \left\{\sum_{\nu=0}^{\infty} (\nu+1)^q |D_\nu|^q r^{\nu q}\right\}^{1/q} \leqslant \left\{\frac{r^p}{2\pi} \int_{-\pi}^{\pi} \left|\frac{F'(re^{i\psi})}{1-re^{i\psi}}\right|^p d\psi\right\}^{1/p},$$

where

$$p = \frac{q}{q-1}, \quad 1 < p < 2.$$

Let

$$\delta = 1 - r.$$

We require the following lemma.

LEMMA 5. *If* $f \geqslant 0$, *then*

$$(2.3.3) \qquad \left|\frac{d}{d\psi} U(r, \psi)\right| \leqslant \frac{C}{\delta} U(r, \psi), \quad \left|\frac{d}{d\psi} V(r, \psi)\right| \leqslant \frac{C}{\delta} U(r, \psi).$$

This result is implicitly contained in Hardy and Littlewood [2], p. 181, where the authors have proved that, if

$$P_r(t) = \frac{1-r^2}{1-2r\cos t+r^2}, \quad Q_r(t) = \frac{r\sin t}{1-2r\cos t+r^2},$$

are respectively the Poisson kernel and the conjugate kernel, then

$$\left|\frac{d}{dt}P_r(t)\right| \leqslant \frac{C}{\delta}P_r(t), \quad \left|\frac{d}{dt}Q_r(t)\right| \leqslant \frac{C}{\delta}P_r(t).$$

These inequalities give (2.3.3) at once, as is seen from the formulae

$$(2.3.4) \qquad \begin{cases} U(r,\psi) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(u)\,P_r(\psi-u)\,du, \\ V(r,\psi) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(u)\,Q_r(\psi-u)\,du. \end{cases}$$

Since $$|zF'(z)| = \left|\frac{d}{d\psi}U(r,\psi)+i\frac{d}{d\psi}V(r,\psi)\right|,$$

the inequalities (2.3.2) and (2.3.3) give

$$(2.3.5) \qquad \left\{\sum_{\nu=1}^{\infty}(\nu+1)^q|D_\nu|^q r^{\nu q}\right\}^{1/q} \leqslant \frac{C}{\delta}\left\{\int_{-\pi}^{\pi}\frac{U^p(r,\psi)}{\Delta_r{}^p(\psi)}\,d\psi\right\}^{1/p},$$

where $$\Delta_r(\psi) = (1-2r\cos\psi+r^2)^{\frac{1}{2}}.$$

Applying the first formula (2.3.4), we may write

$$(2.3.6) \qquad \int_{-\pi}^{\pi}\frac{U^p(r,\psi)}{\Delta_r{}^p(\psi)}\,d\psi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{U^{p-1}(r,\psi)}{\Delta_r{}^p(\psi)}\,d\psi\,\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)\,P_r(\psi-u)\,du$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)\,du\int_{-\pi}^{\pi}\frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi.$$

The last term we decompose into four integrals, extended respectively over the four squares

$$0 \leqslant \pm\psi \leqslant \pi, \quad 0 \leqslant \pm u \leqslant \pi,$$

and it is sufficient to consider *e.g.* the integral

$$S(r) = \int_0^{\pi}f(u)\,du\int_0^{\pi}\frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi,$$

which we break up into three parts

$$S(r) = \int_0^{\delta} f(u)\,du \int_0^{\pi} \frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi$$

$$+\int_{\delta}^{\pi} f(u)\,du \int_0^{\frac{1}{2}u} \frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi$$

$$+\int_{\delta}^{\pi} f(u)\,du \int_{\frac{1}{2}u}^{\pi} \frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi$$

$$= S_1(r)+S_2(r)+S_3(r),$$

say. We estimate the expressions S_1, S_2, S_3 in exactly the same way as we estimated the expressions I_1, I_2, I_3 in sections 1.5 and 1.6.

2.4. The inner integral of $S_1(r)$ is equal to

$$\int_0^{2\delta} \frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi + \int_{2\delta}^{\pi} \frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi - A(r,u) + B(r,u)$$

say. Using the first of the two inequalities

$$\Delta_r(\psi) \geqslant \delta, \quad \Delta_r(\psi) \geqslant C\psi \quad (0 \leqslant \psi \leqslant \pi),$$

and applying Lemma 1 (the inequality (1.3.1)), we obtain

$$A(r,u) \leqslant C\mu^{p-1}\delta^{-p} \int_0^{2\delta} P_r(\psi-u)\,d\psi \leqslant C\mu^{p-1}\delta^{-p}.$$

Similarly, observing that

$$P_r(\psi-u) \leqslant P_r(\tfrac{1}{2}\psi) \leqslant C\delta\psi^{-2} \quad \textit{for} \quad 0 \leqslant u \leqslant \delta, \quad 2\delta \leqslant \psi \leqslant \pi,$$

and again making use of Lemma 1, we obtain

$$B(r,u) \leqslant C\int_{2\delta}^{\pi} \frac{(\mu\psi\,\delta^{-1})^{p-1}\,\delta\psi^{-2}}{\psi^p}\,d\psi \leqslant C\mu^{p-1}\delta^{2-p}\int_{2\delta}^{\infty} \frac{d\psi}{\psi^3} \leqslant C\mu^{p-1}\delta^{-p}.$$

Hence

$$(2.4.1) \qquad S_1(r) \leqslant C\mu^{p-1}\delta^{-p}\int_0^{\delta} f(u)\,du \leqslant C\mu^p\,\delta^{1-p}.$$

We now consider the expression $S_2(r)$. The inner integral of $S_2(r)$ is

$$\int_0^{\frac{1}{2}u} \frac{U^{p-1}(r,\psi)\,P_r(\psi-u)}{\Delta_r{}^p(\psi)}\,d\psi \leqslant P_r(\tfrac{1}{2}u)\int_0^{\frac{1}{2}u} \frac{U^{p-1}(r,\psi)}{\Delta_r{}^p(\psi)}\,d\psi$$

$$\leqslant P_r(\tfrac{1}{2}u)\left(\int_0^{\frac{1}{2}u} U(r,\psi)\,d\psi\right)^{p-1}\left(\int_0^{\frac{1}{2}u} \frac{d\psi}{\Delta_r{}^{p/(2-p)}(\psi)}\right)^{2-p},$$

by Hölder's inequality. The last expression does not exceed

$$(2.4.2)\qquad C\delta u^{-2}\,.\,C\mu^{p-1}u^{p-1}\,.\,\left(\int_0^{\frac{1}{2}\delta}\frac{d\psi}{\Delta_r^{p/(2-p)}(\psi)}+\int_{\frac{1}{2}\delta}^{\frac{1}{2}u}\frac{d\psi}{\Delta_r^{p/(2-p)}(\psi)}\right)^{2-p}.$$

The first integrand does not exceed $\delta^{-p/(2-p)}$, so that the first integral is less than $\delta^{(2-2p)/(2-p)}$. Since the second integrand does not exceed $C^{p/(2-p)}\psi^{-p/(2-p)}$, the second integral is less than

$$\frac{2-p}{2(p-1)}\,C^{p/(2-p)}\,\delta^{-2(p-1)/(2-p)}.$$

It follows that the expression (2.4.2) is less than

$$C\mu^{p-1}\delta u^{p-3}\left(\delta^{-2(p-1)/(2-p)}+\frac{C^{p/(2-p)}\,\delta^{-2(p-1)/(2-p)}}{p-1}\right)^{2-p}\leqslant\frac{C\mu^{p-1}}{(p-1)^{2-p}}\,\delta^{3-2p}\,u^{p-3}.$$

Therefore
$$S_2(r)\leqslant\frac{C\mu^{p-1}\delta^{3-2p}}{(p-1)^{2-p}}\int_\delta^\pi\frac{f(u)}{u^{3-p}}\,du.$$

If we integrate the last integral by parts and make use of the inequality (2.2.5) for $\theta=0$, we easily obtain

$$(2.4.3)\qquad S_2(r)\leqslant\frac{C\mu^p\,\delta^{1-p}}{(2-p)(p-1)^{2-p}}\leqslant\frac{C\mu^p\,\delta^{1-p}}{(2-p)(p-1)}.$$

The integral $S_3(r)$ does not exceed

$$\begin{aligned}\int_\delta^\pi\frac{f(u)}{\Delta_r^p(\frac{1}{2}u)}\,du\int_{\frac{1}{2}u}^\pi U^{p-1}(r,\psi)\,P_r(\psi-u)\,d\psi&\\ \leqslant C\int_\delta^\pi\frac{f(u)}{u^p}\,du\,\frac{1}{2\pi}\int_{-\pi}^\pi U^{p-1}(r,\psi)\,P_r(\psi-u)\,d\psi&\\ \leqslant C\int_\delta^\pi\frac{f(u)}{u^p}\,du\left(\frac{1}{2\pi}\int_\pi^\pi U(r,\psi)\,P_r(\psi-u)\right)^{p-1}*&\\ =C\int_\delta^\pi\frac{f(u)}{u^p}\,U^{p-1}(r^2,u)\,du.&\end{aligned}$$

The last expression is analogous to the right-hand side of (1.5.7) and it may be treated in a similar way. We write

$$\int_\delta^\pi\frac{f(u)}{u^p}\,U^{p-1}(r^2,u)\,du=\int_{\mathfrak{L}(\delta)}+\int_{\mathfrak{F}(\delta)}=J(r)+K(r),$$

* By Hölder's inequality, since $0<p-1<1$.

say, where $\mathfrak{E}(\delta)$ is the portion of $\mathfrak{E}$ contained in the interval (δ, π) and $\mathfrak{F}(\delta) = (\delta, \pi) - \mathfrak{E}(\delta)$. Since

$$U(r^2, u) \leqslant C\mu \quad \text{for} \quad u \in \mathfrak{E}(\delta)$$

we obtain

$$J(r) \leqslant C\mu^{p-1} \int_{\mathfrak{E}(\delta)} \frac{f(u)}{u^p}\, du \leqslant C\mu^{p-1} \int_\delta^\pi \frac{f(u)}{u^p}\, du \leqslant \frac{C}{p-1}\, \mu^p \delta^{1-p},$$

on account of (2.2.5).

The set $\mathfrak{I}(\delta)$ consists of an enumerable sequence of intervals $d_i = (a_i, b_i)$. If we apply Lemma 1 (the inequality (1.3.1) with $k = 1$) to the function $U(r^2, u)$ and each of the intervals d_i, and observe that

$$(\delta + d_i)^{p-1} \leqslant \delta^{p-1} + d_i^{p-1},$$

we obtain

$$K(r) \leqslant C\mu^{p-1} \int_\delta^\pi \frac{f(u)}{u^p}\, du + C\mu^{p-1}\delta^{1-p} \int_0^\pi \frac{\phi^{p-1}(u) f(u)}{u^p}\, du$$

$$\leqslant \frac{C}{p-1}\, \mu^p \delta^{1-p} + C\mu^{p-1}\delta^{1-p} \int_0^\pi \frac{\phi^{p-1}(u)}{u^p}\, f(u)\, du,$$

where the function ϕ has the same meaning as in Lemma 3.

Collecting the results, we see that

$$S(r) \leqslant C_q \mu^p \delta^{1-p} + C\mu^{p-1}\delta^{1-p} \int_\delta^\pi \frac{\phi^{p-1}(u)}{u^p}\, f(u)\, du,$$

where

$$C_q = \frac{C}{(2-p)(p-1)} = \frac{C(q-1)^2}{q-2} \leqslant C_q,$$

if *e.g.* $q \geqslant 3$.

A similar estimate holds for the remaining three of the four integrals into which we decomposed the last integral in (2.3.6). From (2.3.5) and (2.3.6) we thus obtain

$$\left(\sum_{\nu=1}^\infty (\nu+1)^q |D_\nu|^q r^{\nu q}\right)^{1/q} \leqslant \frac{C}{\delta} \left(C_q \mu^p \delta^{1-p} + C\mu^{p-1}\delta^{1-p} \int_{-\pi}^\pi \frac{\phi^{p-1}(u) f(u)}{|u|^p}\, du\right)^{1/p},$$

or, passing to general θ and observing that $1 < p < 2$,

$$(2.4.4) \quad \sum_{\nu=1}^\infty (\nu+1)^q |t_\nu(\theta, f) - \tau_\nu(\theta, f)|^q r^{\nu q}$$

$$\leqslant \left(C_q \mu^p \delta^{1-2p} + C\mu^{p-1}\delta^{1-2p} \int_{-\pi}^\pi \frac{\phi^{p-1}(u+\theta) f(u+\theta)}{|u|^p}\, du\right)^{q/p},$$

where $\theta \in \mathfrak{E}$, $\delta = 1 - r$.

This inequality is an analogue of the inequality (1.4.2). The argument at the end of the section 1.5 shows that the integral on the right-hand side of (2.4.4) is finite at every point of density of $\mathcal{E}$ at which the integral

$$\int_{-\pi}^{\pi} \frac{\phi^{p-1}(u+\theta)}{|u|^p}\,du \tag{2.4.5}$$

is finite.

Now we observe that, for $r = 1-1/(n+1)$,

$$\sum_{\nu=1}^{\infty} (\nu+1)^q |t_\nu(\theta,f)-\tau_\nu(\theta,f)|^q r^{\nu q} \geqslant e^{-q} \sum_{\nu=1}^{n} (\nu+1)^q |t_\nu(\theta,f)-\tau_\nu(\theta,f)|^q.$$

This, in connection with (2.4.4), gives

$$\sum_{\nu=1}^{n} (\nu+1)^q |t_\nu(\theta,f)-\tau_\nu(\theta,f)|^q \leqslant \left(C_q \mu^p n^{2p-1} + C\mu^{p-1} n^{2p-1} \int_{-\pi}^{\pi} \frac{\phi^{p-1}(u+\theta)\, f(u+\theta)}{|u|^p}\,du\right)^{q/p}. \tag{2.4.6}$$

It is sufficient to prove that at almost every point we have

$$\sum_{\nu=1}^{n} (\nu+1)^q |t_\nu(\theta,f)-\tau_\nu(\theta,f)|^q = o(n^{q+1}); \tag{2.4.7}$$

for at every point θ at which we have this relation a summation by parts gives

$$\sum_{\nu=1}^{n} |t_\nu(\theta,f)-\tau_\nu(\theta,f)|^q = o(n), \tag{2.4.8}$$

i.e. the required result (2.2.4).

It is an immediate consequence of (2.4.6) that at almost every point θ we have (2.4.7), and so also (2.4.8), with o replaced by O. In order to obtain the result with o, we may argue as follows.

2.5. For every $A > 0$ we write

$$f(\theta) = f_A(\theta) + g_A(\theta), \tag{2.5.1}$$

where $f_A(\theta) = f(\theta)$, if $f(\theta) \leqslant A$; $f_A(\theta) = 0$ elsewhere;

$g_A(\theta) = 0$, if $f(\theta) \leqslant A$; $g_A(\theta) = f(\theta)$ elsewhere.

When A tends to $+\infty$, the function $g_A(\theta)$ tends to 0 almost everywhere, and

$$\int_{-\pi}^{\pi} g_A(\theta)\,d\theta \to 0.$$

From Lemma 2 it follows that the functions

$$\mu_A(\theta) = \max_{|h| \leqslant \pi} \left| \frac{1}{h} \int_\theta^{\theta+h} g_A(u)\, du \right|$$

tend to 0 almost everywhere as A tends to $+\infty$.

Let $\mathfrak{E} \subset (-\pi, \pi)$ be any perfect set on which $\mu_A(\theta)$ tends to 0 uniformly, and let μ_A denote the upper bound of $\mu_A(\theta)$ on $\mathfrak{E}$, so that $\mu_A \to 0$. We use $\phi(\theta)$ in the sense defined in Lemma 3. Let $\mathfrak{E}_1$ be any perfect subset of $\mathfrak{E}$ such that the integral (2.4.5) is uniformly bounded for $\theta \epsilon \mathfrak{E}_1$, and such that

$$\frac{\text{meas}\, \mathfrak{E}(\theta-h, \theta+h)}{2h} \to 1 \quad \text{uniformly for } \theta \epsilon \mathfrak{E}_1.$$

Let $\xi_n(\theta, f) = \xi_{n,q}(\theta, f)$ denote the left-hand side of the relation (2.4.7). An argument similar to (1.6.3) gives, for $\theta \epsilon \mathfrak{E}_1$ and $n = 1, 2, \ldots$, the following inequality (analogous to (1.6.2))

$$\xi_n(\theta, g_A) \leqslant \eta_A n^{q+1}, \quad \text{wheae} \quad \eta_A \to 0 \quad as \quad A \to +\infty.$$

From (2.5.1) we deduce that

$$(2.5.2) \qquad \xi_n(\theta, f) n^{-q-1} \leqslant 2^{q-1} \xi_n(\theta, f_A) n^{-q-1} + 2^{q-1} \xi_n(\theta, g_A) n^{-q-1}.$$

Here the second term on the right is uniformly small for $\theta \epsilon \mathfrak{E}_1$, provided that A is large enough but fixed. The function $f_A(\theta)$ is bounded. Hence, by the result quoted at the beginning of this section,

$$\sum_{\nu=1}^{n} |t_\nu(\theta, f_A) - \tau_\nu(\theta, f_A)|^q = o(n),$$

and so also

$$\xi_n(\theta, f_A) = o(n^{q+1}),$$

almost everywhere in $\mathfrak{E}_1$. Thus, on account of (2.5.2),

$$(2.5.3) \qquad \xi_n(\theta, f) = o(n^{q+1})$$

almost everywhere in $\mathfrak{E}_1$. Since the measure of $\mathfrak{E}_1$ may be arbitrarily near to 2π, we deduce that (2.5.3) is true at almost every point θ of the interval $(-\pi, \pi)$. This completes the proof of Theorem 2.

3.1. Concerning summability in mean of power series of the class H^λ, Hardy and Littlewood have proved the following theorem†:

If the function $F(z)$ belongs to the class H^λ, where $0 < \lambda < 1$, and if

† Hardy and Littlewood [1]. *Cf.* also Gwilliam [1].

$\alpha = (1/\lambda) - 1$, *then*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |\sigma_n^{\alpha+\epsilon}(\theta, F) - F(e^{i\theta})|^\lambda \, d\theta = 0$$

for every $\epsilon > 0$.

Hardy and Littlewood have also shown that the result does not hold for $\epsilon = 0$.

It seems that there is a theorem which is in a certain sense more precise. The results that I have obtained in this direction are, however, only partial. I state them in the form of

THEOREM 3. *Let the function* $F(z)$ *be regular for* $|z| < 1$ *and let it satisfy the condition*

$$\int_{-\pi}^{\pi} |F(re^{i\theta})|^\lambda \log^+ |F(re^{i\theta})| \, d\theta = O(1),$$

where λ *is one of the numbers*

(3.1.1) $\qquad \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \ldots .$

Let $\qquad \alpha = \dfrac{1}{\lambda} - 1.$

Then the function

$$\max_n |\sigma_n{}^\alpha(\theta)| = \sigma_*{}^\alpha(\theta)$$

satisfies an inequality

(3.1.2) $$\int_{-\pi}^{\pi} \{\sigma_*{}^\alpha(\theta)\}^\lambda \, d\theta \leqslant C_\lambda \int_{-\pi}^{\pi} |F(e^{i\theta})|^\lambda \log^+ |F(e^{i\theta})| \, d\theta + C_\lambda,$$

where C_λ *depends only on* λ.

Moreover,

(3.1.3) $$\lim_{n\to\infty} \int_{-\pi}^{\pi} |\sigma_n{}^\alpha(\theta, F) - F(e^{i\theta})|^\lambda \, d\theta = 0.$$

I do not give the proof of this theorem here since it seems quite unlikely that the restriction of λ to the values (3.1.1) is essential for the truth of the result. For a similar reason I omit the proof of

THEOREM 4. *If* $F(z)$ *belongs to the class* H^λ, *where* λ *is of the form*

(3.1.4) $\qquad \dfrac{2}{k}$ *with* $k = 2, 3, 4, \ldots,$

then the function

$$\tau_*^{\alpha}(\theta) = \max_{n \geqslant 2} \frac{|\sigma_n^{\alpha}(\theta, F)|}{(\log n)^{1/\alpha}}$$

belongs to L^{λ} *and satisfies an inequality*

$$\int_{-\pi}^{\pi} \{\tau_*^{\alpha}(\theta)\}^{\lambda} d\theta \leqslant C_{\lambda} \int_{-\pi}^{\pi} |F(e^{i\theta})|^{\lambda} d\theta,$$

where C_{λ} *depends on* λ *only.*

It is to be observed that the sequence (3.1.4) is denser than the sequence (3.1.1).

§γ.

γ.1. I take this opportunity of supplying an omission in the first part of this paper. An essential argument there was the application of the following inequality, due to Littlewood and Paley:

$$(*) \qquad \left(\int_{-\pi}^{\pi} g^{\lambda}(\theta)\, d\theta\right)^{1/\lambda} \leqslant C_{\lambda} \left(\int_{-\pi}^{\pi} |F(e^{i\theta})|^{\lambda} d\theta\right)^{\lambda},$$

where $$F(z) \in H^{\lambda}, \quad g(\theta) = \left(\int_0^1 (1-\rho)|F'(\rho e^{i\theta})|^2 d\rho\right)^{\frac{1}{2}},$$

λ is any positive number, and C_{λ} depends on λ only.

For some reason that I cannot explain, I omitted to state the following inequality:

$$(**) \qquad \int_{-\pi}^{\pi} g(\theta) \log^+ g(\theta)\, d\theta \leqslant C \int_{-\pi}^{\pi} |F(e^{i\theta})| \log^+ |F(e^{i\theta})|\, d\theta + C,$$

for functions F for which the integral

$$\int_{-\pi}^{\pi} |F(re^{i\theta})| \log^+ |F(re^{i\theta})|\, d\theta$$

is bounded. This inequality is required in the proof of Theorem 2 of the first part of the paper. The proofs of Theorems 1 and 2 of that part differ only in that the inequality (**) must be used† instead of the inequality (*) with $\lambda = 1$.

Since the inequality (*) is true for every $\lambda > 0$, it is apparent that the inequality (**) must also be true; the reader familiar with the work of Littlewood and Paley will in any case have no difficulty in proving it, and there is no need to give a detailed proof here.

† *Cf.* Zygmund [2], 186.

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ON LIMITS OF ALGEBRAIC VARIETIES, IN PARTICULAR OF THEIR INTERSECTIONS AND TANGENTIAL FORMS

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The study of the manner in which the properties of a variable algebraic variety are reflected upon a limiting variety is obviously of great interest from many points of view, but it has never been carried out systematically. In this paper I lay the foundations for such a study, which will be pursued further in other works.

The first two sections are introductory in character, and deal with the notion of limit applied to algebraic varieties and their intersections, introducing the important distinction between general and regular limits, and the consideration of the geometrical and functional contributions of a variety in the intersection of two or more other varieties.

Section III gives a construction for the limit of the intersection group of two curves varying upon a surface, in the case in which the limiting group is apparently indeterminate; in this connection, some previous deductions of Enriques and myself are examined thoroughly (§§ 16, 17), and the impossibility of increasing the genus of a curve by means of a passage to the limit is rigorously established (§ 12). Then some extensions of the construction indicated, to the varieties of higher dimension, follow in section IV.

Section V contains the determination of the limit of the tangent primes of a primal degenerating into (one or more) simple or multiple components. The result is intimately related to some general differential properties concerning the contact between two variable varieties; to this section VI is devoted and, by its means, that result is greatly extended in section VII; there in conclusion many diverse applications are indicated.

SECTION I.

Generalities on limits.

1. We denote by λ a point varying in an infinite set Λ of points of a topological space, and by λ_0 a limit point of Λ. It is well known that $[r]$, considered as a projective complex space, is a topological manifold; a relation of the type

$$\lim_{\lambda\to\lambda_0} P(\lambda) = P_0 \tag{1}$$

has therefore a well-defined meaning, where $P = P(\lambda)$ is a point of $[r]$ depending (uniformly or not) on λ, and P_0 is likewise a point of $[r]$. If V is any fixed algebraic variety containing P, it follows from (1) that P_0 also belongs to V; the relation (1) is moreover covariant with respect to the rational transformations of V which are regular at P_0.

Of course the *ordinary* (or *regular*) *limit* (1) may not exist; but, since $[r]$ is compact, it is always possible to extract from Λ an infinite subset $\bar{\Lambda}$, admitting (like Λ) the limit element λ_0, and to choose for every $\bar{\lambda}$ of $\bar{\Lambda}$ a determination $\bar{P}(\bar{\lambda})$ of P, such that

$$\lim_{\bar{\lambda}\to\lambda_0} \bar{P}(\bar{\lambda})$$

exists. We call the *general limit* of P for $\lambda\to\lambda_0$ the totality $\mathfrak{P}_0$ of the limiting points which can be obtained in such a way, and we write

$$\operatorname{Lm}_{\lambda\to\lambda_0} P(\lambda) = \mathfrak{P}_0;$$

this set of points is always non-null, and it does not necessarily constitute an algebraic variety.

2. If $h(\geqslant 2)$ algebraic varieties $W^1, W^2, \ldots, W^h$ (which may be all or partially coincident) are given, the ordered sets of h points $(P^1, P^2, \ldots, P^h)$ taken respectively from them can be represented (in a well-known manner) by the points P of an algebraic variety V (defined save for a birational transformation without exceptional points). By applying the statements in § 1 to the varieties $V, W^1, \ldots, W^h$, we see at once that the relations

$$P = P(\lambda), \qquad \lim_{\lambda\to\lambda_0} P(\lambda) = P_0$$

are respectively equivalent to sets of h relations of the type:

$$P^i = P^i(\lambda), \quad \lim_{\lambda\to\lambda_0} P^i(\lambda) = P_0^{\,i} \quad (i = 1, 2, \ldots, h),$$

where P_0 represents the set of points $(P_0^1, P_0^2, \ldots, P_0^h)$; moreover we derive the meaning of the ordinary or regular and of the general limit for $\lambda \to \lambda_0$ of the ordered set of points $(P^1, P^2, \ldots, P^h)$ depending arbitrarily on λ.

3. More generally, from the representation of the pure m-dimensional algebraic varieties M of a given order a which belong to a fixed algebraic variety V by means of the points P of another algebraic variety W†, we immediately see the meaning to be attributed to the relations

$$\lim_{\lambda \to \lambda_0} M(\lambda) = M_0, \tag{2}$$

$$\operatorname{Lm}_{\lambda \to \lambda_0} M(\lambda) = \mathfrak{M}_0, \tag{3}$$

in which $M = M(\lambda)$ is a function of λ, M_0 is still a pure m-dimensional algebraic variety of order a belonging to V, to be called the *regular limit* of $M(\lambda)$, and $\mathfrak{M}_0$ is a non-null set of similar varieties, to be called the *general limit of* $M(\lambda)$.

If (2) holds and $Q(\lambda)$ denotes a point of V subject to the one condition that it belongs to $M(\lambda)$, then we prove the following identity (between sets of points):

$$\operatorname{Lm}_{\lambda \to \lambda_0} Q(\lambda) = M_0. \tag{4}$$

In fact, every point of the first member of (4) belongs to the variety M_0, by virtue of the above-mentioned representation; and the problem is to show that, reciprocally, if we fix any point Q_0 of M_0, we can choose on $M(\lambda)$ a point $Q(\lambda)$ having Q_0 as ordinary limit for $\lambda \to \lambda_0$.

For this purpose, let us draw through Q_0, in the $[r]$ which contains V, a linear space $[r-m]$ intersecting M_0 in a finite number of points; this, for λ sufficiently near to λ_0, intersects $M(\lambda)$ also in a finite number of points. In fact, if this were not so, we should have in any neighbourhood of λ_0 some λ such that $M(\lambda)$ intersects $[r-m]$ in an algebraic variety of dimension not less than 1; let us choose then in $[r-m]$ a space $[r-m-1]$ containing none of the intersections of $[r-m]$ and M_0: this $[r-m-1]$ has some point $R(\lambda)$ in common with $M(\lambda)$, and the general limit of $R(\lambda)$ for $\lambda \to \lambda_0$ is a non-null set of points (§ 1) belonging both to M_0 and to $[r-m-1]$; this is impossible, since these have no common points.

† *Cf.* Chow-van der Waerden, **2**. See also van der Waerden, **16**, 153–163.

Thus, effectively, for λ sufficiently near to λ_0, $M(\lambda)$ intersects $[r-m]$ in a group of a points, of which at least one, $Q(\lambda)$, must have Q_0 as ordinary limit† for $\lambda \to \lambda_0$; and so we have established (4) as a consequence of (2).

In the general case, in which the regular limit (2) may well be non-existent, we can consider the general limit (3) (constituted by a non-null set of pure m-dimensional algebraic varieties of order a belonging to V), and instead of (4) we have the relation

$$\operatorname*{Lm}_{\lambda \to \lambda_0} Q(\lambda) = \mathfrak{M}_0;$$

so that, reciprocally, (2) follows from (4) if M_0 is irreducible.

4. An arbitrary algebraic variety, M say, can be expressed uniquely as a sum of a finite number of pure varieties of different dimensions:

$$M = M^1 + M^2 + \ldots + M^h. \tag{5}$$

If M varies continuously upon a fixed algebraic variety V, the same is the case for the M^i's, which therefore maintain constant orders a_i and constant dimensions m_i. We can then represent (as in § 3) the totalities of the pure varieties of V having such characters by the points P^i of h algebraic varieties W^i, so that the variety (5) corresponds to a definite ordered set of h points $(P^1, P^2, \ldots, P^h)$; and, by virtue of § 2, we can define the regular and general limit for $\lambda \to \lambda_0$ of such a set, and therefore also of the variety (5), supposed to be arbitrarily dependent on λ.

Taking § 3 also into account, we have finally:

The general limit $\mathfrak{M}_0$ for $\lambda \to \lambda_0$ of an algebraic variety $M = M(\lambda)$, belonging to a given algebraic variety V, always exists; it belongs to V, and is covariant with respect to the rational transformations of V without exceptional points on $\mathfrak{M}_0$. This general limit $\mathfrak{M}_0$, considered as a locus of points, coincides with the general limit of a point $Q(\lambda)$ of V subject to the one condition that it belongs to $M(\lambda)$; in no case can the dimension of $\mathfrak{M}_0$ be lower than the dimension of $M(\lambda)$, and no part of $\mathfrak{M}_0$ can be of lower dimension if $M(\lambda)$ is pure. If $M(\lambda)$ is irreducible, a sufficient condition for it to possess a regular limit, for $\lambda \to \lambda_0$, is that $\mathfrak{M}_0$ is an irreducible algebraic variety having the same order and the same dimension as $M(\lambda)$; the regular limit for $\lambda \to \lambda_0$ of the sum of a finite number of algebraic varieties exists if, and only if, it is possible to make these varieties dependent on λ in such a way that each of them possesses a regular limit for $\lambda \to \lambda_0$, and then it coincides with the sum of these limits‡.

† The points $Q(\lambda)$ satisfying this condition must have the sum of their multiplicities equal to the multiplicity of intersection of M_0 and $[r-m]$ in Q_0, *cf.* Severi **12,** No. 11.

‡ *Cf.* with results obtained by van der Waerden **14, 15,** on more restricted hypotheses, but from closely related algebraic considerations.

5. Consider two algebraic varieties

$$M = M^1 + M^2 + \ldots + M^p, \qquad N = N^1 + N^2 + \ldots + N^q, \tag{6}$$

where M^i, N^j denote their irreducible components (not all distinct, if some of these are to be counted several times in M or N); we say that the former *belongs* to the latter and that the latter *contains* the former, and write

$$M \subset N \quad \text{or} \quad N \supset M,$$

if it is possible to associate with every M^i (for $i = 1, 2, \ldots, p$) a variety $N^{f(i)}$ coinciding with or containing M^i, with the essential restriction that each N^j that is in this way associated with more than one M^i cannot coincide with any one of them†. We prove that

If $M = M(\lambda)$, $N = N(\lambda)$ *depend on* λ, *from* $M(\lambda) \subset N(\lambda)$ *it follows that*

$$\lim_{\lambda \to \lambda_0} M(\lambda) \subset \lim_{\lambda \to \lambda_0} N(\lambda),$$

when both these regular limits exist.

The conclusion is in fact a simple consequence of § 3, in the case when (for $\lambda \neq \lambda_0$ but sufficiently near to λ_0) the varieties $M(\lambda)$, $N(\lambda)$ are irreducible. In the general case in which M and N are given by expressions of type (6) (with $pq > 1$), the condition $M(\lambda) \subset N(\lambda)$ can be expressed in the manner indicated: but now $f(i)$ may also depend on λ. Since however $f(i)$ can only assume a finite number ($\leqslant q$) of values, it is always possible to make $f(i)$ independent of λ by reducing, if necessary, the aggregate Λ of the points λ for which $M(\lambda)$ and $N(\lambda)$ are defined, without affecting the regular limits of $M(\lambda)$, $N(\lambda)$ for $\lambda \to \lambda_0$; then the general result previously enunciated follows immediately from the particular case already considered.

SECTION II.

Generalities on intersections.

6. Let us consider in a projective space $[r]$ two pure algebraic varieties M, N, of dimensions m, n respectively, belonging to an irreducible algebraic variety V without multiple points, of dimension t; they determine as

† In particular any irreducible variety is to be regarded as containing itself only once, but any sub-variety of itself an arbitrary number of times.

their intersection an algebraic variety, each component L of which has a dimension $d \geqslant m+n-t$ and is to be counted in the intersection a certain number l ($\geqslant 1$) of times. This intersection is a symmetric covariant of M, N with respect to birational transformations without exceptional elements on V; and, since (as we shall show shortly) it depends in fact only on M, N and not on V, we denote† it by the symbol (MN).

In order to determine precisely the *multiplicity* l of an irreducible component L of (MN), we begin by supposing L not to lie upon another component of greater dimension. If P is a simple point of L which does not belong to other components, a variety V' of dimension $t'=t-d$ of V, which passes simply through P and is not particularly related to M, N, intersects M, N along two varieties M', N' passing through P, of dimensions $m'=m-d$, $n'=n-d$ respectively; these varieties have P as an isolated intersection, and l is the multiplicity of their intersection at P‡. In particular, the intersection of a non-singular variety with itself is the variety itself.

If L belongs to other components L_i of (MN), of dimensions $d_i > d$ ($i=1, 2, \ldots, p$), we can suppose that we have already determined the multiplicities l_i of the L_i's in (MN), and that we know the multiplicity k_i with which L_i passes through L. A variety V' of V of dimension $t-d'$

† Many authors use this symbol for the virtual intersection of M, N on V; however, since this virtual intersection depends also on V, we prefer to denote it by the symbol $(MN)_V$, or simply by MN when the meaning of V can be implied without misunderstanding. If G is an (effective or virtual) group of points, we denote by $[G]$ the number of its points.

‡ This multiplicity of intersection has been defined by Severi **12**, Nos. 11, 15, both for $m'+n'=t'$ (*i.e.* $d=m+n-t$) and for $m'+n'<t'$ (*i.e.* $d>m+n-t$).

According to this author, the multiplicity l of intersection of M', N' at P is 1 if, and only if, M', N' *intersect simply* at P, namely if both of them go simply through P and moreover they have no tangent in common at this point. If $m'+n'=t'$, there is on V' some pure n'-dimensional variety H, not passing through P, such that $K=N'+H$ belongs to a continuous system of n'-dimensional varieties lying on V', the generic of which near K has, in the neighbourhood of P, exactly l distinct points in common with M', in each of which it intersects simply this variety. Any two varieties of V' neighbouring M' and N' intersect then in the neighbourhood of P in some points $P_1, P_2, \ldots$ with multiplicities $l_1, l_2, \ldots$, such that $l_1+l_2+\ldots=l$.

If $n^*=t'-m'>n'$, there is on V' some continuous system of n^*-dimensional varieties such that: (i) the generic N^* of these varieties goes through each point of N' having in it the same multiplicity as N', (ii) the varieties of the continuous system have no point in common either outside N' or at infinitesimal distance from N', (iii) the point P is an isolated intersection of M' and the generic N^*; then the multiplicity of intersection of M', N' at P is defined as equal to the multiplicity of intersection of M', N^* at P: this number only depends on M', N', and not on V'.

It follows that, if $m'+n'<t'$ and P is simple for N', then l is also equal to the minimum of the multiplicities of intersection in P of M' with the different varieties containing N' and not intersecting M' apart from P in the neighbourhood of this point.

(where d' is the greatest of the d_i's), passing simply through a generic point P of L and not particularly related to M, N, intersects M, N along two varieties M', N' passing through P, of dimensions $m' = m-d'$, $n' = n-d'$ respectively, and having P as an isolated intersection; if l' is the multiplicity of intersection of M', N' at P, we have

$$l = l' + \sum_{i=1}^{p} k_i l_i.$$

It is worth noticing that, since V, V' have a simple point at P, the multiplicity of intersection of M', N' at this point is the same whether it is determined on V' or on V, or even on $[r]$; so all the numbers l mentioned above, and therefore the variety (MN) itself, are independent of V and depend only on M, N.

We must, moreover, bear in mind that from $L \subset M$, $L \subset N$ it does not always follow that $L \subset (MN)$: thus, for instance, if M, N are two curves intersecting at a point P with the multiplicity l, and l_1 is an integer greater than l, we may have $l_1 P \subset M$, $l_1 P \subset N$, without $l_1 P \subset (MN)$. Likewise, from $L \subset M$ it does not always follow that $(LN) \subset (MN)$: so, for instance, if L is a curve belonging to a surface M, and N is a curve simply intersecting this surface at a point P through which L passes twice, the point P belongs to (LN), (MN) counted twice and once respectively, so that we do not have $(LN) \subset (MN)$; it is, however, easy to prove (taking into account the remark in the last note) that the latter relation certainly follows from the former, in the case in which L passes simply through each of the components of (LN).

7. We can specify exactly the variety $(M^1 M^2 \dots M^h)$ of intersection of $h\,(\geqslant 2)$ pure m_i-dimensional varieties $M^i\,(i = 1, 2, \dots, h)$, upon a non-singular variety W of dimension n, in the following manner.

We consider the variety V, of dimension $t = hn$, the points P of which represent the ordered sets $(P^1 P^2 \dots P^h)$ of h points of W. On it we have the pure varieties M and N, of dimensions

$$m = m_1 + m_2 + \dots + m_h$$

and n respectively, representing the sets $(P^1 P^2 \dots P^h)$ with $P^i \subset M^i$ $(i = 1, 2, \dots, h)$ and with $P^1 = P^2 = \dots = P^h$. Between W and N there is obviously a birational correspondence without exceptional elements, which transforms (MN) into the desired variety $(M^1 M^2 \dots M^h)$. It is not difficult to show that the variety so obtained coincides for $h = 2$ with

that defined in §6†. In any case, its points are the points common to $M^1, M^2, \ldots, M^h$; moreover, $(M^1 M^2 \ldots M^h)$ is a symmetric covariant of $M^1, M^2, \ldots, M^h$ with respect to the birational transformations without exceptional elements on W, which depends only on $M^1, M^2, \ldots, M^h$, and not‡ on W.

The associative property

$$(M^1 M^2 \ldots M^h) = \left(M^1 (M^2 \ldots M^h)\right)$$

does not hold unreservedly; for instance, if M^1 is a curve passing simply through a point P^1, and M^2, M^3 are two surfaces touching each other but not touching M^1 at P^1, this point P^1 appears counted simply

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† By virtue of the last note, the main point is to show that *the varieties* M^1, M^2 *intersect simply at a common point* $P^1 = P^2$ *if, and only if, the varieties* M, N *intersect simply at the point* P *representative of* $(P^1 P^2)$.

To do this, we remark that the result to be proved only affects the neighbourhoods of P^1 on W and of P on V, to each of which we can apply any regular algebraic transformation; so that we can suppose both W and V reduced to linear spaces. If $(x) = (x_1 x_2 \ldots x_n)$ are projective non-homogeneous coordinates in W, all vanishing at P^1, we can define the point of V representing the ordered pair (x^1), (x^2) by means of the $2n$ coordinates $x_1^1 x_2^1 \ldots x_n^1 x_1^2 x_2^2 \ldots x_n^2$ (all 0 at P); then the equations of M^1, M^2, say

$$(*) \qquad \phi_1^1(x) = 0, \ \phi_2^1(x) = 0, \ \ldots \quad \text{and} \quad \phi_1^2(x) = 0, \ \phi_2^2(x) = 0, \ \ldots$$

respectively, are satisfied at P^1, and M, N are represented by the systems

$$(**) \quad \phi_1^1(x^1) = 0, \ \phi_2^1(x^1) = 0, \ \ldots, \ \phi_1^2(x^2) = 0, \ \phi_2^2(x^2) = 0, \ \ldots$$

$$\text{and} \quad x_1^1 - x_1^2, \ x_2^1 - x_2^2, \ \ldots, \ x_n^1 - x_n^2,$$

both holding at P: while N has always a simple point at P, the variety M goes simply through P if, and only if, both M^1 and M^2 have a simple point at P^1. The characteristics c, c' of the Jacobian matrices of the first members of the equations (*) and (**), respectively, calculated for $x_1 = x_2 = \ldots = x_n = 0$ and for

$$x_1^1 = x_2^1 = \ldots = x_n^1 = x_1^2 = x_2^2 = \ldots = x_n^2 = 0,$$

are obviously connected by the relation $c' = c + n$; therefore each of the two conditions of the above made statement is a consequence of the other, since they are respectively equivalent to the simple passage of M^1, M^2 through P^1 and $c = n$, and to the simple passage of M through P and $c' = 2n$.

‡ In fact, if W', W^* are any two non-singular varieties through $M^1, M^2, \ldots, M^h$, we denote by $(M^1 M^2 \ldots M^h)'$ and $(M^1 M^2 \ldots M^h)^*$ the results that we obtain with the above explained process by substituting in it W' and W^* for W. In order to prove that $(M^1 M^2 \ldots M^h) = (M^1 M^2 \ldots M^h)'$, we can suppose W^* to have been chosen to contain both W and W', and establish that one (and likewise also the other) member of this relation is equal to $(M^1 M^2 \ldots M^h)^*$. To see this, we observe that the variety V^* representing the ordered sets of h points of W^* contains a variety V representative of the ordered sets of h points of W, to which the above defined variety M belongs; the locus N^* of the points of V^* which represent sets of h coincident points, intersects on V the locus N analogously related to this variety: thus we have $(MN) = (MN^*)$, whence the equality between $(M^1 M^2 \ldots M^h)$ and $(M^1 M^2 \ldots M^h)^*$ follows at once, by virtue of the definitions of these varieties.

in $(M^1 M^2 M^3)$†, and counted at least twice in $\{M^1(M^2 M^3)\}$. On the other hand, the distributive property

$$(M_1^1+M_2^1, M^2, \ldots, M^h) = (M_1^1 M^2 \ldots M^h)+(M_2^1 M^2 \ldots M^h)$$

holds in every case; and, by means of an obvious extension of this property, we can specify the meaning of the intersection $(M^1 M^2 \ldots M^h)$ even in the case in which some of the M^i are impure.

8. We consider now on V two pure varieties M, N of dimensions m, n, with $n \geqslant m$, and suppose that their intersection (MN) contains as a component [with a certain multiplicity l $(\geqslant 1)$] an irreducible variety L of dimension $m-1$, and that N is the total or partial intersection of $t-n$ varieties $A^1, A^2, \ldots, A^{t-n}$ of dimension $t-1$ of V, with the condition that, if there is a residual intersection, this must not pass through L. Denoting by c the least of the multiplicities with which L must be computed in the intersection of M with each of the A^i's not containing M, we prove that‡

$$c \leqslant l. \tag{1}$$

The primal cone β projecting N from a generic $[r-n-2]$ of $[r]$ intersects M in a pure $(m-1)$-dimensional variety which contains§ L with multiplicity l. On the other hand, since V is non-singular, each A^i can be cut out on V by a primal α_i in such a manner that the residual intersection does not pass through L; then α_i contains M if (and only if) A^i passes through this variety, while, if α_i does not contain M, $(\alpha_i M)$ and $(A^i M)$ contain L with the same multiplicity, and the variety $(V\alpha_1 \alpha_2 \ldots \alpha_{t-n})$ consists of N and another n-dimensional variety not passing through L.

Let us introduce in $[r]$ projective coordinates (x), and denote by $\beta(x) = 0$, $\alpha_i(x) = 0$ the equations of the primals β, α_i $(i = 1, 2, \ldots, t-n)$.

† If (xyz) are local parameters in the neighbourhood of $P^1(000)$ upon a non-singular 3-dimensional variety W containing M^1, M^2, M^3, in which point P^1 the curve M^1 and the surfaces M^2, M^3 respectively touch $x = y = 0$ and $z = 0$, we can refer to the 9-dimensional space V the points $(x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3)$ of which represent the ordered sets of three points $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, $(x_3 y_3 z_3)$, and to the varieties M and N (respectively of 5 and 3 dimensions) defined in it as above. Actually we have $(M^1 M^2 M^3) = 1$, so that M and N intersect simply at the point $P(000000000)$, since both of them go simply through it, having there the tangent spaces

$$x_1 = y_1 = z_2 = z_3 = 0 \quad \text{and} \quad x_1 = x_2 = x_3, \; y_1 = y_2 = y_3, \; z_1 = z_2 = z_3,$$

which do not intersect outside P.

‡ The numeration of the formulae starts afresh in each section.

§ This follows from § 6 and the already quoted paper of Severi, **12**, No. 15.

From elimination theory we get a relation of the type

$$b(x) \cdot \beta(x) = \sum_{i=1}^{t-n} a_i(x) \cdot \alpha_i(x),$$

holding identically upon V, where $a_i(x)$ and $b(x)$ are polynomials the last of which is not identically zero on L. Then the limitation (1) follows immediately, by interpreting this relation upon M.

Simple examples show that we can have either $c = l$ or $c < l$; but in the case in which N passes simply through L we certainly have $c = l$, since $c \geqslant l$ by virtue of the last note of § 6.

9. We consider, upon the non-singular irreducible variety V, two pure m- and n-dimensional varieties $M(\lambda)$ and $N(\lambda)$, both depending on λ and intersecting in a pure d-dimensional variety $L(\lambda)$ of order a; and we suppose that the three regular limits:

$$\lim_{\lambda \to \lambda_0} M(\lambda) = M_0, \quad \lim_{\lambda \to \lambda_0} N(\lambda) = N_0, \quad \lim_{\lambda \to \lambda_0} L(\lambda) = L_0$$

exist. Since $L(\lambda) \subset M(\lambda)$, $L(\lambda) \subset N(\lambda)$, we have (§ 5) $L_0 \subset M_0$, $L_0 \subset N_0$. From this (§ 6) we cannot deduce

$$L_0 \subset (M_0 N_0);$$

but we shall prove that this relation holds on the actual hypotheses, by showing that, if Δ is an irreducible component of $(M_0 N_0)$ not immersed in another component of greater dimension, and having the dimension d and the multiplicity l ($\geqslant 1$), the variety L_0 cannot contain Δ more than l times, and must contain Δ exactly l times when

$$d = m + n - t. \tag{2}$$

We begin by supposing d given by this expression, and choose generically a point P_0 on Δ and a space $[r-d]$ of $[r]$ through P_0, intersecting V, $M(\lambda)$, $N(\lambda)$, $L(\lambda)$, M_0, N_0, L_0, respectively, in V', $M'(\lambda)$, $N'(\lambda)$, $L'(\lambda)$, M_0', N_0', L_0'. $M'(\lambda)$, $N'(\lambda)$ are pure varieties of dimensions

$$m' = m - d, \quad n' = n - d,$$

varying upon the non-singular variety V', of dimension $t' = t - d$; moreover, $L'(\lambda)$ is a group of a points such that

$$L'(\lambda) = \left(M'(\lambda) N'(\lambda)\right),$$

$$\lim_{\lambda \to \lambda_0} M'(\lambda) = M_0', \quad \lim_{\lambda \to \lambda_0} N'(\lambda) = N_0', \quad \lim_{\lambda \to \lambda_0} L'(\lambda) = L_0'.$$

The varieties M_0', N_0' have at P_0 an isolated intersection of multiplicity l (§ 6); since, by virtue of (2), $t'=m'+n'$, when λ is not at λ_0 but conveniently near to λ_0 the group $L'(\lambda)$ must have exactly l of its a points, near to P_0 and tending† to P_0 for $\lambda \to \lambda_0$. Thus L_0' is a group of a points containing P_0 counted precisely l times; and this proves the assertion made above on the assumption (2).

If (2) does not hold, since V is without multiple points, we must have

$$c = d+t-m-n > 0.$$

Let us choose, then, in $[r]$ a linear space of dimension

$$r+d-m-n-1 = r+c-t-1,$$

generically situated with respect to M_0, N_0, and consider the pure varieties $\bar{N}(\lambda)$, $\bar{N}_0$ of dimension $\bar{n} = n+c$ cut out upon V by the cones projecting from it the varieties $N(\lambda)$, N_0. Then we have

$$\lim_{\lambda \to \lambda_0} \bar{N}(\lambda) = \bar{N}_0,$$

and the variety $(M_0 \bar{N}_0)$ does contain Δ as an irreducible component of multiplicity l, non-immersed in other components of greater dimension. On the other hand, the variety

$$\bar{L}(\lambda) = \left(M(\lambda)\, \bar{N}(\lambda)\right),$$

for λ sufficiently near to λ_0, contains $L(\lambda)$, and (§ 1) possesses a definite regular limit $\bar{L}_0$ for $\lambda \to \lambda_0$, provided that, if necessary, the aggregate of the points λ is conveniently reduced in the neighbourhood of λ_0; this does not affect the limits of $M(\lambda)$, $N(\lambda)$, $L(\lambda)$, $\bar{N}(\lambda)$. Therefore (§ 5) from $L(\lambda) \subset \bar{L}(\lambda)$ follows $\bar{L}_0 \subset L_0$; but, since $d = m+\bar{n}-t$, $\bar{L}_0$ must contain Δ exactly l times, by virtue of the part of the theorem already established; thus L_0 cannot contain Δ more than l times, which is precisely what we wished to prove.

Simple examples show that, when the relation (2) does not hold, Δ appears in L_0 counted a number (possibly 0) of times, which actually may be equal to or less than l.

10. The deductions of § 9 can easily be generalized as follows. Taking into account the definition of § 7, by means of which the intersection of h varieties $M^1, M^2, \ldots, M^h$ is reduced to the intersection of only two

† *Cf.* the last note of § 6.

varieties M, N, and applying the results of § 9 to this intersection, we obtain the following result:

If, upon a non-singular variety V of dimension t, $h(\geqslant 2)$ pure or impure variable varieties $M^i(\lambda)$ $(i = 1, 2, \ldots, h)$ are given, having for $\lambda \to \lambda_0$ the regular limits M_0^i, and if the regular limit

$$\lim_{\lambda \to \lambda_0} \left(M^1(\lambda) M^2(\lambda) \ldots M^h(\lambda) \right) = L_0$$

exists, then

$$L_0 \subset (M_0^1 M_0^2 \ldots M_0^h).$$

Moreover, if the $M^i(\lambda)$'s are pure m_i-dimensional varieties, then any component of $(M_0^1 M_0^2 \ldots M_0^h)$ having regular dimension $d = \sum_{i=1}^{h} m_i - (h-1)t$ and not lying in another component of greater dimension appears with the same multiplicity both in $(M_0^1 M_0^2 \ldots M_0^h)$ and in L_0; so that these two varieties coincide if the former is of dimension d.

11. We retain the previous notation and, in particular, we suppose that $\left(M^1(\lambda) M^2(\lambda) \ldots M^h(\lambda) \right)$ is a pure d-dimensional variety, while $(M_0^1 M_0^2 \ldots M_0^h)$ splits up into the sum of two pure varieties, Δ of regular dimension d and Γ of dimension $c > d$, with the condition that no component of Δ belongs to a component of Γ. Then, by virtue of § 10, we have

$$L_0 = \Delta + \gamma,$$

where γ is a pure d-dimensional variety, belonging to Γ, which we call the *geometrical contribution* of Γ to the intersection of the varieties $M_0^1, M_0^2, \ldots, M_0^h$, considered as limiting positions of

$$M^1(\lambda), M^2(\lambda), \ldots, M^h(\lambda)$$

for $\lambda \to \lambda_0$.

This geometrical contribution depends of course on the manner in which the latter varieties approach the former; but it is possible to prove (with general hypotheses) that it simply varies in a system of equivalence on Γ, if the same is the case for the $M^i(\lambda)$'s upon V. In another paper I shall show precisely how γ can then be expressed by means of an equivalence, involving only $\Gamma, M_0^1, M_0^2, \ldots, M_0^h$ and some of their covariant varieties in V; I call such an expression the *functional contribution* of γ to the intersection of $M_0^1, M_0^2, \ldots, M_0^h$.

In the remainder of the present paper we confine ourselves essentially to the problem of giving an effective construction for the above-mentioned geometrical contribution γ, in some particularly interesting cases.

SECTION III.

The limit of the intersection group of two curves variable upon an algebraic surface.

12. Let us consider, upon an algebraic surface V, two curves $A = A(\lambda)$, $A' = A'(\lambda)$, both depending on λ and possessing ordinary limits A_0, A_0' as $\lambda \to \lambda_0$. If, for λ conveniently near to λ_0 but not at λ_0, they intersect in a group $G = G(\lambda)$ of a finite number of points, then (§ 9)

$$\lim_{\lambda \to \lambda_0} G(\lambda) = (A_0 A_0')$$

in the case when A_0, A_0' also intersect in a group of a finite number of points. Our next purpose is to determine the limiting group of G, on the alternative hypothesis that A_0, A_0' have a part in common.

We shall first give, as an application of the previous simple remark, a proof of the theorem (generally assumed without justification) that the genus of a variable algebraic curve is never increased by a passage to the limit; more precisely we prove that:

If Θ is an infinite set of algebraic irreducible curves C in $[r]$, all of order n and genus p, then the genus p_0 of any irreducible limiting curve C_0 of Θ is certainly not greater than p.

We may first of all suppose that $n \geqslant 3$ and $r = 2$, since the theorem enunciated is trivial for $n = 1, 2$, and it is obviously possible to replace the curves C, C_0 in the proof by their generic projections upon a plane. We associate then with each curve C of Θ the linear system Σ_C, of dimension

$$d = n + p - 2, \tag{1}$$

consisting of all the plane curves C' of order $n-2$ adjoint to C; and we consider the ∞^d linear system Σ_{C_0}, the regular limit of Σ_C for $C \to C_0$, which can always be supposed existent by reducing (if necessary) the variability of C in the neighbourhood of C_0.

The curves of Σ_{C_0}, apart from the fixed intersections absorbed by the multiple points of C_0, determine upon this curve the groups of a g_k^d which has the same dimension d as Σ_{C_0}, since no curve of Σ_{C_0} can contain C_0; we have, moreover,

$$k \leqslant n + 2p - 2, \tag{2}$$

since any curve C_0' of Σ_{C_0} can be regarded as the limit for $C \to C_0$ of a curve C' of Σ_C, which intersects C (outside its multiple points) in no more than $n+2p-2$ points, and a simple intersection of C_0, C_0' can only be the limit of a simple intersection of C, C'. From (1), (2) it follows immediately

that $k-d \leqslant p$; hence, if the genus p_0 of C_0 is greater than p, the linear series $g_k{}^d$ is special and we obtain

$$n+2p-2 \geqslant k \geqslant 2d = 2n+2p-4,$$

that is, $n \leqslant 2$, by virtue of (1), (2) and Clifford's theorem applied to $g_k{}^d$. Consequently we can only have $p_0 \leqslant p$.

The result we have just established can be easily put in the following form:

If C_0 is an irreducible curve of $[r]$, of order n and genus p_0, all the curves of $[r]$ (of order n) conveniently near to C_0 have genus not less than p_0.

It follows that

The genus of the generic curve of an irreducible algebraic system of (generally irreducible) algebraic curves is equal to the maximum genus of the irreducible curves of the system.

13. We begin by studying the question stated at the beginning of § 12 in the simplest case, and suppose one (say A') of the two curves considered there to be independent of λ, and therefore coincident with its limit A_0'. If this is composed of h ($\geqslant 1$) distinct irreducible curves A_i' counted r_i times ($i = 1, 2, \ldots, h$), we have

$$\lim_{\lambda \to \lambda_0} G(\lambda) = \sum_{i=1}^{h} r_i \lim_{\lambda \to \lambda_0} (AA_i');$$

the problem thus reduces to the case in which the fixed curve is simple and irreducible. Let us, moreover, suppose that the infinite system Σ described by $A(\lambda)$ belongs to a linear system Λ_r of dimension r ($\geqslant 1$), necessarily containing A_0, and that Σ possesses in A_0 the osculating linear systems Λ_i of dimensions $i = 1, 2, \ldots, k$, where k is a convenient integer less than or equal to r, these systems being defined inductively by the formulae†

$$\Lambda_0 = A_0, \qquad \Lambda_i = \lim_{\lambda \to \lambda_0} \left(A(\lambda) \wedge \Lambda_{i-1} \right).$$

By supposition, the unique curve A_0 constituting Λ_0 contains as a part the irreducible curve A_0', while not every curve of Σ and therefore also not every curve of Λ_r has A_0' as a component: there is consequently a definite integer greater than or equal to 1 and less than or equal to r, which we call k, such that every curve of Λ_{k-1} but

† Considering on a t-dimensional variety V two or more systems $A, B, \ldots, C$ of $(t-1)$-dimensional equivalent varieties, we denote by $A \wedge B \wedge \ldots \wedge C$ the linear system of V which joins $A, B, \ldots, C$.

not every curve of Λ_k contains A_0' as a part. Denoting by A_k any curve of Λ_k not belonging to Λ_{k-1}, and therefore intersecting A_0' in a finite number of points, we prove that

$$\lim_{\lambda \to \lambda_0} G(\lambda) = (A_k A_0').$$

In fact, since $$\Lambda_k = \lim_{\lambda \to \lambda_0} \left(A(\lambda) \frown \Lambda_{k-1} \right),$$

there is a curve $\bar{A}(\lambda)$ of the linear system $A(\lambda) \frown \Lambda_{k-1}$ possessing A_k as limit for $\lambda \to \lambda_0$; $\bar{A}(\lambda)$ (like its limit A_k) intersects A_0' in a group of a finite number of points, which coincides with $G(\lambda)$ by virtue of what we have seen for Λ_{k-1}. Thus

$$\lim_{\lambda \to \lambda_0} G(\lambda) = \lim_{\lambda \to \lambda_0} \left(\bar{A}(\lambda) A_0' \right) = (A_k A_0').$$

The previous hypothesis of the existence of the osculating systems Λ_i is not essential, since it is always possible to satisfy it by reducing the set described by λ in the neighbourhood of λ_0. Also the general case in which Σ does not belong to a linear system can be treated, by adding to $A(\lambda)$ a curve $B(\lambda)$ having for $\lambda \to \lambda_0$ a limit B_0 which intersects A_0' in a finite number of points, and such that

$$\bar{A}(\lambda) = A(\lambda) + B(\lambda)$$

varies upon V within a linear system; hence

$$\lim_{\lambda \to \lambda_0} G(\lambda) = \lim_{\lambda \to \lambda_0} \left(\bar{A}(\lambda) A_0' \right) - (B_0 A_0'),$$

and the limit in the right-hand side can be obtained in the manner already explained.

14. We now suppose both the curves $A(\lambda)$ and $A'(\lambda)$ to be variable on V within linear systems, and to have the regular limits

$$(3) \qquad \lim_{\lambda \to \lambda_0} A(\lambda) = A_0 = \Gamma + C, \qquad \lim_{\lambda \to \lambda_0} A'(\lambda) = A_0' = \Gamma + C'$$

with the common (possibly reducible) part Γ; moreover we assume that:

(i) there is no point common to the three curves Γ, C, C';

(ii) the pencils $A_0 \frown A$, $A_0' \frown A'$ have for $\lambda \to \lambda_0$ definite limits $A_0 \frown A_1$, $A_0' \frown A_1'$ (where A_1, A_1' denote two curves respectively equivalent to and distinct from A_0, A_0');

(iii) the two equivalent curves $C + A_1'$, $C' + A_1$ are distinct, and no curve Ω_0 of their pencil has a part in common with Γ.

Then we prove that *if the regular limit*

$$\lim_{\lambda\to\lambda_0} G(\lambda) = \lim_{\lambda\to\lambda_0} \big(A(\lambda)A'(\lambda)\big) = G_0 \tag{4}$$

exists, it consists precisely of the group (CC') *plus a group* γ *of points of* Γ, *which is the complete intersection of this curve with a curve* Ω_0 *of the pencil considered in* (iii).

In fact, by virtue of (iii), the three equivalent curves

$$\Gamma+C+C', \quad C+A_1', \quad C'+A_1$$

are linearly independent, and therefore determine a net R_0, which can also be defined as the net joining the pencils

$$(C+A_0')\wedge(C+A_1'), \qquad (C'+A_0)\wedge(C'+A_1)$$

(having the common curve $C+A_0' = C'+A_0 = \Gamma+C+C'$). These pencils are the ordinary limits for $\lambda\to\lambda_0$ of the pencils

$$(C+A_0') \quad \big(C+A'(\lambda)\big), \qquad (C'+A_0)\wedge\big(C'+A(\lambda)\big),$$

which, consequently, determine a net $R = R(\lambda)$ having R_0 as limit for $\lambda\to\lambda_0$. The net R contains the pencil

$$\big(C+A'(\lambda)\big)\wedge\big(C'+A(\lambda)\big),$$

for every curve Ω of which simultaneously

$$G\subset(A\Omega), \qquad G\subset(A'\Omega); \tag{5}$$

the general limit of this pencil for $\lambda\to\lambda_0$ consists of at least one pencil of the net R_0, and must therefore have at least one curve Ω_0 in common with the pencil of R_0 considered in (iii). Since Ω_0 is a limiting position of a curve Ω for $\lambda\to\lambda_0$, it follows from (3), (4), (5) by virtue of § 5 that

$$G_0\subset\Gamma+(CC') \tag{6}$$

and

$$\begin{cases} G_0\subset(A_0\Omega_0) = (\Gamma\Omega_0)+(C\Omega_0), \\ G_0\subset(A_0'\Omega_0) = (\Gamma\Omega_0)+(C'\Omega_0). \end{cases} \tag{7}$$

By (6) we may put

$$G_0 = \gamma+\gamma^*, \tag{8}$$

with $$\gamma \subset \Gamma,$$

(9) $$\gamma^* \subset (CC')\dagger;$$

and then from (7) we get

(10) $$\gamma \subset (\Gamma\Omega_0),$$

since no point of γ can belong both to $(C\Omega_0)$ and to $(C'\Omega_0)$, in consequence of (i).

By virtue of (i) and (iii), (CC') and $(\Gamma\Omega_0)$ are well-defined groups of points, in number respectively $[CC']$ and

$$[\Gamma C]+[\Gamma A'] = [A'C]-[CC']+[\Gamma A'] = [AA']-[CC'];$$

thus (8), (9), (10) yield

$$[G_0] = [\gamma]+[\gamma^*] \leqslant [AA']-[CC']+[CC'] = [AA'].$$

Here, however, the sign $\leqslant$ must be replaced by $=$ since

$$[G_0] = [G] = [AA'];$$

therefore (9), (10) can hold only if

$$\gamma^* = (CC'), \qquad \gamma = (\Gamma\Omega_0).$$

15. As a corollary of the theorem just established we see that

With the notation and hypotheses of § 14, *a point of* (ΓC) *belongs to* G_0 *if it is also at the same time a point of* (ΓA_1), *that is* (*with the language of infinitesimals*) *if a curve* $A(\lambda)$ *infinitely near to* A_0 *goes through it*; *this sufficient condition is also necessary, except in the case in which* $\Omega_0 = C+A_1'$ *and therefore* $\gamma = (\Gamma C)+(\Gamma A_1')$.

In fact $C+A_1'$ is the only curve of the pencil considered in § 14 (iii) which contains an arbitrarily chosen point P of (ΓC), unless P is a base point of this pencil; in this case, since every curve Ω_0 of the pencil goes through P, the point P must belong to G_0; such a case can occur if and (by § 14 (i)) only if the curve A_1 contains P.

It may be remarked that the hypothesis (i) assumed in § 14 is not essential, since the case in which some intersection of C, C' falls upon Γ can be derived by means of limiting considerations from the case in which the three curves Γ, C, C' have no points in common.

† Our reasoning incidentally establishes the equality $\gamma^* = (CC')$, which we could write down at once, by § 9.

The question studied in § 14 can be easily discussed even if $A(\lambda)$ and $A'(\lambda)$ are not both variable within linear systems, by adding to them a curve $B(\lambda)$ or $B'(\lambda)$ possessing (for $\lambda \to \lambda_0$) the limits B_0, B_0' not particularly related to A_0, A_0', and such that the same hypotheses as were made in § 14 for $A(\lambda)$, $A'(\lambda)$ and A_0, A_0' now hold for

$$\bar{A}(\lambda) = A(\lambda) + B(\lambda), \qquad \bar{A}'(\lambda) = A'(\lambda) + B'(\lambda)$$

and

$$\bar{A}_0 = A_0 + B_0 = \Gamma + (B_0 + C), \qquad \bar{A}_0' = A_0' + B_0' = \Gamma + (B_0' + C').$$

Then we have

$$(AA') = (\bar{A}\bar{A}') - (\bar{A}B') - (\bar{A}'B) + (BB'),$$

and therefore

$$\begin{aligned} G_0 &= \lim_{\lambda \to \lambda_0} (AA') = \lim_{\lambda \to \lambda_0} (\bar{A}\bar{A}') - (\bar{A}_0 B_0') - (\bar{A}_0' B_0) + (B_0 B_0') \\ &= \bar{\gamma} + (B_0 + C, B_0' + C') - (\Gamma + B_0 + C, B_0') - (\Gamma + B_0' + C', B_0) + (B_0 B_0') \\ &= \bar{\gamma} - (\Gamma, B_0 + B_0') + (CC'), \end{aligned}$$

where (§ 14) $\bar{\gamma}$ is the complete intersection of Γ with a curve $\bar{\Omega}_0$ of a well-defined pencil

$$(B_0 + C + \bar{A}_1') \wedge (B_0' + C' + \bar{A}_1).$$

Since

$$\bar{A}_1 \equiv \bar{A}_0, \qquad \bar{A}_1' \equiv \bar{A}_0' \quad (\text{on } V),$$

we infer at once that the *functional contribution*

$$\gamma = \bar{\gamma} - (\Gamma, B_0 + B_0')$$

of Γ as partial intersection of the two curves A_0 and A_0' (§ 11) is given by the equivalence $\gamma \equiv (\Gamma, C + C' + \Gamma)_V$, holding upon Γ†.

16. The theorem of § 14 and the corollaries of § 15 enable us to analyse completely an interesting example recently given by Enriques‡.

We consider on a plane a variable algebraic curve $A(\lambda)$ of order 8, with 15 variable double points, which for $\lambda \to \lambda_0$ tends to a curve

$$A_0 = \Gamma + C$$

† The much less expressive fact of the validity of this equivalence on V is a simple consequence of (3).

‡ Enriques, 3.

consisting of two quartics Γ, C of genus 3, intersecting in 16 distinct points; and we require the limit of the canonical series

$$|K_{A(\lambda)}| \tag{11}$$

of $A(\lambda)$ for $\lambda \to \lambda_0$. This series is a g_{10}^5, which can be determined on $A(\lambda)$ by the linear system ∞^5 of the quintic curves $A'(\lambda)$ adjoint to $A(\lambda)$; every $A'(\lambda)$ cuts on $A(\lambda)$ a canonical group $K_{A(\lambda)}$, in addition to the intersections absorbed by the 15 double points of $A(\lambda)$: so that, denoting the group of these points by $H(\lambda)$, we have

$$G(\lambda) = \big(A(\lambda)\,A'(\lambda)\big) = K_{A(\lambda)} + 2H(\lambda). \tag{12}$$

If λ tends in a well-defined way to λ_0, then the group $H(\lambda)$ tends to a group H_0 formed by 15 of the 16 points of (ΓC), the remaining point O constituting a so-called connecting-point of the reducible curve $A_0 = \Gamma + C$ considered as limiting position of the irreducible curve $A(\lambda)$; and the ∞^5 linear system of the curves $A'(\lambda)$ tends to the ∞^5 linear system of the quintics A_0' passing simply through H_0. All these curves A_0' also go through O; and they determine on Γ and on C [besides the fixed points of (ΓC)] the respective canonical g_4^2's $|K_\Gamma|$ and $|K_C|$, so that

$$(A_0\,A_0') = K_\Gamma + K_C + 2O + 2H_0. \tag{13}$$

Thus apparently, according to Enriques, we have the result that for $\lambda \to \lambda_0$ the linear series (11), of dimension 5, tends to the series of equivalence

$$|K_\Gamma| + |K_C| + 2O \tag{14}$$

having dimension 4 only; this is contrary to a property which we have proved and used in previous papers† and which, on the other hand, is a particular case of the theorem here established in § 4.

As we shall see immediately, the fact is that *the limit of* (11) *is no longer a series of equivalence on the limiting curve* $A_0 = \Gamma + C$, *but is broken up into the two* g_{10}^5*'s*

$$|K_\Gamma + 2O| + |K_C| \quad \text{and} \quad |K_\Gamma| + |K_C + 2O| \tag{15}$$

[*connected through the* g_{10}^4 (14)], *and therefore it has still dimension* 5.

For, while it is true that when $A'(\lambda)$ tends to a generic A_0' the group (12) tends to a group (13), we get other limiting groups G_0, in correspondence with the particular A_0' which have a component (Γ or C) in common

† B. Segre, 7, No. 4, and 8.

with A_0: so we find, as the limiting position of the groups of (11), the groups of the first or of the second of the g_{10}^5's (15) which do not belong to the g_{10}^5 (14)†. In fact, if we assume for instance that

$$A_0' = \Gamma + C',$$

C' being a straight line intersecting C in a generic canonical group

$$(CC') = K_C,$$

using the notations of § 14 we obtain

$$G_0 = \lim_{\lambda \to \lambda_0} G(\lambda) = \gamma + K_C,$$

where γ is a group of points of Γ, which (§ 15) upon this curve is equivalent to

$$(\Gamma,\ C + C' + \Gamma) - K_\Gamma + 2O + 2H_0$$

and which must, moreover, contain $2H_0$, since $G(\lambda)$ contains $2H(\lambda)$. The point O, which is a point of $(C\Gamma)$, does not, by § 15, in general belong to γ since in general a curve $A(\lambda)$ infinitely near to A_0 does not go through it, and the group $(\Gamma C) + (\Gamma A_1')$ does not contain $2H_0$ and is therefore distinct from γ. Thus, in the limit, $K_{A(\lambda)}$ becomes K_C plus a group of points of Γ equivalent to and different from $K_\Gamma + 2O$; and we immediately arrive at the conclusion in the manner enunciated above, taking into account that both the complete series of equivalence (15) are exactly of the same dimension 5 as the series (11).

17. The preceding example, and others of an even simpler character‡, show that, if on an irreducible curve $A = A(\lambda)$, variable with continuity upon a surface V, we consider the linear series cut by a variable linear system $|A'(\lambda)|$, it may well happen that, when A tends to a reducible curve $A_0 = \Gamma + C$, the limit of that linear series exceeds the series cut on A_0 by the linear system $|A_0'|$ which is the limit of $|A'(\lambda)|$. There are, however, some important cases in which such a possibility can be excluded, and we shall now examine one of them.

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† These limiting groups depend not only on that A_0', but also on the manner in which $A'(\lambda)$ approaches it; and that explains why the former are ∞^5 while the latter are only ∞^2.

‡ B. Segre, 8; another similar result is the following, concerning a curve A arbitrarily variable in a linear system $|A|$ whose effective multiple base points are all assigned. If a plane curve A of order 6 with 8 double points tends to a curve $A_0 = \Gamma + C$ composed of two elliptic cubics, then *the canonical g_2^1 of A tends to the two g_2^1 determined on Γ and on C by the connecting-point of A_0 counted twice.* This conclusion contradicts a theorem enunciated, without proof, by Zariski (in **17**, 56); I am, however, indebted to him for some criticisms of this and the preceding paragraph.

We suppose that $A(\lambda)$ has a group $H(\lambda)$ of double points which are simple base-points of $|A'(\lambda)|$ and that the curves of this linear system intersect residually $A(\lambda)$ in the (not necessarily canonical) groups $K_{A(\lambda)}$, so that (12) still holds. Further, when $\lambda \to \lambda_0$ $A(\lambda)$ tends to the curve A_0 composed of two curves Γ, C, intersecting in a group (ΓC) of distinct points, and $H(\lambda)$ tends to some of these points constituting a group H_0; thus $|A'(\lambda)|$ tends to a linear system of curves A_0' passing through H_0. We suppose, moreover, that the limiting linear system $|A_0'|$ possesses all the points of (ΓC) as simple base-points, and intersects Γ, C in the further groups K_Γ, K_C of two complete linear series; so that, if A_0' contains neither Γ nor C as a component, we can still write (13), denoting by O the group $(\Gamma C) - H_0$ of the connecting-points of the reducible curve A_0 considered as a limiting position of the irreducible curve $A(\lambda)$. We assume that the curves $A(\lambda)$ and $A'(\lambda)$ are variable on V within fixed linear systems, that no curve A_0' contains the whole curve A_0, and that all the curves $A(\lambda)$ infinitely near to A_0 pass through O.

By virtue of § 15, we have then the result that each point of this group O must be contained in every possible limit of $G(\lambda)$, and therefore also of $K_{A(\lambda)}$, for $\lambda \to \lambda_0$. Since (as in § 16) a limit of $K_{A(\lambda)}$ must belong to one at least of the two series (15), so, in the present circumstances, it ought more precisely to be contained in $|K_\Gamma + O| + |K_C| + O$ or in $|K_\Gamma| + |K_C + O| + O$; thus, with the additional hypothesis that on Γ, C respectively

$$|K_\Gamma + O| = |K_\Gamma| + O, \qquad |K_C + O| = |K_C| + O,$$

such a limit is necessarily a group of the linear series (14), and can certainly be cut on $A_0 = \Gamma + C$ by a curve A_0' containing neither Γ nor C.

This conclusion clears up and completes a deduction which I made in a previous paper, concerning a case in which all the hypotheses here made are satisfied†.

SECTION IV.

On limits of complete intersections of primals variable upon an algebraic variety.

18. Generalizations of the developments of section III to varieties of greater dimension can be made in several directions. Here we content ourselves with giving some simple extensions of § 14.

† B. Segre, 7, No. 7.

We begin by considering, upon a t-dimensional non-singular variety V, t primals

$$A^i = A^i(\lambda) \qquad (i = 1, 2, \ldots, t)$$

each variable within a linear system, which intersect in a finite number $[A^1 A^2 \ldots A^t]$ of points, constituting a variable group $G(\lambda)$. We suppose, moreover, that the ordinary limits

$$\lim_{\lambda \to \lambda_0} A^i(\lambda) = A_0^i \quad (i = 1, 2, \ldots, t) \tag{1}$$

exist; and that these primals intersect in a group δ of points and, further, in a pure h-dimensional non-singular variety Γ (with $1 \leqslant h \leqslant t-1$), in every point P of which they have the ∞^{h-1} tangent lines of Γ as their only common tangents: so that a suitably chosen set of $t-h$ primals A_0^i have no intersection other than Γ in the neighbourhood of P.

Our purpose is to construct the group

$$G_0 = \lim_{\lambda \to \lambda_0} G(\lambda), \tag{2}$$

by means of intersection operations. Of course we assume the existence of this ordinary limit, which we can always obtain also in the contrary case, by conveniently reducing the variability of λ in the neighbourhood of λ_0. Similar considerations are to be understood for the other limits which we have to consider later.

19. First of all, we have $G_0 \subset \Gamma + \delta$, by virtue of § 10, so that we can put

$$G = \gamma + \gamma^*, \tag{3}$$

where

$$\gamma \subset \Gamma,$$

$$\gamma^* \subset \delta\dagger. \tag{4}$$

Next we denote by Φ_0^i the limits, for $\lambda \to \lambda_0$, of the pencils

$$\Phi^i(\lambda) = A_0^i \wedge A^i(\lambda) \quad (i = 1, 2, \ldots, t);$$

we suppose, for the sake of simplicity, that the bases of these t pencils Φ_0^i have no points in common, and moreover that one at least of these bases, *e.g.* the base of Φ_0^1, does not contain any point of G_0. Then we fix a primal B_0^i distinct from A_0^i in every pencil Φ_0^i, and choose in $\Phi^i(\lambda)$ a primal $B^i(\lambda)$ having B_0^i as limit for $\lambda \to \lambda_0$.

When λ is not equal to λ_0, but is sufficiently near it, the three primals A_0^i, $A^i(\lambda)$ and $B^i(\lambda)$, all belonging to $\Phi^i(\lambda)$, are distinct. So we can

† Our reasoning incidentally proves the equality $\gamma^* = \delta$, which, by §10, we could have written down at once.

define projective correspondences among the t pencils $\Phi^i(\lambda)$ taken two by two, which associate their primals A_0^i, their primals $A^i(\lambda)$ and their primals $B^i(\lambda)$; and finally consider the locus of the points of V which are common to t corresponding varieties of these pencils. This locus consists of Γ and of a residual curve $\Omega(\lambda)$, such that

$$G(\lambda) \subset \left(\Omega(\lambda) A^i(\lambda)\right) \quad (i = 1, 2, \ldots, t). \tag{5}$$

By proceeding to the limit for $\lambda \to \lambda_0$, we obtain projective correspondences among the t pencils Φ_0^i taken two by two, associating their primals A_0^i and their primals B_0^i; the locus of the points of V which are common to t corresponding varieties of those pencils is composed of Γ, and of a further variety which we suppose (as will generally be the case) to be a curve Ω_0, so that

$$\Omega_0 = \lim_{\lambda \to \lambda_0} \Omega(\lambda). \tag{6}$$

From (1), (2), (5), (6) it follows (§§ 5, 10) that

$$G_0 \subset (\Omega_0 A_0^i) \quad (i = 1, 2, \ldots, t). \tag{7}$$

Therefore any point P of G_0 belonging to Γ, that is any point P of γ, is common to Ω_0 and Γ; since, on the present hypotheses, the multiplicity of intersection of Ω_0 and Γ in P is equal to the multiplicity of intersection in P of Ω_0 and at least one of the A_0^i's (§ 8), from (7) we find at once that

$$\gamma \subset (\Omega_0 \Gamma). \tag{8}$$

We remark now that the primals F_0^1 of the pencil Φ_0^1 determine on the curve Ω_0 the groups of ν variable points of a g_ν^1, the generic group of which is precisely the intersection of F_0^1 and its corresponding primals $F_0^2, \ldots, F_0^t$ of the pencils $\Phi_0^2, \ldots, \Phi_0^t$. In fact, when F_0^1 varies in Φ_0^1, a point of the last group can neither remain fixed (since the bases of the pencils Φ_0^i have no points in common) nor can it vary on the base of Φ_0^1, since otherwise a component of the curve Ω_0 would lie upon this base, which would therefore have some point in common with G_0. Hence

$$\nu = [F_0^1 F_0^2 \ldots F_0^t].$$

When F_0^1, varying in Φ_0^1, tends to A_0^1, the ν above-mentioned intersections of F_0^1 and Ω_0 can only fall in points of the base of Φ_0^1, or in points of δ, or in points of Γ; since in any point of the latter type the multiplicity of intersection of A_0^1 and Ω_0 is not less than the multiplicity of intersection of Γ and Ω_0 (§ 6), we have

$$[F_0^1 F_0^2 \ldots F_0^t] \geqslant [\Omega_0 \Gamma] + [\delta].$$

From this relation and (3), (4), (8) it follows that

$$[G_0] = [\gamma] + [\gamma^*] \leqslant [\Omega_0 \Gamma] + [\delta] \leqslant [F_0^1 F_0^2 \dots F_0^t];$$

here, however, all the signs $\leqslant$ must be replaced by $=$, since

$$[G_0] = [G] = [A^1 A^2 \dots A^t] = [F_0^1 F_0^2 \dots F_0^t],$$

and therefore (4) and (8) can hold only if

$$\gamma^* = \delta, \quad \gamma = (\Omega_0 \Gamma).$$

The last equality gives the required construction for the geometrical contribution γ.

20. Let us now consider more generally, upon the t-dimensional non-singular variety V, $t-h$ primals

$$A^i = A^i(\lambda) \quad (i = 1, 2, \dots, t-h;\ 0 \leqslant h \leqslant t-2),$$

variable within linear systems, and intersecting in a pure h-dimensional variety $G(\lambda)$; we suppose, moreover, that the ordinary limits

$$\lim_{\lambda \to \lambda_0} A^i(\lambda) = A_0^i \quad (i = 1, 2, \dots, t-h)$$

exist, and intersect in a pure h-dimensional variety Δ and also in a pure k-dimensional variety Γ, with $h+1 \leqslant k \leqslant t-1$.

We then have (§ 11) the result that the limit G_0 of $G(\lambda)$ for $\lambda \to \lambda_0$ (supposed of course to exist) breaks up into Δ and into another pure h-dimensional variety γ belonging to Γ. By applying the results of § 19 to the intersection of the above-considered figure with any $(t-h)$-dimensional variety of V, not particularly related to it, we see that

The geometrical contribution γ of Γ to the intersection of the A_0^i's [considered as limiting positions of the $A^i(\lambda)$'s] can in general be constructed as the effective h-dimensional intersection of Γ with a convenient variety Ω_0 of dimension $h+1$.

In particular, if we suppose that

(i) Γ has at most ∞^{h-1} multiple points;

(ii) the points of Γ in which the A_0^i's possess more than ∞^{k-1} tangent lines in common are at most ∞^{h-1};

(iii) the $t-h$ pencils

$$\Phi^i(\lambda) = A_0^i \wedge A^i(\lambda) \quad (i = 1, 2, \dots, t-h)$$

have regular limits Φ_0^i for $\lambda \to \lambda_0$, and the bases of these pencils have at most ∞^{h-1} points in common; then Ω_0 can be obtained as the locus, residual to Γ, of the intersections of $t-h$ corresponding primals of the pencils Φ_0^i, referred two by two by suitable projectivities associating the A_0^i's one with another. And we can, without restriction upon γ, impose on these projectivities the additional conditions of associating with one another $t-h$ primals distinct from the A_0^i's, arbitrarily chosen one in each of the pencils Φ_0^i.

21. The variety Ω_0 just defined, and therefore also the geometrical contribution γ, depend upon the pencils Φ_0^i, that is, upon the $A^i(\lambda)$'s infinitely near to the A_0^i's, rather than on the whole systems Σ^i described by the $A^i(\lambda)$'s when λ varies. Once these systems Σ^i and, consequently, also the pencils Φ_0^i are fixed, γ may still vary on Γ in dependence on the projective correspondences (considered in § 20) among the latter, which, for their part, depend upon the correspondences determined among the former by associating their primals arising from the same λ. These projective correspondences depend rationally on $t-h-1$ essential parameters; also the pencils Φ_0^i can be determined within the complete linear systems $|A^i|$ containing the Σ^i's as rational functions of a certain number of parameters. Thus we see that, in fixing only Γ and the linear systems $|A^i|$, the varying positions which γ can assume on Γ constitute an unirational system, and therefore a *system of equivalence upon* Γ†.

Leaving to another paper the problem, which thus arises, of giving an equivalence expressing γ as a function of $\Gamma, A^1, A^2, \ldots, A^{t-h}$, we now devote ourselves to the problem of determining more closely the geometrical contribution γ in some particularly interesting cases, related to the important problem of finding the limit of the linear tangent spaces of an algebraic variety tending to a given limiting position.

SECTION V.

The limit of the tangent primes of a primal degenerating into one or more (simple or multiple) components.

22. Let us consider, in a projective space $[t]$, a non-singular primal $A = A(\lambda)$ of order n, variable in a pencil Σ, and tending for $\lambda \to \lambda_0$ to a primal

$$A_0 = \sum_{i=1}^{s} r_i A_i$$

† Severi, 13.

broken up into s ($\geqslant 1$) parts $A_1, A_2, \ldots, A_s$ of orders $n_1, n_2, \ldots, n_s$, counted $r_1, r_2, \ldots, r_s$ times respectively, so that

$$n = \sum_{i=1}^{s} r_i n_i. \tag{1}$$

Our present purpose is to determine the limit for $\lambda \to \lambda_0$ of the ∞^{t-1} primes of $[t]$ touching A.

For the sake of simplicity, we suppose that the A_i's taken h by h (for $h = 1, 2, \ldots, p$, p being the smaller of the numbers s, t) intersect in the non-singular varieties

$$A_{i_1 i_2 \ldots i_h} = (A_{i_1} A_{i_2} \ldots A_{i_h}) \tag{2}$$

of regular dimension $t-h$; we also suppose that the varieties (independent of λ)

$$A_{0 j_1 j_2 \ldots j_k} = (A A_{j_1} A_{j_2} \ldots A_{j_k}) \tag{3}$$

(for $k = 1, 2, \ldots, q$, q being the smaller of the numbers s, $t-1$) are without multiple points, and of regular dimension $t-k-1$. Then, if we denote generally by M^* the totality of the ∞^{t-1} tangent primes of a given non-singular variety M (of any dimension) immersed in $[t]$, the answer to our question is given by the relation

$$\lim_{\lambda \to \lambda_0} A^* = \sum_{h=1}^{p} \sum_{(i)} (r_{i_1} + r_{i_2} + \ldots + r_{i_h}) A^*_{i_1 i_2 \ldots i_h} + \sum_{k=1}^{q} \sum_{(j)} (r_{j_1} + r_{j_2} + \ldots + r_{j_k} - 1) A^*_{0 j_1 j_2 \ldots j_k}, \tag{4}$$

where the inner sums are to extend over all the combinations $(i) = (i_1, i_2, \ldots, i_h)$ or $(j) = (j_1, j_2, \ldots, j_k)$ of the numbers $1, 2, \ldots, s$ taken h or k at a time.

23. Choosing in $[t]$ $t-1$ independent points $P', P'', \ldots, P^{(t-1)}$ joined by a space S_{t-2} unrelated to each of the varieties (2), (3), we denote generically by γ_M the group of the points of contact of a variety M of $[t]$ not touching S_{t-2} with the tangent primes passing through S_{t-2} (so that the group of these tangent primes is simply the projection of γ_M from S_{t-2}), and by $M^{(i)}$ the first polar of $P^{(i)}$ with respect to M if, of course, M is a primal. Then:

$$\gamma_A = (A A' A'' \ldots A^{(t-1)}); \tag{5}$$

and the question considered in § 22, being equivalent to the problem of finding the limit of the group (5) for $\lambda \to \lambda_0$, is of the same type as that

dealt with in § 20, save that not all the hypotheses stated there are now to hold.

Even now we can, however, follow the same leading principles, and consider the locus described by γ_A while A describes the pencil Σ. This locus is a curve $\mathfrak{C}$ not belonging to A_0, and we have evidently

$$(\mathfrak{C}A) = \gamma_A + \delta,$$

where A is generic in Σ and δ is the group of the fixed points of $\mathfrak{C}$ which fall on the base (AA_0) of the pencil Σ, each point counted with the multiplicity which is due to it as intersection of $\mathfrak{C}$ and A. From this relation it follows immediately that

$$\lim_{\lambda \to \lambda_0} \gamma_A = (\mathfrak{C}A_0) - \delta,$$

and the whole question is thus reduced to the correct determination of the points which are common to $\mathfrak{C}$ and A_0.

With regard to this, we shall show later (§§ 24–27) that

The curve $\mathfrak{C}$ has in common with the primal A_0 precisely the points of the different groups

$$\gamma_{A_{i_1 i_2 \dots i_h}} \quad \textit{and} \quad \gamma_{A_{0 j_1 j_2 \dots j_k}}, \tag{6}$$

through each of which it passes simply, without touching either A_0 or A there.

A point of the first or of the second group (6) has respectively multiplicity $r_{i_1}+r_{i_2}+\dots+r_{i_h}$ or $r_{j_1}+r_{j_2}+\dots+r_{j_k}$ for A_0, and multiplicity 0 or 1 for A; hence

$$(\mathfrak{C}A_0) = \sum_{h=1}^{p} \sum_{(i)} (r_{i_1}+r_{i_2}+\dots+r_{i_h}) \gamma_{A_{i_1 i_2 \dots i_h}} + \sum_{k=1}^{q} \sum_{(j)} (r_{j_1}+r_{j_2}+\dots+r_{j_k}) \gamma_{A_{0 j_1 j_2 \dots j_k}},$$

$$\delta = \sum_{k=1}^{q} \sum_{(j)} \gamma_{A_{0 j_1 j_2 \dots j_k}},$$

and therefore

$$\lim_{\lambda \to \lambda_0} \gamma_A = \sum_{h=1}^{p} \sum_{(i)} (r_{i_1}+r_{i_2}+\dots+r_{i_h}) \gamma_{A_{i_1 i_2 \dots i_h}} + \sum_{k=1}^{q} \sum_{(j)} (r_{j_1}+r_{j_2}+\dots+r_{j_k}-1) \gamma_{A_{0 j_1 j_2 \dots j_k}}. \tag{7}$$

Hence (4) follows by projection from the space S_{t-2} generically fixed in $[t]$ at the beginning, since the projections of γ_A, $\gamma_{A_{i_1 i_2 \dots i_h}}$, $\gamma_{A_{0 j_1 j_2 \dots j_k}}$ are precisely the tangent primes of A, $A_{i_1 i_2 \dots i_h}$, $A_{0 j_1 j_2 \dots j_k}$ passing through S_{t-2},

and the projection of the limit of γ_A coincides with the limit of the projection of this group of points.

24. In order to establish the result enunciated above, we now introduce in $[t]$ a system of projective homogenous coordinates (x), and consider the equations (respectively of degrees n and n_i)

$$a(x) = 0 \tag{8}$$

and

$$a_i(x) = 0 \quad (i = 1, 2, \ldots, s) \tag{9}$$

of any particular primal of Σ, distinct from A_0, and of A_i. Thus, putting for the sake of brevity

$$a_0(x) = [a_1(x)]^{r_1}[a_2(x)]^{r_2} \ldots [a_s(x)]^{r_s},$$

we can represent the general primal $A = A(\lambda)$ of Σ by the equation

$$a_0(x) + \lambda a(x) = 0, \tag{10}$$

from which comes the equation of the primal A_0 for the value $\lambda_0 = 0$ of the parameter λ.

If we denote by $a^{(l)}(x)$, $a_i^{(l)}(x)$ the first polar forms of $P^{(l)}$ with respect to $a(x)$, $a_i(x)$, then the first polar $A^{(l)}$ of $P^{(l)}$ with respect to A has the equation

$$a_0(x) \sum_{i=1}^{s} r_i a_i^{(l)}(x)/a_i(x) + \lambda a^{(l)}(x) = 0 \quad (l = 1, 2, \ldots, t-1), \tag{11}$$

and the system formed by the t equations (10), (11) represents the group (5); it follows that the curve $\mathfrak{C}$, considered in § 23, is the intersection not on A_0, that is not on the primals (9), of the primals

$$a(x)\, a_1(x)\, a_2(x) \ldots a_s(x) \sum_{i=1}^{s} r_i a_i^{(l)}(x)/a_i(x) - a^{(l)}(x)\, a_1(x)\, a_2(x) \ldots a_s(x) = 0 \tag{12}$$
$$(l = 1, 2, \ldots, t-1).$$

These $t-1$ primals have in common, upon A_0, the $\binom{s+1}{2}$ varieties A_{ij} $(i, j = 0, 1, \ldots, s)$, the intersections in pairs of the $s+1$ primals (8), (9); more precisely, each of the former goes simply through each of the latter, and therefore contains each of the varieties (2), (3), with the single exception of $A_1, A_2, \ldots, A_s$.

The problem is to determine the points of $\mathfrak{C}$ which belong to A_0, that is, to one at least of its s components A_i.

A point P belonging to a single one of these components, A_i say, and not lying at the same time on the primal (8), satisfies the equations (12) if, and only if, in it we have

$$\alpha_i^{(l)}(x) = 0 \quad (l = 1, 2, \ldots, t-1), \tag{13}$$

i.e., if P is a point of γ_{A_i}. The primes touching at such a point P the varieties (12) and (13) corresponding to the same l intersect the tangent prime of A_i at P along the same space of dimension $t-2$; thus the $t-1$ varieties (12), like the varieties (13), have at the point P no tangent in common with A_i, and therefore intersect in a curve passing simply through P and not touching at P the variety A_i, and hence not touching A_0.

The theorem enunciated in § 23 is thus only partially proved; in order to complete the proof, we have still to examine the possibility of a point P of $\mathfrak{C}$ belonging to two or more of the $s+1$ primals (8), (9) simultaneously.

25. If P does not belong to the primal (8), but belongs to exactly h of the primals (9) (with $2 \leqslant h \leqslant p$), say to

$$A_{i_1}, A_{i_2}, \ldots, A_{i_h}, \tag{14}$$

then P is a point of $A_{i_1 i_2 \ldots i_h}$ (§ 22).

Each of the $t-1$ primals (12) passes $h-1$ times through $A_{i_1 i_2 \ldots i_h}$; the cone touching such a primal at the point P has as vertex the space S'_{t-h} touching $A_{i_1 i_2 \ldots i_h}$ at P, is exactly of order $h-1$ (since the points $P^{(l)}$ are generic in $S_{t\ 2}$) and goes through the $\binom{h}{2}$ spaces, of dimension $t-2$, along which the h (linearly independent) tangent primes of the primals (14) at P intersect two by two. A fact easy to establish, which we shall use afterwards, is that the totality of the cones of $[t]$ of vertex S'_{t-h} and order $h-1$ containing those $\binom{h}{2}$ spaces constitutes an ∞^{h-1} homaloidal system within the star of centre S'_{t-h}; so that h of such cones linearly independent have no points in common not on the fixed spaces indicated, while $h-1$ of them linearly independent intersect further in exactly one space of dimension $t-h+1$, which is generic through S'_{t-h} if the cones considered are generic.

We remark now that the ∞^{t-2} linear system

$$\sum_{l=1}^{t-1} \lambda_l [a(x)\, a_1(x) \ldots a_s(x) \sum_{i=1}^{s} r_i \alpha_i^{(l)}(x)/a_i(x) - a^{(l)}(x)\, a_1(x) \ldots a_s(x)] = 0, \tag{15}$$

determined by the primals (12), is projectively related to the space S_{t-2} described by the point

$$\sum_{l=1}^{t-1} \lambda_l P^{(l)}. \tag{16}$$

If $t-z-1$ among the cones of order $h-1$ which touch at P the $t-1$ varieties (12) are linearly independent, then exactly ∞^{z-1} among their linear combinations (15) have at P a singular point of multiplicity greater than $h-1$, and in correspondence we get in S_{t-2} a space $[z-1]$, locus of the homologous points (16). We shall prove that this space $[z-1]$ is precisely the intersection of S_{t-2} and S'_{t-h}.

For this purpose we consider any point of $[z-1]$, which we may suppose without restriction to be coincident with P', so that the primal

$$a(x)\,a_1(x)\dots a_s(x)\sum_{i=1}^{s} r_i a_i'(x)/a_i(x) - a'(x)\,a_1(x)\dots a_s(x) = 0$$

has at P a point of multiplicity greater than or equal to h. Since this also happens for the primal $a'(x)\,a_1(x)\dots a_s(x) = 0$, the same can consequently be said for

$$a(x)\,a_1(x)\dots a_s(x)\sum_{i=1}^{s} r_i a_i'(x)/a_i(x) = 0.$$

But at P the equation (8) is not satisfied by hypothesis; therefore the primal

$$[a_1(x)]^{r_1}\,[a_2(x)]^{r_2}\dots[a_s(x)]^{r_s}\sum_{i=1}^{s} r_i a_i'(x)/a_i(x) = 0,$$

namely the first polar of P' with respect to A_0, has at P a point of multiplicity not less than the multiplicity $r_{i_1}+r_{i_2}+\dots+r_{i_h}$ with which A_0 itself passes through P. It follows that the line PP' must have at least this multiplicity for the tangent cone of A_0 at P; since this cone breaks into the tangent primes of the h primals (14) counted respectively r_{i_1}, r_{i_2}, …, r_{i_h} times, P' must lie not only on S_{t-2}, but also upon the space S'_{t-h}, the intersection of the h primes. The reasoning being reversible without difficulty, the space $[z-1]$ turns out to be the intersection of the space S_{t-2}, fixed at the beginning in a generic position, with the space S'_{-h}, touching $A_{i_1 i_2 \dots i_h}$ at P.

Thus the only two possible cases are those in which $z-1$ has the values $t-h-2$ or $t-h-1$, the latter being equivalent to the hypothesis that P is a point of $\gamma_{A_{i_1 i_2 \dots i_h}}$.

In the former case, there are exactly $t-z-1=h$ linearly independent cones among the tangent cones of the primals (12) at the point P. These cones have consequently no points in common, outside the $\binom{h}{2}$ spaces of dimension $t-2$ touching at P the intersections of the primals (14) taken two by two; so that the primals (12) do not intersect in the neighbourhood of P outside those varieties, through each of which they pass simply, and $\mathfrak{C}$ cannot contain the point P.

In the latter case, there are precisely $t-z-1=h-1$ linearly independent cones among the tangent cones of the primals (12) at the point P. These cones, outside the above-mentioned spaces of dimension $t-2$, intersect therefore in exactly one space of dimension $t-h+1$ passing through their common vertex S'_{t-h}; and this space does not touch at P any of the primals (14), since we have chosen S_{t-2} generically. It follows that $\mathfrak{C}$ passes through each point of $\gamma_{A_{i_1 i_2 \dots i_h}}$ without touching at it any of the primals (14) and hence not even A_0; presently (§ 27), we shall rule out the possibility that $\mathfrak{C}$ has such a point as a multiple one.

26. The remaining alternative possibility of a point P belonging to the primal (8) and also to a certain number k of primals (9) (with $1 \leqslant k \leqslant q$), can be investigated by means of considerations which we do not explain in detail, since they are only slightly different from those which we have developed in § 25.

In this way it can be shown that, if the k primals (9) containing P are

$$A_{j_1}, A_{j_2}, \dots, A_{j_k}, \tag{17}$$

so that P belongs to $A_{0 j_1 j_2 \dots j_k}$, the curve $\mathfrak{C}$ goes through P if, and only if, P is a point of $\gamma_{A 0 j_1 j_2 \dots j_k}$; more precisely, $\mathfrak{C}$ contains each point of this group without touching at it any one of the primals (8), (17), that is without touching at it either A_0 or the generic primal A of the pencil Σ.

27. We are not yet able to deduce the relations (4) or (7) of §§ 22, 23; but from §§ 24–26 follows at once the relation obtainable from (7) by substituting in it for the different groups $\gamma_{A i_1 i_2 \dots i_h}$, $\gamma_{A 0 j_1 j_2 \dots j_k}$ the groups constituted by their respective points each counted with the multiplicity ($\geqslant 1$) with which the curve $\mathfrak{C}$ passes through it. Therefore, if we prove that the two groups represented by the two sides of (7) contain the same number of points, then we infer at once that the above-mentioned multiplicities are all equal to 1, so that the theorem of § 23 and its consequences (7), (4) are completely established.

In order to obtain the required verification of the numerical content of the equation (7), we put

$$n = m_0+1, \quad n_i = m_i+1 \ (i = 1, 2, \ldots, s), \quad r_0 = -1,$$

so that the identity (1) becomes

$$\sum_{i=0}^{s} r_i(m_i+1) = 0. \tag{18}$$

It is then easy to show that the equality to be proved can be written in the compact form

$$\sum_{k=1}^{q+1} \sum_{(j)} (r_{j_1}+r_{j_2}+\ldots+r_{j_k})(m_{j_1}+1)(m_{j_2}+1)\ldots(m_{j_k}+1)\,\mathfrak{X}^{t-k}_{j_1 j_2 \ldots j_k} = 0, \tag{19}$$

where the inner sum has to extend over all the combinations

$$(j) = (j_1, j_2, \ldots, j_k)$$

of the numbers 0, 1, ..., s taken k at a time, q is the smaller of the numbers s, $t-1$, and $\mathfrak{X}^{t-k}_{j_1 j_2 \ldots j_k}$ represents the sum of all the monomials of degree $t-k$ which can be formed with $m_{j_1}, m_{j_2}, \ldots, m_{j_k}$, each monomial being taken with coefficient 1.

We remark now that the left-hand side of (19) is symmetrical both in $r_0, r_1, \ldots, r_s$ and in $m_0, m_1, \ldots, m_s$ and that it can be put in the form

$$\sum_{i=0}^{s} r_i\,(m_i+1)\,P_i(m),$$

where $P_i(m)$ is a polynomial in $m_0, m_1, \ldots, m_s$, symmetrical in all these variables except possibly m_i. The equality (19) can then be deduced from (18), by proving that $P_0(m) = P_1(m) = \ldots = P_s(m)$, that is, that *e.g.* $P_0(m)$ is a symmetric function of all the $s+1$ variables $m_0, m_1, \ldots, m_s$.

For this purpose we endeavour to determine when the monomial

$$m_0^{\mu_0} m_1^{\mu_1} \ldots m_h^{\mu_h} \text{ (with } 0 \leqslant h \leqslant q, \text{ and } \mu_0 \geqslant 0, \mu_1 \geqslant 1, \ldots, \mu_h \geqslant 1 \text{ integers)} \tag{20}$$

effectively occurs in $P_0(m)$, and then to obtain its coefficient c. The problem is evidently equivalent to that of finding the coefficient c of

$$r_0(m_0+1)\,m_0^{\mu_0} m_1^{\mu_1} \ldots m^{\mu_h}, \tag{21}$$

in the expansion of the left-hand side of (19).

The terms of this left-hand side from which (21) can be derived are necessarily of the type

$$(r_0+r_1+\ldots+r_h+r_{j_1}+r_{j_2}+\ldots+r_{j_l})(m_0+1)(m_1+1)\ldots(m_h+1) \times (m_{j_1}+1)(m_{j_2}+1)\ldots(m_{j_l}+1)\,\mathfrak{X}^{t-h-l-1}_{01\ldots h j_1 j_2 \ldots j_l}, \tag{22}$$

where $0 \leqslant l \leqslant q-h$ and $j_1, j_2, \ldots, j_l$ are any l of the numbers $h+1$, $h+2, \ldots, s$; on the whole, for a given l ($= 0, 1, \ldots, q-h$), we have therefore $\binom{s-h}{l}$ terms like (22). In order to obtain (21) by expanding this term, we must multiply suitable summands of its $h+l+3$ factors; more precisely, we must take r_0 from the first factor, m_0+1 from the second, $+1$ from the following $h+l$ factors except perhaps a certain number i (with $0 \leqslant i \leqslant h$) of the first h:

$$m_1+1, m_2+1, \ldots, m_h+1, \tag{23}$$

in each of which the summand m_j can be chosen. The quotient of (21) and the product of the summands so selected in all the factors of (22) but the last, is a monomial of degree $\mu_0+\mu_1+\ldots+\mu_h-i$ in $m_0, m_1, \ldots, m_h$, with coefficient unity; and this monomial occurs (also with coefficient unity) in the last factor $\mathfrak{X}^{t-h-l-1}_{01\ldots h j_1 j_2 \ldots j_l}$ if, and only if, we have

$$\mu_0+\mu_1+\ldots+\mu_h-i = t-h-l-1. \tag{24}$$

Finally, with the additional condition (24) determining i, the expansion of (22) gives (21) a number of times exactly equal to the number $\binom{h}{i}$ of the ways in which it is possible to choose the first summand in i of the h factors (23).

If, for the sake of brevity, we put

$$\mu_0+\mu_1+\ldots+\mu_h = d, \tag{25}$$

the condition (24) becomes

$$d = t+i-h-l-1,$$

and the above limitations $0 \leqslant i \leqslant h$, $0 \leqslant l \leqslant q-h$ impose on the degree d of the monomial (20) the double limitation $t-q-1 \leqslant d \leqslant t-1$. If this restriction is satisfied, then from the previous analysis it follows that $P_0(m)$ contains the monomial (20), with coefficient

$$c = \sum_{l=0}^{q-h} \binom{s-h}{l}\binom{h}{i} = \sum_{l=0}^{q-h} \binom{s-h}{l}\binom{h}{l+h+d-t+1}.$$

We shall now prove that the last sum can be extended from $l=0$ to $l=s-h$. This is evident if $s \leqslant t-1$, since then $q=s$; if, on the contrary, $s > t-1$, we have $q=t-1$, and also $d \geqslant h$ in consequence of (25) and (20): so that, supposing $l > q-h$, we obtain

$$l+h+d-t+1 > d \geqslant h,$$

and therefore

$$\binom{h}{l+h+d \;\; -t+1} = 0.$$

Thus, in any case†,

$$c = \sum_{l=0}^{s-h} \binom{s-h}{l} \binom{h}{l+h+d-t+1} - \binom{s}{s+d-t+1} = \binom{s}{t-d-1}$$

is independent of h; and $P_0(m)$ is actually a symmetric function of $m_0, m_1, \ldots, m_s$, given explicitly by the expression

$$\sum_{d-t-q-1}^{t-1} \sum_{(\mu)} \binom{s}{t-d-1} m_0^{\mu_0} m_1^{\mu_1} \ldots m_s^{\mu_s},$$

in which the inner sum has to be extended over all the solutions in positive integers (zero included) $\mu_0, \mu_1, \ldots, \mu_s$ of the equation $\mu_0+\mu_1+\ldots+\mu_s = d$.

28. The relation (4) has a lump character; but its proof rests mainly on the theorem of § 23, which is substantially an infinitesimal one, even though the developments of § 27, which were used in completing the proof, have no infinitesimal character.

In the next section we shall obtain a wide generalisation, interesting in itself, of this infinitesimal property, from which we shall then deduce several further extensions of the results of the present section.

SECTION VI.

On the contact of two variable primals upon an analytical variety.

29. In the present paragraph we leave the algebraic field, and assume only that each of the geometrical entities considered is analytic. We might indeed suppose merely that some convenient hypotheses of continuity and derivability are satisfied; but, since this extension does not offer any particular difficulty, we disregard it.

We begin by considering, upon an analytic variety V of dimension t, an ∞^1 analytic system Σ of primals F, one F_0 of which admits at a simple point O of V a simple point or an algebroid singularity, say a k-fold point, where k is an integer greater than or equal to 1. If an analytic branch $\mathfrak{L}$ of a curve is taken, having the origin in O and not lying on F_0, we have

† We use here the identity

$$\sum_{l=0}^{\alpha} \binom{\alpha}{l} \binom{\beta}{\gamma+l} = \binom{\alpha+\beta}{\alpha+\gamma}.$$

in general through each point P of ξ, distinct from O, some primal F_P of Σ, which has P as a simple point; and the general limit for $P \to O$ of the $(t-1)$-dimensional linear space π touching F_P at the point P consists of a finite number $(\geqslant 1)$ of $(t-1)$-dimensional spaces π_0. Precisely:

(i) *π_0 has only one position, coinciding with the tangent space of F_0 at the point O, for $k = 1$;*

(ii) *if $k > 1$ and O does not belong to the envelope of Σ†, then π_0 has still only one position, coinciding with the linear polar space of the tangent line l of ξ at O with respect to the algebraic cone ϕ (of order k) touching F_0 at O* (this space‡ being of course supposed not to be indeterminate, that is supposing that l is not a multiple generator of ϕ);

(iii) *if $k > 1$, and O belongs to the envelope of Σ and the primal $\bar{F}$ of Σ infinitely near to F_0 passes simply through O, we consider the $(t-1)$-dimensional linear space ψ touching $\bar{F}$ at the point O§. Then π_0 coincides with ψ, if l is a multiple generator of ϕ; otherwise every limiting space π_0 must belong to the pencil determined by ψ and the polar space of l with respect to ϕ, if these two spaces are distinct.*

It is important to notice that all the results just enunciated are covariant with respect to the analytic point-transformations of V into another variety of the same dimension t, which are regular at the point P. We can therefore always reduce (for instance by means of a convenient projection) to the case in which V is a linear space; it is this unrestrictive hypothesis which we make for the proof given in § 30.

30. In the linear t-dimensional space V we introduce a system of projective non-homogeneous coordinates $(x) = (x_1, x_2, \ldots, x_t)$, having the origin at O. Then the branch of the curve ξ can be represented by the parametric equations

$$x_i = x_i(\lambda) = l_i \lambda^m + \ldots \qquad (i = 1, 2, \ldots, t), \tag{1}$$

where the l_i's are constants not all zero, m is an integer greater than or equal to 1, and the $\ldots$ represent terms of degree greater than m in λ. Likewise Σ is given by an analytic equation of the type

$$F[x_1, x_2, \ldots, x_t; \mu] \equiv \phi(x) + \mu^n \psi(x) + \ldots = 0, \tag{2}$$

† That is if O does not belong to the primal $\bar{F}$ of Σ infinitely near to F_0.

‡ Which is clearly defined within the star of centre O belonging to the t-dimensional linear space touching V at this point.

§ ψ intersects ϕ precisely in the cone touching at O the characteristic variety $(F_0 \bar{F})$ of the system Σ upon F_0.

regular in the neighbourhood of $x_1 = x_2 = \ldots = x_t = \mu = 0$; in this equation we have explicitly indicated the first two coefficients which do not vanish identically in the expansion in power series of the parameter μ.

If the primal F_0 corresponds to $\mu = 0$, so that its equation is $\phi(x) = 0$, and the correspondence between the primals of Σ near to F_0 and the values of μ near to 0 is one-to-one, we say that Σ *contains* F_0 *regularly* when $n = 1$; this notion has an intrinsic character, and can be at once extended to the case in which V is an arbitrary variety.

Since F_0 has a k-fold point at O, $\phi(x)$ must be of the form

$$\phi(x) \equiv \phi_k(x) + \ldots, \tag{3}$$

where $\phi_k(x)$ is a non-vanishing homogeneous polynomial of degree k and the remaining terms are of higher degree in the (x)'s. Similarly, in case (ii) of § 29 we have

$$\psi(x) \equiv \psi_0 + \ldots, \tag{4}$$

with ψ_0 a non-zero constant; and in case (iii)

$$\psi(x) \equiv \psi_1(x) + \ldots, \tag{5}$$

where $\psi_1(x) = 0$ represents the linear space ψ.

The equation of the correspondence between a point $P(\lambda)$ of $\mathfrak{x}$ and a primal $F_P(\mu)$ of Σ, passing through it, is

$$F[x_1(\lambda), x_2(\lambda), \ldots, x_t(\lambda); \mu] = 0. \tag{6}$$

This analytic equation is satisfied for $\lambda = \mu = 0$, but not identically for $\mu = 0$, and therefore defines implicitly in the neighbourhood of $\lambda = 0$ a finite number (greater than or equal to 1) of analytic functions $\mu = \mu(\lambda)$, vanishing for $\lambda = 0$ and having at the most an algebroid singularity at this point. We fix one of these functions, and consider the tangent prime π to $F_P(\mu)$ at the point $P(\lambda)$; on our hypothesis the equation of this space, given by

$$\sum_{i=1}^{t} \left(\frac{\partial F[x_1, x_2, \ldots, x_t; \mu(\lambda)]}{\partial x_i} \right)_{x_1 = x_1(\lambda), \ldots, x_t = x_t(\lambda)} [x_i - x_i(\lambda)] = 0, \tag{7}$$

does not vanish identically, as we shall see that it has a non-zero limit for $\lambda \to 0$; hence F_P generally possesses P as a simple point.

We now observe that the limit (finite or infinite)

$$\lim_{\lambda \to 0} [\mu(\lambda)]^n / \lambda^{m(k-1)} = a$$

certainly exists. We have, for instance, $a = 0$ if $k = 1$; and the same conclusion is true if $k > 1$ and (4) holds, as we see by dividing the two

sides of (6) by $\lambda^{m(k-1)}$, and passing to the limit for $\lambda \to 0$ taking into account (1)–(4). In both cases, if we divide the two sides of (7) by $\lambda^{m(k-1)}$ and proceed to the limit for $\lambda \to 0$, we obtain the equation

$$\sum_{i=1}^{t} \frac{\partial \phi_k(l)}{\partial l_i} x_i = 0;$$

for $k = 1$ this represents the tangent prime of F_0 at the point O, while for $k > 1$ it represents the polar space of the line l touching $\mathfrak{L}$ at O (and therefore having the equations $x_1 : l_1 = x_2 : l_2 = \ldots = x_t : l_t$) with respect to the cone ϕ which touches F_0 at O.

The results enunciated in § 29, (i), (ii) are thus established. In the case (iii), if a is finite we can proceed as before, and deduce from (7) the equation of π_0 in the form

$$\sum_{i=1}^{t} \frac{\partial \phi_k(l)}{\partial l_i} x_i + a\psi_1(x) = 0,$$

representing a prime of the type stated; if $a = \infty$, by dividing the two sides of (7) by $[\mu(\lambda)]^n$ and proceeding to the limit for $\lambda \to 0$, taking into account (1), (2), (3), (5), we obtain the equation $\psi_1(x) = 0$, showing that π_0 coincides with ψ.

31. If the cone ϕ, of order $k > 1$, touching F_0 at the point O possesses as vertex an infinite linear space ω, necessarily passing through O, from § 29 (ii) follows the noteworthy result that the spaces π_0 there considered cannot be arbitrary, since each of them must contain ω, at least if l is not a multiple generator of ϕ. We shall see (§ 32) that π_0 must always contain ω, without possibility of exception, if the singularity of F_0 at O is of a certain particular type.

In the case (iii) of § 29, if ϕ possesses as vertex a space ω which is intersected by ψ in an infinite space ω_1, we see at once that π_0 cannot assume every position through O, since it must contain ω_1.

32. We say that a primal F_0 on an analytic t-dimensional variety V has a *quasi-normal singularity* at a simple point O of V, if the neighbourhood of O upon F_0 breaks up into a finite number s ($\geqslant 1$) of analytic $(t-1)$-dimensional branches $F_1, F_2, \ldots, F_s$ of the first order, possibly to be counted multiply, which have at O s linearly independent tangent spaces $\alpha_1, \alpha_2, \ldots, \alpha_s$ (so that necessarily $s \leqslant t$). These branches therefore intersect along an analytic $(t-s)$-dimensional branch Ω of the first order, having at O the intersection ω of $\alpha_1, \alpha_2, \ldots, \alpha_s$ as $(t-s)$-dimensional tangent

space; and, if $r_1, r_2, \ldots, r_s$ are respectively the multiplicities ($\geqslant 1$) of $F_1, F_2, \ldots, F_s$ for the primal F_0, this passes through Ω with the multiplicity

$$k = \sum_{i=1}^{s} r_i. \tag{8}$$

In particular, the tangent cone ϕ of F_0 at the point O is of order k and consists of the s spaces α_i taken with the respective multiplicities r_i, so that it has the space ω as vertex.

The notion of quasi-normal singularity is evidently covariant with respect to the regular analytic transformations of V.

We are able to complete what we have said in § 31 for the case (ii), by showing that

On the hypothesis of § 29 (ii), *if F_0 has a quasi-normal singularity at O, for which we maintain the preceding notation, then π_0 must always assume a well-defined position through ω, even if l is a multiple generator of ϕ.*

In order to establish this result, by a preliminary analytic transformation, we may restrict ourselves to the case in which V and $F_1, F_2, \ldots, F_s$ are linear spaces, so that the equations (1), (2), (4) of § 30 hold with

$$\phi(x) \equiv x_1^{r_1} x_2^{r_2} \ldots x_s^{r_s}. \tag{9}$$

Since $\mathfrak{x}$ does not lie upon F_0, the right-hand side of any one of the first s equations (1) cannot vanish identically; we can therefore write these equations in the more convenient form

$$x_i = x_i(\lambda) = c_i \lambda^{m_i} + \ldots \quad \text{with } c_i \neq 0,\ m_i \geqslant m,\ i = 1, 2, \ldots, s, \tag{10}$$

and suppose the indices arranged so that

$$m_i = m_1 \text{ for } i = 1, 2, \ldots, s'; \quad m_i < m_1 \text{ for } i > s' \quad (1 \leqslant s' \leqslant s),$$

where the second set of relations has to be suppressed if $s' = s$.

By virtue of (2), (4), (8), (9), the equation (6) gives at once:

$$\lim_{\lambda \to 0} [\mu(\lambda)]^n / \lambda^{r_1 m_1 + r_2 m_2 + \ldots + r_s m_s - m_1} = 0.$$

Hence, by dividing both sides of (7) by

$$c_1^{r_1} c_2^{r_2} \ldots c_s^{r_s} \lambda^{r_1 m_1 + r_2 m_2 + \ldots + r_s m_s - m_1}$$

and proceeding to the limit for $\lambda \to 0$, taking into account (2), (9), (10), we see that π_0 has a well-defined position, given by the equation

$$\sum_{i=1}^{s'} r_i x_i / c_i = 0,$$

which passes through ω.

33. Consider upon an analytic t-dimensional variety V two ∞^1 analytic systems of primals, Σ and Σ' say; the locus of points P of contact between a primal F_P on Σ and a primal F_P' of Σ' (the two primals and V passing simply through P) is in general an analytic curve, covariant of Σ, Σ' with respect to the analytic regular transformations of V. When P, moving on a branch ξ of that curve, tends to a limiting position O through which V passes simply, F_P, F_P' and their common $(t-1)$-dimensional tangent space at P have the limiting positions F_0, F_0', π_0; these, on the hypotheses for the applicability of the results of §§ 29, 31, 32, are related among themselves and the line l touching ξ at P in the manner explained there. From this it follows, in many cases, that some conditions are imposed on F_0 and F_0' at the point O.

Thus, for instance, F_0 and F_0' must touch at O if both pass simply through this point (§ 29, i).

Two less obvious consequences concern the particular case in which F_0, F_0' have at O a quasi-normal singularity and a simple point respectively. Keeping the notation of § 32 for F_0, we see at once that

(i) *if O does not belong to the envelope of Σ, then the primal F_0' must touch at O the $(t-s)$-dimensional variety Ω (the locus of k-fold points of F_0)*;

(ii) *if O belongs to the envelope of Σ, that is, if the primal $\bar{F}$ of Σ infinitely near to F_0 contains O, and we suppose that $\bar{F}$ intersects Ω in a $(t-s-1)$-dimensional variety Ω_1 passing simply through O, then F_0' must touch Ω_1 at the point O.*

34. The last results of § 33, with a slightly restrictive condition added, can be reversed in the following manner.

We consider, upon the t-dimensional analytic variety V, two ∞^1 analytic systems of primals Σ, Σ', and suppose that the former contains regularly (§ 30) a primal F_0 having a quasi-normal singularity (§ 32) at a point O, simple for V, and that the latter contains regularly a primal F_0' passing simply through O, without possessing O as a point of its envelope. Then we can affirm that, in each of the following two cases, *the locus of the points of contact outside F_0 between a primal of Σ and a primal of Σ' consists, in the neighbourhood of O, of one analytic curve, which passes simply through O without touching F_0 at this point.*

(i) The point O does not belong to the envelope of Σ, that is, the primal $\bar{F}$ of Σ infinitely near to F does not contain O; moreover, if $s < t$,

the analytic variety $\Omega = (F_1\, F_2 \ldots F_s)$ and the primal F_0' have a simple contact at O†, but (if $s > 1$) F_0' does not touch at O any of the varieties along which intersect $s-1$ of the $F_1, F_2, \ldots, F_s$. In the present case we may also have $k = 1$, *i.e.* F_0 can pass simply through O.

(ii) The primal $\bar{F}$ goes through O, and intersects Ω along a variety $\Omega_1 = (\bar{F}\, F_1 F_2 \ldots F_s)$, of dimension $t-s-1 \geqslant 0$, which passes simply through O and, if $s+1 < t$, has at this point a simple contact with the primal F_0'; this, however, does not touch at O any of the varieties along which intersect s among the $\bar{F}, F_1, F_2, \ldots, F_s$. In the present case we have, moreover, to suppose that $k \geqslant 2$. *If $k \geqslant 3$, the curve ξ can never touch $\bar{F}$ at O; and the same result holds also for $k = 2$, unless Σ satisfies a particular condition, affecting the primal $\bar{F}$ of Σ infinitely near to $\bar{F}$.*

All these results are covariant with respect to the analytic transformations of V, regular at the point O. We can, therefore, as in §§ 30, 32, introduce upon V local coordinates $(x) = (x_1, x_2, \ldots, x_t)$, vanishing at O, so that Σ and Σ' are represented by the analytic equations

$$F[x_1, x_2, \ldots, x_t;\ \mu] \equiv \phi(x) + \mu\psi(x) + \mu^2\chi(x) + \ldots = 0, \tag{11}$$

$$F'[x_1, x_2, \ldots, x_t;\ \nu] \equiv f(x) + \nu g(x) + \ldots = 0, \tag{12}$$

regular in the neighbourhood of

$$x_1 = x_2 = \ldots = x_t = \mu = \nu = 0, \tag{13}$$

and reducing for $\mu = 0$ and $\nu = 0$ respectively to the equations

$$\phi(x) \equiv x_1^{r_1} x_2^{r_2} \ldots x_s^{r_s} = 0, \tag{14}$$

$$f(x) \equiv \sum_{i=1}^{t} a_i x_i + \sum_{i=1}^{t}\sum_{j=1}^{t} b_{ij} x_i x_j + \ldots = 0 \quad (b_{ij} = b_{ji}), \tag{15}$$

representing F_0 and F_0'. The conditions of contact between a primal $F(\mu)$ of Σ and a primal $F'(\nu)$ of Σ', at a point $P(x)$ not situated on F_0, can be written in the form

$$\phi(x)\,\frac{\partial F'[x_1, x_2, \ldots, x_t;\ \nu]}{\partial x_i} = \rho\,\frac{\partial F\,[x_1, x_2, \ldots, x_t;\ \mu]}{\partial x_i} \quad (i = 1, 2, \ldots, t). \tag{16}$$

We have to study the solutions, in the neighbourhood of (13), of the system of $t+2$ analytic equations (11), (12), (16), which do not satisfy (14).

In the following §§ 35 36 we prove completely the results enunciated in cases (i) and (ii), by assuming tentatively the existence of such a solution

† This means that F_0' intersects Ω along a $(t-s-1)$-dimensional variety having at O an ordinary double point.

depending on a parameter λ:

$$(17)\quad x_1 = x_1(\lambda),\ x_2 = x_2(\lambda),\ \ldots,\ x_t = x_t(\lambda),\ \mu = \mu(\lambda),\ \nu = \nu(\lambda),\ \rho = \rho(\lambda),$$

where $x_1, x_2, \ldots, x_t, \mu, \nu$ reduce to zero for $\lambda \to 0$. We make no hypothesis about the functions (17); we suppose merely that the $x_i(\lambda)$'s which are not identically zero are of definite infinitesimal orders with respect to λ; this can always be effected by suitable choice of the parameter λ and possibly reducing its field of variability near $\lambda = 0$.

Before proceeding, we observe that, by (11), (12), (14), the equations (16) can be arranged in the systems

$$(18)\quad \frac{\partial f(x)}{\partial x_i} + \nu \frac{\partial g(x)}{\partial x_i} + \ldots = \rho \left[\frac{r_i}{x_i} + \frac{\mu}{\phi(x)} \frac{\partial \psi(x)}{\partial x_i} + \frac{\mu^2}{\phi(x)} \frac{\partial \chi(x)}{\partial x_i} + \ldots \right], \qquad (i = 1, 2, \ldots, s),$$

$$(19)\quad \frac{\partial f(x)}{\partial x_j} + \nu \frac{\partial g(x)}{\partial x_j} + \ldots = \rho \left[\frac{\mu}{\phi(x)} \frac{\partial \psi(x)}{\partial x_j} + \frac{\mu^2}{\phi(x)} \frac{\partial \chi(x)}{\partial x_j} + \ldots \right], \qquad (j = s+1, \ldots, t),$$

the second of which, of course, must be omitted for $s = t$. Moreover, since O does not belong to the envelope of Σ', we have

$$(20)\quad g(x) \equiv g_0 + \sum_{i=1}^{t} g_i x_i + \ldots$$

with $g_0 \neq 0$.

35. We begin with the case (i) of § 34. As $\bar{F}$ does not contain O, in the equation (11) we have

$$(21)\quad \psi(x) \equiv d_0 + \sum_{i=1}^{t} d_i x_i + \ldots$$

with $d_0 \neq 0$, so that

$$(22)\quad \lim_{\lambda \to 0} \phi(x)/\mu = -d_0;$$

in this relation, as in the others that we consider later, all the variables have to be expressed as functions of λ by means of (17): and we notice that neither $\phi(x)$ nor μ can then be identically zero. Moreover, since the primal F_0' represented by (15) has in O a simple contact with the variety $\Omega = (F_1\ F_2\ \ldots\ F_s)$ (whose equations are $x_1 = x_2 = \ldots = x_s = 0$) without touching at O any of the intersections of $s-1$ among the $F_1, F_2, \ldots, F_s$,

we must have

$$a_{s+1} = a_{s+2} = \ldots = a_t = 0, \tag{23}$$

$$\begin{vmatrix} b_{s+1,\,s+1} & \cdots & b_{s+1,\,t} \\ \cdots & \cdots & \cdots \\ b_{t,\,s+1} & \cdots & b_{tt} \end{vmatrix} \neq 0, \tag{24}$$

$$a_1 a_2 \ldots a_s \neq 0, \tag{25}$$

where (23) and (24) are to be suppressed for $s = t$.

None of the functions $x_1(\lambda)$, $x_2(\lambda)$, ..., $x_s(\lambda)$ can be identically zero, since the equation (14) cannot be satisfied identically; and the equations (18), by virtue of (15), (25) and (22), (21), show that those functions and $\rho(\lambda)$ have the same infinitesimal order, h say, with respect to λ. We shall prove that, in all cases, h is the least of the infinitesimal orders with respect to λ of the functions $x_1(\lambda)$, $x_2(\lambda)$, ..., $x_t(\lambda)$ which are not identically zero. Let us in fact suppose the contrary, so that necessarily $t > s$, and let us consider the least $h_1 < h$ of those infinitesimal orders; then we have

$$\lim_{\lambda \to 0} \rho/\lambda^{h_1} = 0. \qquad \lim_{\lambda \to 0} x_i/\lambda^{h_1} = 0 \quad (i = 1, 2, \ldots, s);$$

also the limits $$\lim_{\lambda \to 0} x_j/\lambda^{h_1} = l_j \qquad (j = s+1, \ldots, t)$$

exist, are finite and not all zero. If we divide both sides of the equation (12) by λ^{h_1} and pass to the limit for $\lambda \to 0$, using (15), (23), (20), we obtain

$$\lim_{\lambda \to 0} \nu/\lambda^{h_1} = 0;$$

so that, by proceeding in the same manner with the equations (19) and using (15), (23), (22), (21), we get the relations:

$$\sum_{j=s+1}^{t} b_{ij} l_j = 0 \quad \text{for} \quad i = s+1, \ldots, t,$$

which are incompatible, by virtue of (24).

We have now to establish the existence of the (finite) limits:

$$\lim_{\lambda \to 0} x_i/\lambda^h = c_i \quad \text{for} \quad i = 1, 2, \ldots, t, \tag{26}$$

$$\lim_{\lambda \to 0} \mu/\lambda^{hk} = m, \tag{27}$$

$$\lim_{\lambda \to 0} \nu/\lambda^h = n, \tag{28}$$

$$\lim_{\lambda \to 0} \rho/\lambda^h = p \neq 0. \tag{29}$$

In fact, (26), (29) are consequences of the above remarks; (27) follows from (22), (14), (26), (8), with $m = -c_1^{r_1} c_2^{r_2} \dots c_s^{r_s}/d_0$; (28) follows from (12), (15), (23), (20), (26), with

$$n = -\sum_{i=1}^{s} a_i c_i / g_0. \tag{30}$$

Moreover, (18) shows at once that in the first s equations (26) we must have

$$c_i = p r_i / a_i \neq 0 \quad (i = 1, 2, \dots, s); \tag{31}$$

and finally, if $t > s$, by dividing both sides of (19) by λ^h and passing to the limit for $\lambda \to 0$, taking into account (15), (23), (26), (20), (28), (30), (29), (21), (22), we see that

$$\sum_{i=1}^{t} b_{ij} c_i - g_j/g_0 \sum_{i=1}^{s} a_i c_i = -p d_j/d_0 \quad (j = s+1, \dots, t).$$

The last relations, by virtue of (31), (8), give

$$\sum_{i=s+1}^{t} b_{ij} c_i = -p \left(\sum_{i=1}^{s} b_{ij} r_i/a_i - k g_j/g_0 + d_j/d_0 \right) \quad (j = s+1, \dots, t),$$

so that also $c_{s+1}, \dots, c_t$ are uniquely determinate, since (24) holds.

We have thus expressed all the limits (26)–(29) as functions of known elements and of the quantity p; this may assume any value different from zero, by a multiplicative transformation upon the parameter λ; we can, for instance, suppose $p = a_1/r_1$, so that $c_1 = 1$, after which all the limits are fully determined.

From the above analysis it follows that, by changing the variables to $x_1', x_2', \dots, x_t', \mu', \nu', \rho'$ from $x_1, x_2, \dots, x_t, \mu, \nu, \rho$ by means of the relations

$$\begin{cases} x_1 = x_1', \quad x_i = (x_i' + c_i) x_1' \quad \text{for} \quad i = 2, \dots, t, \\ \mu = (\mu' + m) x_1'^k, \quad \nu = (\nu' + n) x_1', \quad \rho = (\rho' + p) x_1', \end{cases} \tag{32}$$

the required solutions of our initial system (11), (12), (18), (19) become the solutions of the transformed system having $x_1' \not\equiv 0$ and belonging to the neighbourhood of

$$x_1' = x_2' = \dots = x_t' = \mu' = \nu' = \rho' = 0. \tag{33}$$

Now we see by means of a simple calculation that the transforms of the $t+2$ equations (11), (12), (18), (19), divided respectively by $x_1'^k, x_1', 1, x_1'$, are satisfied at the point (33), at which point, moreover, they are analytic,

regular and with a non-vanishing Jacobian with respect to the $t+2$ variables $x_2', \ldots, x_t', \mu', \nu', \rho'$; these variables are therefore defined implicitly and uniquely in that neighbourhood as analytic functions of x_1' vanishing for $x_1' = 0$. The substitution of such functions in (32) gives the required solution (17), with the parameter $\lambda = x_1' = x_1$; and we see immediately, taking into account (31), that the first t of the $t+3$ equations (17) so obtained represent an analytic curve which passes simply through O, without touching F_0 at this point.

36. In case (ii) of § 34, we can choose the coordinates (x) in such a manner that

$$\psi(x) \equiv x_{s+1}. \tag{34}$$

Then the equations of Ω_1 are $x_1 = \ldots = x_s = x_{s+1} = 0$; and, since the primal F_0', represented by (15), must have in O a simple contact with that variety without touching there any of the intersections of s among the $\bar{F}$, F_1, F_2, ..., F_s, we have

$$a_{s+2} = a_{s+3} = \ldots = a_t = 0, \tag{35}$$

$$\begin{vmatrix} b_{s+2,\,s+2} & \ldots & b_{s+2,\,t} \\ \ldots & \ldots & \ldots \\ b_{t,\,s+2} & \ldots & b_{tt} \end{vmatrix} \neq 0, \tag{36}$$

$$a_1 a_2 \ldots a_s a_{s+1} \neq 0, \tag{37}$$

with the convention of suppressing (35) and (36) for $s+1 = t$.

As in § 35, the equations (18) still make $x_1, x_2, \ldots, x_s$ and ρ have the same infinitesimal order, h say, with respect to λ. The equation given by (19) for $j = s+1$, by virtue of (14), (15), (34), (37), shows therefore that μ has the infinitesimal order $h(k-1)$; so that, from (11), combined with (14) and (34), it follows that x_{s+1} is at least of infinitesimal order h. Finally, by an argument analogous to that developed in § 35, if $t > s+1$, we deduce from the remaining $t-s-1$ equations (19) that the $x_{s+2}, \ldots, x$ also are at least of infinitesimal order h.

In conclusion, on our present hypotheses, the limits

$$\lim_{\lambda \to 0} x_i/\lambda^h = c_i \quad \text{for} \quad i = 1, 2, \ldots, t, \tag{38}$$

$$\lim_{\lambda \to 0} \mu/\lambda^{h(k-1)} = m, \tag{39}$$

$$\lim_{\lambda \to 0} \nu/\lambda^h = n, \tag{40}$$

$$\lim_{\lambda \to 0} \rho/\lambda^h = p \neq 0, \tag{41}$$

exist. In fact, (38), (41) are consequences of the above remarks; (39) follows easily from the equation given by (19) for $j = s+1$, with

$$m = a_{s+1} c_1^{r_1} c_2^{r_2} \dots c_s^{r_s}/p \neq 0; \tag{42}$$

(40) follows from (12), (15), (35), (20), (38), with

$$n = -\sum_{i=1}^{s+1} a_i c_i / g_0.$$

Moreover, the s equations (31) still hold as a consequence of (18); while from (11), recalling (14), (34), (38), (39), (42) and putting

$$\chi(x) = d_0 + \sum_{i=1}^{t} d_i x_i + \dots, \tag{43}$$

we deduce without difficulty

$$c_{s+1} = \begin{cases} -p/a_{s+1} & \text{if } k \geqslant 3, \\ -p/a_{s+1} - m d_0 & \text{if } k = 2. \end{cases} \tag{44}$$

Finally, if $t > s+1$, by dividing by λ^h both sides of the equations given by (19) for $j = s+2, \dots, t$ and passing to the limit for $\lambda \to 0$, using (15), (35), (38), (20), (40), (14), (41), (42), (34), (43), we see that

$$\sum_{i=1}^{t} b_{ij} c_i + n g_j = \begin{cases} 0 & \text{if } k \geqslant 3 \\ m a_{s+1} d_j & \text{if } k = 2 \end{cases} \qquad (j = s+2, \dots, t);$$

this system defines $c_{s+2}, \dots, c_t$ uniquely in every case, since (36) holds.

In the manner just explained we have expressed all the limits (38)–(41) as functions of known elements and of the quantity p, which may assume any value different from zero by a multiplicative transformation upon the parameter λ; we can thus, for instance, suppose that $p = a_1/r_1$, so that $c_1 = 1$ and thereafter all those limits are completely determined. In order to prove the results enunciated in § 34 (ii), we have only to effect the transformation

$$x_1 = x_1', \qquad x_i = (x_i' + c_i) x_1' \quad \text{for} \quad i = 2, \dots, t,$$

$$\mu = (\mu' + m) x_1'^{k-1}, \qquad \nu = (\nu' + n) x_1', \qquad \rho = (\rho' + p) x_1',$$

and then to argue as in § 35. It is easy to show that the curve $\mathfrak{L}$ at which we thus arrive possesses all the properties stated (§ 34, ii); in particular, $\mathfrak{L}$ touches at O the primal $\bar{F}$ of Σ infinitely near to F_0 if and only if $c_{s+1} = 0$, which, by virtue of (44), (42), (31), may only happen if $k = 2$ and moreover $d_0 = -(a_1/r_1)^{r_1}(a_2/r_2)^{r_2} \dots (a_s/r_s)^{r_s}/a_{s+1}^2$: and, by (11), (43), we can effectively say that the last condition affects the primal $\bar{\bar{F}}$ of Σ infinitely near to $\bar{F}$.

SECTION VII.

Some extensions and further applications.

37. The result of § 22 can now be immediately extended to the case of a $(t-1)$-dimensional algebraic variety A varying on a non-singular t-dimensional variety V, and describing on it an ∞^1 analytic system Σ which contains regularly (§ 30) a $(t-1)$-dimensional variety B, such that, while A is in general non-singular, B possesses some multiple points which are all quasi-normal singularities (§ 32). Each point O of B, of multiplicity $k(\geqslant 1)$, is then a simple point of an algebraic variety ϕ well defined by it, of dimension $t-s\,(1 \leqslant s \leqslant t)$, in the neighbourhood of which B is decomposed into s independent linear branches $\Phi_1, \Phi_2, \ldots, \Phi_s$ passing through ϕ and counted with certain multiplicities $r_1, r_2, \ldots, r_s\,(\geqslant 1)$, the sum of which is precisely k. We suppose moreover (for simplicity) that, if O is any point common to the variety B and its consecutive $\bar{A}$ in Σ, the latter goes simply through it with a branch independent of $\Phi_1, \Phi_2, \ldots, \Phi_s$. For each variety ϕ of dimension $t-s>0$, we consider its intersection ϕ_1 with $\bar{A}$ (namely the limit of ϕA for $A \to B$); and we denote by A^*, ϕ^*, $\phi_1{}^*$ the ∞^{t-1} algebraic systems constituted by the $[t-1]$'s the generic one of which simultaneously touches V and A, ϕ, ϕ_1 at a simple point of the latter variety.

Then *we have the relation*

$$\lim_{A \to B} A^* = \Sigma k\phi^* + \Sigma(k-1)\,\phi_1{}^*, \tag{1}$$

where the sums are extended to all the algebraic varieties ϕ and ϕ_1 defined above.

In order to establish (1), we fix within the linear space $[r]$ containing V an $[r-2]$ generically situated with respect to the varieties V, B, ϕ_1, and consider the pencil Σ' of the $(t-1)$-dimensional varieties A' cut on V by the primes through $[r-2]$. Moreover, M being any pure m-dimensional variety without multiple components belonging to V and not particularly related to $[r-2]$, we denote by γ_M the group of the (simple) points P of M the tangent $[m]$ at which is in a prime through $[r-2]$; at such a point P (which can be supposed to be neither a base-point nor a point of the Jacobian group of Σ') there is a well-defined $[t-1]$ touching both V and M, namely the space joining P and the $[t-2]$ cut on $[r-2]$ by the t-space touching V at P.

When A varies in Σ, the group γ_A describes a curve $\mathfrak{k}$, the locus of the contact points of a variety A of Σ with a variety A' of Σ'. If $A \to B$,

each point P of γ_A, moving on $\mathfrak{L}$, tends to a simple point O of a well-defined variety ϕ or ϕ_1; and, correspondingly, the prime joining P and $[r-2]$ tends to a well-defined $[r-1]$ through $[r-2]$, intersecting V along a variety A_0' of Σ' passing through O.

Such a point O cannot belong to $[r-2]$. In fact, if O were a point of $[r-2]$, it would be a simple point of A_0', since otherwise $[r-2]$ would touch V at the simple point O, which cannot occur owing to the generality of $[r-2]$ in $[r]$; then, by virtue of § 33, A_0' must touch at O the considered variety ϕ or ϕ_1, so that the same would occur for $[r-1]$ and therefore for $[r-2]$ too, while the contact between $[r-2]$ and any variety ϕ or ϕ_1 can be excluded on account of the generality of $[r-2]$. We can moreover suppose, for the same reason, that no point of the Jacobian group of Σ' belongs to B. By virtue of § 33, the point O just considered is necessarily a point of γ_ϕ or γ_{ϕ_1}; and there is a well-defined $(t-1)$-dimensional space touching at it both V and ϕ or ϕ_1 (namely the space which joins O and the $[t-2]$ intersected on $[r-2]$ by the t-space touching V at O), a space which is the limit for $A \to B$ of the $(t-1)$-dimensional tangent space of A at P.

Reciprocally, in consequence of the theorem of § 34, we obtain that any point O of a group γ_ϕ or γ_{ϕ_1} (the multiplicity k of which for B is supposed to be greater than or equal to 1 in the former case and greater than or equal to 2 in the latter) is certainly the limit for $A \to B$ of at least one point of γ_A; moreover, on account of the generality of $[r-2]$ in $[r]$, we see that $\mathfrak{L}$ goes simply through O, without touching at this point either B or A. The exact multiplicity of O in the limiting group of γ_A can easily be deduced, by means of infinitesimal considerations, from the particular case already treated in § 23; and we conclude therefore that

$$\lim_{A \to B} \gamma_A = \Sigma k \gamma_\phi + \Sigma (k-1) \gamma_{\phi_1},$$

whence (1) follows immediately.

The proof of (1) just given holds, save for slight alterations, if the variety V, instead of being non-singular, has quasi-normal singularities.

38. Consider the particular case in which B breaks up into s ($\geqslant 1$) $(t-1)$-dimensional non-singular parts $A_1, A_2, \ldots, A_s$, counted $r_1, r_2, \ldots, r_s$ times respectively. If we put

$$A_0 = A, \qquad r_0 = -1,$$

so that we can write

$$-A_0+B=\sum_{i=0}^{s} r_i A_i \to 0$$

instead of $A \to B$, the equation (1) can be written as (4) of § 22, or in the more compact form

$$(2)\qquad \lim_{\sum_{i=0}^{s} r_i A_i \to 0} \sum_{k=1}^{q} \sum_{(j)} (r_{j_1}+r_{j_2}+\ldots+r_{j_k}) A^{*}_{j_1 j_2 \ldots j_k} = 0.$$

Here q is the smaller of the numbers $s+1, t$; $(j)=(j_1, j_2, \ldots, j_k)$ is any combination of the numbers $0, 1, \ldots, s$ taken k at a time; $A_{j_1 j_2 \ldots j_k}$ is the intersection of $A_{j_1}, A_{j_2}, \ldots, A_{j_k}$, and $A^{*}_{j_1 j_2 \ldots j_k}$ denotes the ∞^{t-1} totality of the $[t-1]$'s which at the different points of it touch both this variety and V.

As an immediate corollary of § 37 we have thus obtained a wide extension of the result of § 22, the generality of which is emphasized by the fact that a $(t-1)$-dimensional algebraic variety describing an ∞^1 algebraic system always generates a t-dimensional algebraic variety. The formula (2) implies in particular some of the numerical results obtained by Semple on composite surfaces, and it also gives a general answer to a question which he put†.

39. It is worth noticing that the right-hand side of the equation (1) of § 37 [and also of the particular form (4) of § 22] consists of two parts: the first is uniquely determined by the limit B towards which the variable variety A is tending; while the second part depends on the manner in which A is approaching B, being the sum of the ϕ_1*'s, each counted a certain number $k-1$ (generally greater than 1) of times and determined by a $(t-s-1)$-dimensional variety ϕ_1, which varies upon a fixed $(t-s)$-dimensional variety ϕ when the manner of approach of A towards B changes.

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† Semple, **9**; see specially Nos. 6.2 and 6.3. A first attempt at determining in some particular cases the nature and multiplicity of the tangent planes to a degenerate surface in [3] led Zeuthen (in **17**, p. 48) to erroneous conclusions, as Roth pointed out in **5**. Before Roth, Chisini had determined (in **1**) the tangential envelope of a pair of surfaces of [3], regarded as the limit of a surface without singularities; then Roth went further (in **6**) into the method of Chisini, and made application of it to solve the analogous problem for surfaces in [4]. Other interesting contributions to similar questions have been given very recently by Semple, who (in **10**) treated the case of an irreducible curve or surface degenerating into a composite one, consisting respectively of several multiple curves or of a single multiple surface; and by Macpherson, who (in **4**) determined the limiting envelope of a non-singular variety degenerating into two simple components.

If A varies arbitrarily upon V within the linear system $|B|$, all the varieties ϕ_1 obtainable in such a way on ϕ can be rationally expressed as functions of a certain number of parameters, so that they belong to a fixed linear system depending only upon B and V. In § 40 we show, in a particular case, that the algebraic-geometric determination of the above-considered varieties ϕ_1 can be explicitly obtained as a function only of B (and independently of V).

Another question which can be asked in connection with the observation made above concerns the possibility of approaching B in such a particular way that the second part of the limiting tangential form contains only simple components. A partial answer to this question, for the very particular case of the degeneration of a curve into one with one or more multiple components, can be found in a paper by Semple†; but, with this author's process, the fixed components of the limiting tangential form (which depend only on the limiting curve) possess in general greater multiplicity than that belonging to our general manner of approach. Below (§ 41), an answer to the question stated is incidentally given in another particular case.

40. Let us consider, upon a t-dimensional non-singular algebraic variety V of $[r]$, a $(t-1)$-dimensional variety A varying in a linear system and tending to a variety $B = \sum_{i=1}^{s} A_i$ broken up into s distinct $(t-1)$-dimensional components. Putting for the sake of brevity

$$(3) \qquad A_{i_1 i_2 \dots i_h} = (A_{i_1} A_{i_2} \dots A_{i_h}), \quad A_{0 j_1 \dots j_k} = \lim_{A \to B} (A A_{j_1} \dots A_{j_k}),$$

where $(i) = (i_1, i_2, \dots, i_h)$, $(j) = (j_1, j_2, \dots, j_k)$, and p, q have the same meanings as in § 22, and supposing all these varieties to be non-singular and of regular dimensions $(t-h$ and $t-k-1)$, we have (§ 38)

$$(4) \qquad \lim_{A \to B} A^* = \sum_{h=1}^{p} \sum_{(i)} h A^*_{i_1 i_2 \dots i_h} + \sum_{k=2}^{q} \sum_{(j)} (k-1) A^*_{0 j_1 \dots j_k},$$

where the stars have the usual meaning.

In each point of a variety (3) having at least two distinct indices $l, m > 0$, the tangent space of V is perfectly defined as the t-space joining the tangent spaces of A_l, A_m at it; so that the single varieties appearing in the right-hand side of (4) are completely known (independently of V) when the variety B and the $A_{0 j_1 \dots j_k}$'s are given. We shall now obtain

† Semple, **10**, No. 2.

an algebraic-geometric expression of the latter varieties as functions only of the components of B; so that *the right-hand side of* (4) *is functionally determined by these components, independently of the manner in which* A *approaches* B *and of the variety* V *described by* A.

We denote by K, S and T respectively a (virtual) canonical variety of V, a prime section of V and the locus of the points of V in which the tangent space is incident to a given generic $[r-t-1]$; and we introduce similar notations $K_{j_1 j_2 \dots j_k}$, $S_{j_1 j_2 \dots j_k}$, $T_{j_1 j_2 \dots j_k}$ for $A_{j_1 j_2 \dots j_k}$. Then we have

$$T \equiv (t+1)\,S + K \quad (\text{on } V),$$

whence

$$TA_{j_1 j_2 \dots j_k} \equiv (t+1)\,S_{j_1 j_2 \dots j_k} + (KA_{j_1 j_2 \dots j_k})_V \quad (\text{on } A_{j_1 j_2 \dots j_k}),$$

and, moreover,

$$T_{j_1 j_2 \dots j_k} \equiv (t-k+1)\,S_{j_1 j_2 \dots j_k} + K_{j_1 j_2 \dots j_k} \quad (\text{on } A_{j_1 j_2 \dots j_k});$$

it follows that

$$TA_{j_1 j_2 \dots j_k} - T_{j_1 j_2 \dots j_k} \equiv kS_{j_1 j_2 \dots j_k} + (KA_{j_1 j_2 \dots j_k})_V - K_{j_1 j_2 \dots j_k} \quad (\text{on } A_{j_1 j_2 \dots j_k}),$$

and therefore, since

$$K_{j_1 j_2 \dots j_k} \equiv (KA_{j_1 j_2 \dots j_k})_V + (A_{j_1} + A_{j_2} + \dots + A_{j_k},\ A_{j_1 j_2 \dots j_k})_V \quad (\text{on } A_{j_1 j_2 \dots j_k}),$$

we obtain

$$(A_{j_1} + A_{j_2} + \dots + A_{j_k},\ A_{j_1 j_2 \dots j_k})_V \equiv kS_{j_1 j_2 \dots j_k} + T_{j_1 j_2 \dots j_k} - TA_{j_1 j_2 \dots j_k} \quad (\text{on } A_{j_1 j_2 \dots j_k}).$$

For $k \geqslant 2$ the right-hand side of this equivalence is fully determined, independently of V, by the k varieties $A_{j_1}, A_{j_2}, \dots, A_{j_k}$; in fact both $S_{j_1 j_2 \dots j_k}$ and $T_{j_1 j_2 \dots j_k}$ are already defined by the variety $A_{j_1 j_2 \dots j_k}$, their intersection, while $TA_{j_1 j_2 \dots j_k}$ is the locus of the points of $A_{j_1 j_2 \dots j_k}$ for which the t-space containing the k $(t-1)$-spaces touching $A_{j_1}, A_{j_2}, \dots, A_{j_k}$ is incident to the fixed $[r-t-1]$. The functional expression required is now immediate, being, upon $A_{j_1 j_2 \dots j_k}$,

$$\begin{aligned} A_{0 j_1 \dots j_k} &= \lim_{A \to B} (AA_{j_1} \dots A_{j_k}) \equiv (BA_{j_1} \dots A_{j_k})_V \\ &= (A_{j_1} + A_{j_2} + \dots + A_{j_k},\ A_{j_1 j_2 \dots j_k})_V + \sum_{(i)} A_{i j_1 j_2 \dots j_k} \\ &\equiv kS_{j_1 j_2 \dots j_k} + T_{j_1 j_2 \dots j_k} - TA_{j_1 j_2 \dots j_k} + \sum_{(i)} A_{i j_1 j_2 \dots j_k}, \end{aligned}$$

where the index i takes all the values (if any) $1, 2, \dots, s$ different from $j_1, j_2, \dots, j_k$.

41. An important problem is that of determining the simplest limiting tangential form which a given algebraic variety can assume, by means of continuous deformations within another fixed variety of greater dimension. Here we content ourselves with a very simple example, suggestive of some difficult questions concerning possible extensions.

In a linear space $[r]$, a primal U of order ρ_0 can always be continuously deformed into an $[r-1]$ counted ρ_0 times; for this purpose we can, for instance, apply to it a homology having as (fixed) fundamental spaces such an $[r-1]$ and a generic point P, and of characteristic varying continuously between 1 and 0. U may have any singularity, but, for simplicity, it is supposed not to have multiple components; so the same is also true for the $(r-2)$-dimensional variety U_1, the apparent contour of U from P on to $[r-1]$. It is then easy to see that the ∞^{r-1} algebraic system U^*, constituted by the single (generally simple) points of U each together with its tangent prime, has a regular limit when U tends, in the manner indicated, to $[r-1]$ counted ρ_0 times; a limit which is decomposed into the ∞^{r-1} system $[r-1]^*$ (formed by the points of $[r-1]$ each together with this prime) counted ρ_0 times, and into the ∞^{r-1} system U_1^* (formed by the ∞^{r-2} points of U_1 coupled with any one of the ∞^1 primes of $[r]$ touching there U_1)†.

If we denote by $\rho_1, \rho_2, \ldots, \rho_{r-1}$ the successive ranks of U, we see that U_1 has the order ρ_1 and the successive ranks $\rho_2, \rho_3, \ldots, \rho_{r-1}$; and, by means of the repeated application of the above process (to U_1, to its generic apparent contour on to an $[r-2]$, etc.), we see that U^* can be continuously deformed into a sum of elementary varieties. More precisely

$$U^* \succ\succ \sum_{i=1}^{r} \rho_{i-1}[r-i]^*,$$

where the symbol $\succ\succ$ indicates that the expression on the left can be reduced to that on the right through regular passages to the limit, and $[r-i]^*$ denotes the ∞^{r-1} system formed by the points of $[r-i]$ each coupled with any one of the ∞^{i-1} primes of $[r]$ containing $[r-i]$.

42. From § 41 we obtain at once a simple limiting tangential form for the intersection $W = (UV)$ of a primal U and a t-dimensional variety $V = V_t$, situated in a general mutual position within a linear space $[r]$; a form which is interesting even if it is not the simplest possible.

† It is worth noticing that the second part of the limiting tangential form, which depends upon the particular manner in which U approaches its limit, has no multiple components; a different result would be obtained by using a general manner of approach (§ 39).

Keeping for U the notation of § 41, and denoting respectively by V_{t-i}, W^* and V^*_{t-i} the section of V by a generic $[r-i]$ $(i = 1, 2, \ldots, t)$, the system of the ∞^{t-1} tangent $[t-1]$'s of W and the ∞^{t-1} system constituted by the $[t-1]$'s touching at the single points of V_{t-i} both this variety and V_t, we have

$$W^* \twoheadrightarrow \sum_{i=1}^{t} \rho_{i-1} V^*_{t-i}.$$

Hence the projective characters of W can be deduced as functions of those of U and V; we thus immediately obtain the well-known relation†

$$\tau_h = \sum_{l=0}^{h} \rho_l \sigma_{h-l} \quad (h = 0, 1, \ldots, t-1),$$

expressing the h-th rank τ_h of W in terms of the ranks ρ and σ of U and V.

43. Consider in $[r]$ a non-singular primal A of order n, approaching, in a pencil, an irreducible primal B having only quasi-normal singularities (§ 32). These will therefore constitute $(r-i)$-dimensional varieties ϕ_{r-i} through which B passes with the multiplicity i $(i = 2, 3, \ldots, r)$.

By virtue of § 37 we have

$$A^* \twoheadrightarrow B^* + \sum_{i=2}^{r} i\phi^*_{r-i} + \sum_{i=2}^{r-1} (i-1)(\phi_{r-i}A)^*.$$

Now, if we denote by $\phi_{r-i,j}$ the section of ϕ_{r-i} by a generic $[r-j]$ $(j = 1, 2, \ldots, r-i)$, it follows from § 42 that

$$(\phi_{r-i}A)^* \twoheadrightarrow \sum_{j=1}^{r-i} n(n-1)^{j-1} \phi^*_{r-i,j},$$

so that

$$A^* \twoheadrightarrow B^* + \sum_{i=2}^{r} i\phi^*_{r-i} + \sum_{i=2}^{r-1} \sum_{j=1}^{r-i} (i-1)\, n(n-1)^{j-1} \phi^*_{r-i,j}.$$

In all these formulae the symbol $\twoheadrightarrow$ has the same meaning as in § 41, and the starred varieties are to be considered as the envelopes of their tangent primes.

The projective characters of A being well known and those of the varieties ϕ_{r-i} being supposed given, it is possible to deduce the projective characters of B from the last relation. Thus, for instance, the h-th rank τ_h of B $(h = 1, 2, \ldots, r-1)$ is expressible in terms of the orders $\sigma_{r-i,0}$ and

† *Cf.* F. Severi, **11**, No. 12. The proof there given is of a completely different kind, and supposes U, V to have singularities of a restricted type.

of the successive ranks $\sigma_{r-i,j}$ of the varieties ϕ_{r-i} $(i=2, 3, \ldots, r;$ $j=1, 2, \ldots, r-i)$, by means of the equality

$$\tau_h = n(n-1)^h - \sum_{i=2}^{h+1} i\sigma_{r-i,h-i+1} - \sum_{i=2}^{h} \sum_{j=1}^{h-i+1} (i-1)\, n(n-1)^{j-1} \sigma_{r-i,h-i-j+1}.$$

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THE LOGARITHMIC DERIVATIVES AND FLAT REGIONS OF ANALYTIC FUNCTIONS

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1. *Introduction.*

In a classical paper* published in 1903 Boutroux discussed the asymptotic behaviour of the logarithm of an integral function $F(z)$ and its derivatives. It will be sufficient for the moment to quote his results for $\log|F(z)|$ and $F'(z)/F(z)$, and indeed in an improved form, to avoid a tedious discussion of his awkward notation. For the majority of values of $r<R$, with R sufficiently large, the integral function $F(z)$ satisfies the inequality

$$\log|F(z)|>-K\log M(kR) \tag{1}$$

on the whole circle $|z|=r$, where $k>1$ is prescribed, $K=K(k)$, and $M(r)$ is the maximum modulus of $F(z)$ on the circle $|z|=r$. Its logarithmic derivative satisfies the inequality

$$\left|\frac{F'(z)}{F(z)}\right|<K\frac{\log M(kR)\log_2 M(kR)}{R} \tag{2}$$

under the same conditions. Boutroux extended his analysis further in connection with the latter inequality and although his results were rather vague they suggested that it is valid except for a set of circles surrounding the zeros of $F(z)$, the sum of whose radii is relatively small.

The Nevanlinna potential-theory analysis of functions shows clearly that the behaviour of $F(z)$ inside a circle $|z|=R$ depends essentially on the behaviour of polynomials of high degree and (in a relatively simple manner) on the modulus of $F(z)$ on the circumference of a larger circle $|z|=kR$.

* **2**; for more readable accounts see **19** and **20**. See also **27**.

This makes it possible to deal with all integral functions at once and to express the results simply, whereas Boutroux, working with infinite products, had to express his results in terms of a complicated theory of "proximate orders". This theory was indeed an important discovery, but in the present connection it is inappropriate and particularly so in relation to functions of infinite order.

A considerable advance was made by Cartan* in 1928 when he solved the problem concerning polynomials corresponding to the inequality (1). As a result this inequality was established throughout the whole plane except for a set of circles the sum of whose radii is relatively small.

J. M. Whittaker† has drawn attention to the integrated form of (2) and especially to the question of regions in which it is possible to assert the inequality

$$\left|\log|F(z_1)| - \log|F(z_2)|\right| < \alpha \log M(|z_1|) \tag{3}$$

or this inequality in association with

$$\log|F(z_1)| > \beta \log M(|z_1|), \tag{4}$$

and he has given various applications of this kind of theory‡.

Similar results with $|F(z_1)|$ actually equal to $M(|z_1|)$ have been obtained by Wiman§, who used entirely different methods. The later results of Wiman's type, as well as some of Whittaker's special problems and the consideration of examples, suggest that the factor $\log_2 M(kR)$ in (2) is unnecessary‖. The removal of this factor has been our main ambition, and although we have not quite succeeded, we can remove it if z is restricted to a prescribed radius and so establish a number of long-standing conjectures made by one of us concerning the coefficients of certain power series.

The implications of Cartan's lemma have been examined in detail, both for meromorphic functions and for functions regular or meromorphic in an angle, by Miss Cartwright¶. We set out to do the same for some recent extensions** appropriate to the discussion of the logarithmic derivatives of regular or meromorphic functions. Before

* **4**, 272–276; **11**.
† **23**, **24**, **25**, **26**.
‡ **26**, which contains further references.
§ **28**.
‖ **10**, **18**, **26**, 72 for example.
¶ **5**, **6**, **7**, **8**, **22**.
** **12**.

describing our results in detail it is convenient to indulge in some generalities. We propose to prove that certain inequalities are satisfied in a certain part of the plane. Clearly, the more drastic the inequality the less extensive will be its region of validity. We consider as regions of validity "an appreciable fraction" of the plane and "almost all" the plane. The term "almost all" is used in several senses. The exceptional points where our inequality is not valid may be enclosed in a set of circles of relatively small total area when we say that the inequality holds "almost everywhere A", or, a stronger restriction, when the sum of the radii is relatively small, in which case the inequality holds "almost everywhere B". We shall find better results to hold in the former circumstance, but a result holding "almost everywhere B" is of special interest in that its region of validity includes the majority of points on every radius $\arg z = \text{const.}$ and indeed the whole circumference of the majority of circles $|z| = r$. Unfortunately the generalisations of Cartan's lemma at present available do not give us the inequality in which we are most interested "almost everywhere B" as we should expect, but only in a region somewhat less precisely defined. This region is characterised by the fact that it includes the greater part of every radius $\arg z = \text{const.}$

By using Nevanlinna's characteristic $T(r)$ in place of $\log M(r)$ we can state our results for functions meromorphic in the whole plane as follows. We find that the inequality

$$\left|\frac{F'(z)}{F(z)}\right| \leqslant K\,\frac{T(kR)}{R} \quad [K = K(h, k)] \tag{5}$$

holds for all z in $|z| < R$ except for a set of circles of total area less than $h\pi R^2 (0 < h < 1,\ k > 1)$, that is to say almost everywhere A. The weaker inequality

$$\left|\frac{F'(z)}{F(z)}\right| \leqslant K\,\frac{T(kR)\log T(kR)}{R} \tag{6}$$

holds almost everywhere B, but we can show that the better inequality (5) holds almost everywhere in the nearly related third sense. The natural generalisation of (4) to meromorphic functions involves the assumption that the poles are defective, and with the use of known results we obtain some advance in the theory of the regions in which a meromorphic function is both "flat" and "large"*.

* 9, 21.

As regards higher derivatives we have

$$(7)\quad |D^2 \log F(z)| < \frac{KT(kR)\log T(kR)}{R^2}, \quad |D^q \log F(z)| < K\frac{T(kR)^{\frac{1}{2}q}}{R^q} \qquad (q>2)$$

almost everywhere A, and

$$(8)\quad |D^q \log F(z)| < K\frac{T(kR)^q}{R^q} \qquad (q>1)$$

almost everywhere B. With the qualifications we have mentioned, the inequalities (5), (6), (7) were obtained by Boutroux*. The factor $\log T(kR)$ in the first part of (7) may be unnecessary.

We extend these results to functions regular or meromorphic in an angle, but there is then an additional source of complication, for the discussion of functions analytic in an angle has to be divided into several different cases† and the number of our results is multiplied thereby. The interaction of the two features is fairly straightforward and we content ourselves with stating here only the simplest results. Supposing $F(z)$ to be regular and to satisfy the inequality

$$(9)\quad \log|F(z)| \leqslant H|z|^\rho \quad (\rho>0,\ |z|>r_0)$$

in the half-plane $I(z)\geqslant 0$, then, if $\rho>1$, it follows that

$$(10)\quad \left|\frac{F'(iy)}{F(iy)}\right| < KR^{\rho-1}$$

for the majority of values y in $0<y<R$ for *all* sufficiently large R. If $\rho=1$ the right-hand side of (10) is replaced by $K\log R$ and if $\rho<1$ by K. In this last case we can, however, give the more exact inequality

$$(11)\quad \left|\frac{F'(iy)}{F(iy)} - i\eta\right| < \epsilon$$

η being a positive constant depending on the function.

The inequality (10) can, however, be asserted for *some* large R provided that we make an additional assumption of the form

$$(12)\quad \overline{\lim}\,\frac{\log|F(iy)|}{y^\rho} > -\infty$$

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* **2**, 167, 150, 164, 166, respectively.

† **5, 6, 7, 8, 13, 14.**

even when $\rho < 1$. Instead of considering only the imaginary axis we can, of course, discuss the function in the region

$$\delta R \leqslant |z| \leqslant (1-\delta)R, \quad \alpha \leqslant \arg z \leqslant \pi - \alpha, \tag{13}$$

and in relation to inequalities generalising (5), (6), (7) and (8), but it does not seem appropriate to enumerate the results apart from their proofs. We should perhaps remark that the consideration of a half-plane instead of an angle of arbitrary opening involves no real loss of generality.

Although we do not propose to discuss the influence of more special hypotheses in full, we do include a final section on a more particular theorem. This section has a certain claim to be regarded as the next step, but its main justification is in its applications to gap theorems and the like.

2. *Preliminary results.*

The results on polynomials and their logarithmic derivatives which we require can be stated as follows*. $P(z) = (z-a_1)(z-a_2)\ldots(z-a_n)$ being a polynomial of the nth degree and $\alpha = \frac{1}{2}$ or 1, H an arbitrary positive number, there exists a set of at most n circles (not necessarily the same set for all the inequalities), whose radii h_k satisfy

$$\Sigma h_k^{1/\alpha} \leqslant (2H)^{1/\alpha}, \tag{14}$$

outside which

$$\Sigma \log|z-a_k| \geqslant n \log(H/e) \quad (\alpha = 1) \tag{15}$$

and

$$\Sigma \frac{1}{|z-a_k|^m} \leqslant \frac{n^{m\alpha}}{H^m} \sum_{p=1}^{n} \frac{1}{p^{m\alpha}} \quad (\alpha = \tfrac{1}{2} \text{ or } 1). \tag{16}$$

The right-hand side of this inequality can evidently be replaced by five different approximations according to the values of m and α, viz.:

$$\left\{\begin{array}{l} \dfrac{n}{H}(1+\log n) \quad (\alpha = 1,\ m = 1); \qquad B_m\left(\dfrac{n}{H}\right)^m \quad (\alpha = 1,\ m > 1); \\ A\,\dfrac{n}{H} \quad (\alpha = \tfrac{1}{2},\ m = 1); \qquad \dfrac{n}{H^2}(1+\log n) \quad (\alpha = \tfrac{1}{2},\ m = 2); \\ A_m \dfrac{n^{\frac{1}{2}m}}{H^m} \quad (\alpha = \tfrac{1}{2},\ m > 2). \end{array}\right. \tag{17}$$

* For (15) 4, 272–276; 22, 11–12. For (16), (17), (18), 12.

Also, when z is restricted to a particular straight line of the complex plane,

$$\left|\Sigma \frac{1}{z-a_k}\right| \leqslant \lambda, \tag{18}$$

except for a set of at most n intervals* of total length not exceeding $2en/\lambda$. We also need several of Nevanlinna's formulae for functions $F(z)$ meromorphic in a circle and set them out here for purposes of reference as†

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\phi})| \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\phi-\theta)}\,d\phi + \sum_{|b_\nu|<R} \log\left|\frac{R^2-\bar{b}_\nu z}{R(z-b_\nu)}\right| - \sum_{|a_\mu|<R} \log\left|\frac{R^2-\bar{a}_\mu z}{R(z-a_\mu)}\right|, \tag{19}$$

where the zeros and poles of $F(z)$ are at a_μ and b_ν respectively and $z = re^{i\theta}$;

$$\log F(z) = iC + \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\phi})| \frac{Re^{i\phi}+z}{Re^{i\phi}-z}\,d\phi + \sum_{|b_\nu|<R} \log\frac{R^2-\bar{b}_\nu z}{R(z-b_\nu)} - \sum_{|a_\mu|<R} \log\frac{R^2-\bar{a}_\mu z}{R(z-a_\mu)}; \tag{20}$$

$$D^q \log F(z) = \sum_{|a_\mu|<R} \frac{(-)^{q-1}(q-1)!}{(z-a_\mu)^q} - \sum_{|b_\nu|<R} \frac{(-)^{q-1}(q-1)!}{(z-b_\nu)^q} + S_R(z) + I_R(z), \tag{21}$$

where

$$\begin{cases} S_R(z) = (q-1)! \sum\limits_{|a_\mu|<R} \left(\dfrac{\bar{a}_\mu}{R^2-\bar{a}_\mu z}\right)^q - (q-1)! \sum\limits_{|b_\nu|<R} \left(\dfrac{\bar{b}_\nu}{R^2-\bar{b}_\nu z}\right)^q, \\ I_R(z) = \dfrac{q!}{2\pi}\displaystyle\int_0^{2\pi} \log|F(Re^{i\phi})| \frac{2Re^{i\phi}}{(Re^{i\phi}-z)^{q+1}}\,d\phi. \end{cases} \tag{22}$$

We make the special restriction $F(0) = 1$ which will clearly involve us in no loss of generality and then (19) with $z = 0$ can be written

$$m(R, F)+N(R, F) = m(R, 1/F)+N(R, 1/F) = T(R, F) = T(R), \tag{23}$$

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* These intervals cannot exceed n in number, since they are terminated by real roots of the equation

$$\left(\Sigma \frac{1}{z-a_k}\right)\left(\Sigma \frac{1}{z-\bar{a}_k}\right) = \lambda^2.$$

† 15, 3, 4, 27, or 16, 154, 155, 211.

where

$$(24)\quad m(R,\,F)=\frac{1}{2\pi}\int_0^{2\pi}{}^{+}\log|F(Re^{i\phi})|\,d\phi,\qquad N(R,\,F)=\sum_{|b_\nu|<R}\log\frac{R}{|b_\nu|}.$$

The corresponding formulae for functions meromorphic in an angle will also be required, but it is convenient to postpone their statement until later.

3. *The logarithmic derivatives of meromorphic functions.*

The behaviour of the last two terms of (21) is, so far as we are concerned, comparatively simple. We restrict z to lie inside the circle $|z|=r$ and take $R=\sqrt{k}\,r$ where $k>1$ is given. We need a restriction on the number of poles and zeros of $F(z)$ in $|z|<R$, and, writing

$$(25)\qquad N=N(R)=n(R,\,0)+n(R,\,\infty),$$

where $n(R,\,0)$ is the number of zeros, $n(R,\,\infty)$ the number of poles, we have the elementary inequality*

$$(26)\qquad N(R)<KT(\sqrt{k}R)=KT(kr).$$

Each term in $S_R(z)$ is in modulus less than Kr^{-q} and so

$$(27)\qquad |S_R(z)|<K\frac{T(kr)}{r^q},$$

A similar inequality for $I_R(z)$ is evident, indeed

$$(28)\quad |I_R(z)|\leqslant\frac{K}{r^q}\{m(\sqrt{k}r,\,F)+m(\sqrt{k}r,\,1/F)\}\leqslant\frac{K}{r^q}T(\sqrt{k}r)\leqslant\frac{K}{r^q}T(kr).$$

Both these inequalities are as good as those that we hope to obtain, and so the essential difficulty is in connection with the sums

$$(29)\qquad \sum_{|a_\mu|<R}\frac{1}{(z-a_\mu)^q},\qquad \sum_{|b_\nu|<R}\frac{1}{(z-b_\nu)^q}.$$

For these we refer to the inequalities quoted in § 2, and, putting $2H=hr$, we immediately have the results described in the introduction which we state here as formal theorems.

* 15, 26.

THEOREM 1. *$F(z)$ being a meromorphic function, and $h>0$, $k>1$ being given, then*

$$(30)\qquad \begin{cases} \left|\dfrac{F'(z)}{F(z)}\right| < \dfrac{KT(kr)}{r}, \qquad |D^2 \log F(z)| < \dfrac{KT(kr)\log T(kr)}{r^2}, \\ |D^q \log F(z)| < \dfrac{KT(kr)^{\frac{1}{2}q}}{r^q} \quad (q>2) \end{cases}$$

for all z in $|z|<r$, (r sufficiently large), except for a set of at most $KT(kr)$ circles, the sum of whose areas is at most $h^2\pi r^2$.

Here we have applied the inequalities (16) and (17) with $\alpha=\frac{1}{2}$. In the same way, with $\alpha=1$ we obtain

THEOREM 2. *$F(z)$ being a meromorphic function, and $h>0$ and $k>1$ being given, then*

$$(31)\qquad \left|\frac{F'(z)}{F(z)}\right| < K\frac{T(kr)\log T(kr)}{r}, \qquad |D^q \log F(z)| < K\frac{T(kr)^q}{r^q} \quad (q>1)$$

for all z in $|z|<r$, (r sufficiently large), except for a set of at most $KT(kr)$ circles, the sum of whose radii is at most hr.

Similarly, using (18), we have

THEOREM 3. *For z on a given radius* $\arg z = \text{const.}$ *the meromorphic function $F(z)$ satisfies*

$$(32)\qquad \left|\frac{F'(z)}{F(z)}\right| < K\frac{T(kr)}{r}$$

for all $|z|<r$, (r sufficiently large), except for a set of at most $KT(kr)$ intervals, the sum of whose lengths is at most hr.

We have omitted details of the calculations which would give approximate values for the K's. These are complicated and are of secondary interest since exact values are not to be expected by our methods. In the formulation of our theorems we have omitted the condition $F(0)=1$, and in consequence our results are only valid "for sufficiently large r". To retain $F(0)=1$ we have only to modify $F(z)$ by a rational factor. The contribution of this factor to the logarithmic derivatives is negligible in comparison with our approximations if r is sufficiently large. The change in $T(r)$ is also negligible under the same condition. The resulting "asymptotic" theorems are, of course, of interest only when $F(z)$ is transcendental and not if it is rational. We shall, however, use the "finite" form of these theorems later, if only as a temporary expedient,

and so we now state them here, giving at the same time the crude values for the K's which are available by more detailed consideration of our chains of inequalities. We can replace (26), (27) and (28) by the more precise forms

$$N(R) \leqslant \frac{4}{\log k}\, T(kr), \tag{26a}$$

$$|S_R(z)| \leqslant \frac{8(q-1)!}{(k^{\frac{1}{2}}-1)^q \log k}\, \frac{T(kr)}{r^q}, \tag{27a}$$

$$|I_R(z)| \leqslant \frac{2q!\, k^{\frac{1}{2}}}{(k^{\frac{1}{2}}-1)^{q+1}}\, \frac{1}{r^q}\, \{m(R, F)+m(R, 1/F)\} \tag{28a}$$

$$\leqslant \frac{4q!\, k^{\frac{1}{2}}}{(k^{\frac{1}{2}}-1)^{q+1}}\, \frac{T(kr)}{r^q}.$$

Now, since we propose to assume $F(z)$ to be meromorphic only in a circle, we apply these inequalities with θr in place of r and $\theta = k^{-1}$, so that the required circle of meromorphy is $|z| \leqslant r$ and we have, assuming always in the three following theorems $F(z)$ to be meromorphic in $|z| \leqslant r$ and $F(0) = 1$,

THEOREM 1a. *Given positive numbers h and θ (< 1), then*

$$\left|\frac{F'(z)}{F(z)}\right| \leqslant \left\{\frac{16}{h\theta \log(1/\theta)} + \frac{8}{(\theta^{\frac{1}{2}}-\theta)\log(1/\theta)} + \frac{4}{\theta^{\frac{1}{2}}(1-\theta^{\frac{1}{2}})^2}\right\} \frac{T(r)}{r},$$

$$|D^2 \log F(z)| \leqslant \left\{\frac{16}{\theta^2 h^2 \log(1/\theta)}\left(1+\overset{+}{\log}\frac{4T(r)}{\log(1/\theta)}\right) + \frac{8}{(\theta^{\frac{1}{2}}-\theta)^2 \log(1/\theta)} + \frac{8\theta^{\frac{1}{2}}}{(\theta^{\frac{1}{2}}-\theta)^3}\right\} \frac{T(r)}{r^2},$$

$$|D^q \log F(z)| \leqslant \frac{1}{r^q}\left\{\frac{2^q A_q}{\theta^q h^q}\left(\frac{4T(r)}{\log(1/\theta)}\right)^{\frac{1}{2}q} + \frac{8(q-1)!}{(\theta^{\frac{1}{2}}-\theta)^q \log(1/\theta)}\, T(r) + \frac{4q!\,\theta^{\frac{1}{2}}\, T(r)}{(\theta^{\frac{1}{2}}-\theta)^{q+1}}\right\} \quad (q > 2),$$

each inequality holding in $|z| < \theta r$ except for a set of at most $\dfrac{4}{\log(1/\theta)}\, T(r)$ circles the sum of whose areas is at most $h^2\pi^2 r^2$.

THEOREM 2a. *Given positive numbers h and θ (< 1), then*

$$\left|\frac{F'(z)}{F(z)}\right| \leqslant \left\{\frac{8}{h\theta \log(1/\theta)}\left(1+\overset{+}{\log}\frac{4T(r)}{\log(1/\theta)}\right) + \frac{8}{(\theta^{\frac{1}{2}}-\theta)\log(1/\theta)} + \frac{4\theta^{\frac{1}{2}}}{(\theta^{\frac{1}{2}}-\theta)^2}\right\} \frac{T(r)}{r},$$

$$|D^q \log F(z)| \leqslant \left\{B_q\left(\frac{8}{h\theta \log(1/\theta)}\right)^q \left(\frac{T(r)}{r}\right)^q + \frac{8(q-1)!}{(\theta^{\frac{1}{2}}-\theta)^q \log(1/\theta)}\, \frac{T(r)}{r^q} + \frac{4q!\,\theta^{\frac{1}{2}}}{(\theta^{\frac{1}{2}}-\theta)^{q+1}}\, \frac{T(r)}{r^q}\right\} \quad (q > 1),$$

each inequality holding in $|z|<\theta r$ *except for a set of at most* $\dfrac{4}{\log(1/\theta)}T(r)$ *circles the sum of whose radii is at most* hr.

THEOREM 3a. *Given positive numbers* h *and* θ (<1), *then*

$$\left|\frac{F'(z)}{F(z)}\right|\leqslant\left\{\frac{8e}{h\theta\log(1/\theta)}+\frac{8}{(\theta^{\frac{1}{2}}-\theta)\log(1/\theta)}+\frac{4\theta^{\frac{1}{2}}}{(\theta^{\frac{1}{2}}-\theta)^2}\right\}\frac{T(r)}{r}$$

along each radius $\arg z=\text{const.}$ *in* $|z|<\theta r$ *except for at most* $\dfrac{4}{\log(1/\theta)}T(r)$ *intervals of total length not exceeding* hr.

4. *Circles, annuli, and intervals throughout which the inequalities hold.*

We are interested in circles $|z-z_0|<C(r)$ throughout which the inequalities of Theorem 1 can be asserted. If, concentric with each circle $|z-x_n|=h_n$ enclosing values of z for which they are invalid, we draw a concentric circle of radius $h_n+h'r/\sqrt{T(kr)}$, the total area of such circles does not exceed (with $2H=hr$)

$$\pi\Sigma\{h_n+h'r/\sqrt{T(kr)}\}^2\leqslant 2\pi\Sigma h_n^2+2\pi K(k)h'^2r^2 \tag{33}$$
$$\leqslant 2\pi\{h^2+K(k)h'^2\}r^2.$$

If z' is taken outside these circles and z is inside $|z-z'|<h'r/\sqrt{T(kr)}$ then z is inside the region to which Theorem 1 applies; and integrating the first of its inequalities we have

THEOREM 4. *Given* $h>0$, $h'>0$ *and* $k>1$, *then for* $|z'|<r$, *(*r *sufficiently large), and outside a set of at most* $KT(kr)$ *circles, the sum of whose areas does not exceed* $2\pi\{h^2+K(k)h'^2\}r^2$, *the inequality*

$$|\log F(z)-\log F(z')|<K\frac{T(kr)}{r}|z-z'|<K\sqrt{T(kr)} \tag{34}$$

is satisfied by the meromorphic function $F(z)$ *throughout the circle*

$$|z-z'|<h'r/\sqrt{T(kr)} \tag{35}$$

provided that $|z|<r$.

This result is considerably sharper than the similar one obtained from Schottky's theorem*.

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* 9, 21.

The corresponding device applied to Theorem 2 gives whole annuli. Surround each circle $|z-x_n| = h_n$ of exception in Theorem 2 with a concentric one of radius $h_n+h'r/T(kr)$. The sum of the radii of all such circles does not exceed $[h+K(k)h']r$, and with $||z|-\xi| < h'r/T(kr)$, z is inside the region of validity of Theorem 2 provided that $\xi < r$ is outside a set of intervals of total length not exceeding $2[h+K(k)h']r$. We thus have

THEOREM 5. *Given $h > 0$, $h' > 0$ and $k > 1$, then, if ξ is outside a set of at most $KT(kr)$ intervals of total length at most $2[h+K(k)h']r$, and r is sufficiently large, the meromorphic function $F(z)$ satisfies the inequality*

$$|\log F(z)-\log F(z')| < K\frac{T(kr)\log T(kr)}{r}|z-z'| \tag{36}$$

throughout the whole annulus

$$\xi-\frac{h'r}{T(kr)} < |z| < \xi+\frac{h'r}{T(kr)}, \quad \xi-\frac{h'r}{T(kr)} < |z'| < \xi+\frac{h'r}{T(kr)}. \tag{37}$$

The inequality (36) could probably be improved by the omission of the factor $\log T(kr)$, but we have a proof only in the restricted circumstances in which Theorem 3 applies. From it we evidently have

THEOREM 6. *Given $h > 0$, $h' > 0$ and $k > 1$, then, provided that x' is outside a set of at most $KT(kr)$ intervals of total length not exceeding $\{h+K(k)h'\}r$, and r is sufficiently large, the meromorphic function $F(z)$ satisfies*

$$|\log F(x)-\log F(x')| < K\frac{T(kr)}{r}|x-x'| \tag{38}$$

throughout the interval

$$x'-\frac{h'r}{T(kr)} < x < x'+\frac{h'r}{T(kr)}.$$

As remarked at the end of the last section, the restriction r sufficiently large can be dropped if we add to the hypotheses the condition $F(0) = 1$.

5. *Regions in which a meromorphic function of finite order and defective poles is large.*

We can expect a meromorphic function to be large in an appreciable fraction of the plane only if its poles are defective*. Also, even an

* *Cf.* (23) and (24). The main result of this section, expressed in the lemma, is to be found in **9**, where, unfortunately, some errors occur following (11). A different argument leading to a similar conclusion is used in **21**.

integral function can be small almost everywhere if its order is infinite. Accordingly we suppose that $F(z)$ is of finite order and that its poles are of permanent defect so that

$$\varliminf_{r\to\infty} \frac{m(r, F)}{T(r)} > 0 \tag{39}$$

and obtain the following result:

LEMMA. *If $F(z)$ is a meromorphic function of finite order whose poles are of permanent defect, and if b is any positive number less than the defect then there are arbitrarily large values of R such that*

$$\log|F(z)| > bT(Re^{-1}) \tag{40}$$

for a set of z in the annulus $Re^{-1} < |z| < R$ of area at least $c\pi R^2$, where c is positive and depends only on the defect, b and the order. Further, for the same values of R

$$T(re^{-1}) > BT(eR), \tag{41}$$

where B is positive and depends only on the order.

From (39) there exists a positive number a such that

$$m(r, F) > a\,T(r) \geqq a\,T(Re^{-1}) \tag{42}$$

for all r in $Re^{-1} < r < R$, and hence

$$\int_{Re^{-1}}^{R} \int_0^{2\pi} {}^{+} \log|F(te^{i\theta})|\,t\,dt\,d\theta > a\pi R^2(1-e^{-2})\,T(Re^{-1}). \tag{43}$$

The contribution to the integral (43) from those parts of the annulus for which $\log|F(z)| < bT(Re^{-1})$ is evidently at most

$$b\pi R^2(1-e^{-2})\,T(Re^{-1}); \tag{44}$$

and so, if $b < a$, a substantial contribution, at least

$$(a-b)\,\pi R^2(1-e^{-2})\,T(Re^{-1}), \tag{45}$$

must accrue from those parts of the annulus for which

$$\log|F(z)| \geqq bT(Re^{-1}). \tag{46}$$

The next step is to show that the contribution to (43) from any area cannot be as large as (45) unless the area occupies an appreciable fraction

of the annulus. From (19) we clearly have, with $z = te^{i\theta}$ $(t < R)$,

$$(47) \qquad \overset{+}{\log}\,|F(te^{i})| \leqslant \frac{\sqrt{e}+1}{\sqrt{e}-1}\, m(R\sqrt{e}) + \sum_{|b_\nu|\leqslant R\sqrt{e}} \log\left|\frac{eR^2-\bar{b}_\nu z}{R(z-b_\nu)\sqrt{e}}\right|.$$

Now, if A is a set of points of area $A(R)$ contained in the annulus $Re^{-1} < |z| < R$, we have the inequality

$$(48) \qquad S = \iint_A \log\left|\frac{eR^2-\bar{b}_\nu z}{R(z-b_\nu)\sqrt{e}}\right| t\,dt\,d\theta$$

$$\leqslant A(R)\log(2R\sqrt{e}) + \iint_A \log\left|\frac{1}{z-b_\nu}\right| t\,dt\,d\theta$$

for the double integral S. The integral on the right attains its maximum when A is a circle of centre b_ν, and hence

$$(49) \quad S \leqslant A(R)\log(2R\sqrt{e}) + A(R)\log\frac{e\sqrt{\pi}}{\sqrt{A(r)}} = A(R)\log\frac{2e^{\frac{3}{2}}R\sqrt{\pi}}{\sqrt{A(r)}}.$$

Thus, from (47), we have

$$(50) \quad \iint_A \overset{+}{\log}|F(te^{i\theta})|\,t\,dt\,d\theta$$

$$\leqslant A(R)\left\{\frac{\sqrt{e}+1}{\sqrt{e}-1}\, m(R\sqrt{e}) + n(R\sqrt{e})\log\frac{2e^{\frac{3}{2}}R\sqrt{\pi}}{\sqrt{A(r)}}\right\},$$

and, on using the inequalities

$$(51) \quad m(R\sqrt{e}) \leqslant T(R\sqrt{e}) \leqslant T(Re), \qquad n(R\sqrt{e}) \leqslant 2N(Re) \leqslant 2T(Re),$$

we get

$$(52) \quad \iint_A \overset{+}{\log}|F(te^{i\theta})|\,t\,dt\,d\theta \leqslant A(R)\left\{\frac{\sqrt{e}+1}{\sqrt{e}-1} + 2\log\frac{2e^{\frac{3}{2}}R\sqrt{\pi}}{\sqrt{A(r)}}\right\} T(Re)$$

$$= A(R)\left\{\frac{\sqrt{e}+1}{\sqrt{e}-1} + 3 + \log 4 + \log\frac{\pi R^2}{A(R)}\right\} T(Re).$$

Let A be the area of the annulus $Re^{-1} < |z| < R$ in which (46) is satisfied. Then, comparing (52) and (45), we have

$$(53) \quad A(R)\left\{\frac{\sqrt{e}+1}{\sqrt{e}-1} + 3 + \log 4 + \log\frac{\pi R^2}{A(R)}\right\} T(Re)$$

$$\geqslant (a-b)\,\pi R^2(1-e^{-2})\,T(Re^{-1}).$$

Now, if $A(R)$ is equal to $C\pi R^2$, with C greater than zero, we have

$$C\left\{\frac{\sqrt{e}+1}{\sqrt{e}-1}+3+\log 4+\log\frac{1}{C}\right\} \geqslant (a-b)(1-e^{-2})\,T(Re^{-1})/T(Re), \tag{54}$$

and, if $F(z)$ is of finite order ρ, a positive constant B exists such that

$$\overline{\lim_{R\to\infty}}\,\frac{T(Re^{-1})}{T(Re)} > B. \tag{55}$$

Hence, provided that the annuli are chosen from those values of R for which

$$T(Re^{-1}) > BT(Re), \tag{56}$$

there is a positive lower bound c to the set of numbers C which satisfy (54), and so $A(R) \geqslant c\pi R^2$.

This completes the proof of the lemma.

If in Theorem 4 we take $k = e$ and h and h' sufficiently small the areas in which this theorem and the lemma are applicable overlap and we can assert

THEOREM 7. *If $F(z)$ is a meromorphic function of finite order whose poles are of permanent defect, then there exist arbitrarily large values z_1 such that*

$$|\log F(z)-\log F(z_1)| < A\,\frac{T(r)}{r}\,|z-z_1| \leqslant AC\sqrt{T(r)}, \tag{57}$$

$$\log|F(z_1)| > BT(r), \tag{58}$$

where $|z_1| = r$ and z is any point in the circle $|z-z_1| < Cr/\sqrt{T(r)}$, and A, B and C are positive constants depending only on the order of $F(z)$ and the defect of the poles.

6. *Nevanlinna's formula and characteristic for functions meromorphic in an angle.*

We suppose $F(z)$ to be meromorphic in an angle $0 \leqslant \arg z \leqslant \pi/k$. The consideration of $F(z^k)$ in place of $F(z)$ reduces the angle concerned to the half-plane $I(z) \geqslant 0$ and we assume throughout that $F(z)$ is meromorphic in this half-plane.

Nevanlinna* has given the formula

$$(59)\quad \log|F(z)|$$
$$=\frac{1}{\pi}\int_{-R}^{R}\log|F(t)|\,r\sin\theta\left\{\frac{1}{r^2+t^2-2rt\cos\theta}-\frac{1}{R^2+(r/R)^2t^2-2rt\cos\theta}\right\}dt$$
$$+\frac{1}{2\pi}\int_0^{\pi}\log|F(Re^{i\phi})|\,\frac{4Rr(R^2-r^2)\sin\theta\sin\phi\,d\phi}{(R^2+r^2-2Rr\cos\overline{\phi-\theta})(R^2+r^2-2Rr\cos\overline{\phi+\theta})}$$
$$-\sum_{|a_\mu|<R}\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right|$$
$$+\sum_{|b_\nu|<R}\log\left|\frac{z-\bar{b}_\nu}{z-b_\nu}\,\frac{R^2-\bar{b}_\nu z}{R^2-b_\nu z}\right|.$$

In this formula $z=re^{i\theta}$, and the a_μ and b_ν are the zeros and poles of $F(z)$ in the half-plane. It is not difficult to express the right-hand side of (59) as the real part of an analytic function, a device familiar in the case of the corresponding formula for a circle†. The result is

$$(60)\quad \log F(z)=\frac{1}{\pi i}\int_{-R}^{R}\log|F(t)|\left\{\frac{R^2}{t(tz-R^2)}-\frac{1}{z-t}\right\}dt$$
$$+\frac{1}{2\pi}\int_0^{\pi}\log|F(Re^{i\phi})|\left\{\frac{Re^{i\phi}+z}{Re^{i\phi}-z}-\frac{Re^{-i\phi}+z}{Re^{-i\phi}-z}\right\}d\phi$$
$$-\sum_{|a_\mu|<R}\log\left(\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right)+\sum_{|b_\nu|<R}\log\left(\frac{z-\bar{b}_\nu}{z-b_\nu}\,\frac{R^2-\bar{b}_\nu z}{R^2-b_\nu z}\right)$$

the verification of which is elementary.

Formula (60) may be differentiated with respect to z, giving

$$(61)\quad D^q\log F(z)$$
$$=\frac{1}{\pi i}\int_{-R}^{R}\log|F(t)|\left\{\frac{(-)^q q!\,R^2t^{q-1}}{(tz-R^2)^{q+1}}-\frac{(-)^q q!}{(z-t)^{q+1}}\right\}dt$$
$$+\frac{1}{\pi}\int_0^{\pi}\log|F(Re^{i\phi})|\left\{\frac{q!\,Re^{i\phi}}{(Re^{i\phi}-z)^{q+1}}-\frac{q!\,Re^{-i\phi}}{(Re^{-i\phi}-z)^{q+1}}\right\}d\phi$$
$$+\sum_{|a_\mu|<R}(q-1)!\left\{\frac{(-)^{q-1}}{(z-a_\mu)^q}-\frac{(-)^{q-1}}{(z-\bar{a}_\mu)^q}-\frac{a_\mu{}^q}{(R^2-a_\mu z)^q}+\frac{\bar{a}_\mu{}^q}{(R^2-\bar{a}_\mu z)^q}\right\}$$
$$-\sum_{|b_\nu|<R}(q-1)!\left\{\frac{(-)^{q-1}}{(z-b_\nu)^q}-\frac{(-)^{q-1}}{(z-\bar{b}_\nu)^q}-\frac{b_\nu{}^q}{(R^2-b_\nu z)^q}+\frac{\bar{b}_\nu{}^q}{(R_2-\bar{b}_\nu z)^q}\right\},$$

which corresponds to (21). For the moment we are concerned with the

* **3, 14,** 29. The formula is effectively given in **13.**

† **15,** 4; **16,** 155.

case $q=1$ and we set it out specially as

$$(62)\quad \frac{F'(z)}{F(z)}=\frac{1}{\pi i}\int_{-R}^{R}\log|F(t)|\left\{\frac{1}{(z-t)^2}-\frac{R^2}{(tz-R)^2}\right\}dt$$
$$+\frac{1}{\pi}\int_0^{\pi}\log|F(Re^{i\phi})|\left\{\frac{Re^{i\phi}}{(Re^{i\phi}-z)^2}-\frac{Re^{-i\phi}}{(Re^{-i\phi}-z)^2}\right\}d\phi$$
$$+\sum_{|a_\mu|<R}\left\{\frac{1}{z-a_\mu}-\frac{1}{z-\bar{a}_\mu}-\frac{a_\mu}{R^2-a_\mu z}+\frac{\bar{a}_\mu}{R^2-\bar{a}_\mu z}\right\}$$
$$-\sum_{|b_\nu|<R}\left\{\frac{1}{z-b_\nu}-\frac{1}{z-\bar{b}_\nu}-\frac{b}{R^2-b_\nu z}+\frac{\bar{b}_\nu}{R^2-\bar{b}_\nu z}\right\}.$$

If we wish to put $z=0$ in this formula we clearly need $|F(0)|=1$ with $F'(0)/F(0)$ not real, so that the first of the integrals on the right may converge at $t=0$. It is, however, evident that by taking out a factor of the form $z^k(a+bz)$ we can ensure that $F(0)=1$ and that $F'(0)=0$, and we suppose that $F(z)$ has been treated so as to satisfy these conditions. The effect of the missing factor is entirely trivial. We usually state theorems in a form not involving this condition and make alterations in hypothesis and conclusion to allow for this. We do not always make comments on this process. With $z=0$, and after multiplication by i, (62) may be written

$$(63)\quad \frac{1}{\pi}\int_{-R}^{R}\overset{+}{\log}|F(t)|\left(\frac{1}{t^2}-\frac{1}{R^2}\right)dt+\frac{2}{\pi R}\int_0^{\pi}\overset{+}{\log}|F(Re^{i\phi})|\sin\phi\,d\phi$$
$$+2\sum_{|b_\nu|<R}\left\{1-\frac{|b_\nu|^2}{R^2}\right\}\frac{\sin\beta_\nu}{|b_\nu|}$$
$$=\frac{1}{\pi}\int_{-R}^{R}\overset{+}{\log}|1/F(t)|\left(\frac{1}{t^2}-\frac{1}{R^2}\right)dt+\frac{2}{\pi R}\int_0^{\pi}\overset{+}{\log}|1/F(Re^{i\phi})|\sin\phi\,d\phi$$
$$+2\sum_{|a_\mu|<R}\left\{1-\frac{|a_\mu|^2}{R^2}\right\}\frac{\sin\alpha_\mu}{|a_\mu|},$$

where $$a_\mu=|a_\mu|e^{i\alpha_\mu},\qquad b_\nu=|b_\nu|e^{i\beta_\nu}.$$

This is a modification of a formula given by Carleman and Nevanlinna*. By using the device of multiplying the original function by an appropriate factor we have avoided the necessity of omitting a portion of the integral

$$\int_{-R}^{R}\log|F(t)|\left(\frac{1}{t^2}-\frac{1}{R^2}\right)dt$$

near the origin.

As proved by Nevanlinna†, each side of (63) is an increasing function

* **3, 14, 5.** It is ascribed to F. Nevanlinna, **14, 6,** footnote.

† **14, 7.**

of R and we denote their common value by $S(R) = S(R, F)$. Following Nevanlinna we write

$$
(64)\quad \begin{cases}
A(R, F) = \dfrac{1}{\pi}\displaystyle\int_{-R}^{R} {}^{+}\log|F(t)|\left(\frac{1}{t^2}-\frac{1}{R^2}\right)dt,\\[2ex]
B(R, F) = \dfrac{2}{\pi R}\displaystyle\int_{0}^{\pi} {}^{+}\log|F(Re^{i\phi})|\sin\phi\, d\phi,\\[2ex]
c(R, F) = c(R, \infty) = \displaystyle\sum_{|b_\nu|<R}\sin\beta_\nu,\\[2ex]
c(R, 1/F) = c(R, 0) = \displaystyle\sum_{|a_\mu|<R}\sin\alpha_\mu,\\[2ex]
C(R, F) = C(R, \infty) = 2\displaystyle\sum_{|b_\nu|<R}\left\{1-\frac{|b_\nu|^2}{R^2}\right\}\frac{\sin\beta_\nu}{|b_\nu|}\\[2ex]
\qquad\qquad = 2\displaystyle\int_0^R\left(1-\frac{t^2}{R^2}\right)\frac{dc(t, \infty)}{t} = 2\int_0^R\left(\frac{1}{t^2}+\frac{1}{R^2}\right)c(t, \infty)\,dt,\\[2ex]
S(R) = A(R, F)+B(R, F)+C(R, F)\\[2ex]
\qquad\qquad = A(R, 1/F)+B(R, 1/F)+C(R, 1/F).
\end{cases}
$$

7. *Generalities on the Nevanlinna formulae*.*

The formulae (60) and (61) do not differ greatly in their general nature from the corresponding formulae (20) and (21) for a circle. Instead of restricting z to lie in a smaller concentric circle we must evidently restrict it to a region of the kind

$$
(65)\qquad D(\alpha, \delta),\quad \delta R \leqslant |z| \leqslant (1-\delta)R,\quad \alpha \leqslant \arg z \leqslant \pi-\alpha,\ (\delta > 0,\ \alpha > 0).
$$

It must be remembered, however, that the formulae for a circle involved the value of $F(0)$ which, being a constant, was of little subsequent interest. For the angle formulae the corresponding element may be taken as the value of $F(z)$ at some point near the centre of the region $D(\alpha, \delta)$. Before we can exploit this analogy, it is therefore necessary to make some hypothesis about the asymptotic behaviour of $F(z)$ inside the angle. It will also be observed that a special hypothesis needs to be made about the poles of $F(z)$ and we restrict ourselves to the case of a regular function with none. Our results must be expressed not in terms of Nevanlinna's characteristic $T(kr)$, or perhaps we should say $\log M(kr)$, but in terms of an increasing function $V(r)$ with the properties

$$
(66)\qquad \log\left|\frac{F(z)}{F(z_1)}\right| \leqslant V(R)
$$

* *Cf.* **14, 5, 6, 7, 8.**

for all z on the boundary of the semicircle $I(z) > 0$, $|z| < R$, where z_1 is a suitably chosen point in $D(\alpha, \delta)$. Evidently unless the behaviour of $F(z)$ is exceptionally regular we shall have a better inequality (66) for some values of R than others. We are interested in the strongest possible inequalities of this form and can expect them to hold only for a special sequence of values of R. The analogy with § 3 is fairly straightforward, but instead of our final results being valid for a set of z of large "density" they will only be available for a set of large "upper density".

There is, however, another possibility. Suppose that we attempt to use (60) for a fixed value of z. As R tends to infinity this is practically equivalent to differentiating with respect to z and putting $z = 0$. In other words, we are led to (63) and Nevanlinna's characteristic $S(R)$. The relation between (59) and (63) is, however, not so simple as that between (19) and (23). The difference relates principally to the difference between the kernels of the integrals $\int_{-R}^{R}$ in (59) and (63). When z is in $D(\alpha, \delta)$ this term in (59) is of the nature of

$$\mathcal{A}(R) = \int_{-R}^{R} \log^{+} |F(t)| \left(1 - \frac{t^2}{R^2}\right) dt. \tag{67}$$

The relation between $\mathcal{A}(R)$ and $A(R)$ can be judged from

$$\begin{cases} \mathcal{A}(R) = \int_0^R t^2 \, dA(t) = R^2 A(R) - 2\int_0^R t\, A(t)\, dt, \\ A(R) = \int_0^R t^{-2} \, d\mathcal{A}(t) = R^{-2}\, \mathcal{A}(R) + 2 \int_0^R t^{-3}\, \mathcal{A}(t)\, dt. \end{cases} \tag{68}$$

It will be seen that if $A(R)$ tends to infinity then $\mathcal{A}(R) = O[R^2 A(R)]$ and that this is a fairly complete result. If, however, $A(R)$ is bounded we simply get $\mathcal{A}(R) = o(R^2)$ by such simple methods. There is a similar hiatus in the process of finding an order of magnitude of $A(R)$ from an assumption about the order of magnitude of $\log|F(z)|$. If this is like $|z|^{\rho}$ with $\rho > 1$ the relation is quite simple, but if $\rho < 1$ then $A(R)$ is bounded and we get nothing more however small ρ may be. Further, we shall need the value of $A(R, 1/F)$ from (63) and here we have only $A(R, 1/F) < S(R)$. It thus appears that, if $S(R)$ tends to infinity (or if $F(z)$ is regular and of order greater than unity), we get a second set of results quite similar to those of § 3 but with $RS(kR)$ replacing $T(kR)$. Apart from these two kinds of result the possibilities appear to be very complex. We may, for example, make special assumptions about $F(z)$ on the arms of the angle, or about the sequence of R for which (66) holds,

or about each component of $S(R)$. We give a selection of the resulting theorems.

There is evidently no complete analogy with the developments of § 5, for even an integral function may tend to zero inside a particular angle. Some information about this matter has been obtained in the case of regular functions by Bernstein and Miss Cartwright*.

8. *The upper density results. First method*†.

Let us suppose that $F(z)$ is regular in $I(z) > 0$ and that for some arbitrarily large values of r the inequality

$$\log\left|\frac{F(z)}{F(ir)}\right| < HV(r) \tag{69}$$

is satisfied inside the circle $|z-ir| = r$. We can clearly apply Theorems 1a, 2a, and 3a to $G(z) = \dfrac{F(z+ir)}{F(ir)}$ inside the circle $|z| = r$, remembering that $T(r, G) < HV(r)$ from (23). We evidently have

THEOREM 8. *The results of Theorems* 1a, 2a *and* 3a *apply to a function* $F(z+ir)$, *if* $F(z)$ *is regular in* $I(z) > 0$ *and satisfies* (69), *except that we write* $HV(r)$ *in place of* $T(Vr)$ *and restrict* z *to the circle* $|z| \leqslant k^{-1}r$.

The region to which these results apply is evidently not so extensive as $D(\alpha, \delta)$. We could use a more elaborate theory based on the fact that (59) is simply (19) together with the conformal transformation from a circle to a semicircle, but it seems on the whole preferable to consider the explicit form of (59); this we do in the next paragraph.

The condition (69) is satisfied in particular if $V(r)$ is a "proximate order" and if $\log|F(z)| < HV(|z|)$, while

$$\overline{\lim_{r\to\infty}}\, V(r)^{-1} \log|F(ir)| > -\infty.$$

9. *The upper density results. Second method.*

We write the formula (59), $F(z)$ being assumed regular, in the form

$$\log|F(z)| = \int_\Gamma \log|F(\zeta)|\,\omega(z, \zeta)\lambda(\zeta)|d\zeta| - \sum_{|a_\mu|<R} \log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right| \quad (z = re^{i\theta}), \tag{70}$$

* 1, 5, 6, 7, 8.

† 6, a generalization of Theorem 1, 164–166.

where Γ is the boundary of the semicircle $|\zeta| < R$, $I(\zeta) > 0$, and $\omega(z, \zeta)$ and $\lambda(\zeta)$ can be written out explicitly as

$$(71)\quad \omega(z,\zeta)=\frac{r\sin\theta(R^2-r^2)}{(r^2+t^2-2rt\cos\theta)(R^2+[r/R]^2t^2-2rt\cos\theta)},\quad \lambda(\zeta)=\frac{1}{\pi}\left(1-\frac{t^2}{R^2}\right),$$

if $\zeta = t$ in $-R < t < R$, the corresponding expressions for $\zeta = Re^{i\phi}$ being

$$(72)\quad \begin{cases} \omega(z,\zeta)=\dfrac{r\sin\theta(R^2-r^2)}{\{r^2+R^2-2rR\cos(\phi-\theta)\}\{r^2+R^2-2rR\cos(\phi+\theta)\}}, \\ \lambda(\zeta)=\dfrac{2\sin\phi}{\pi}. \end{cases}$$

It is evident that so long as z is in $D(a, \delta)$, $\omega(z, \zeta)$ satisfies the inequalities

$$(73)\qquad \omega R^{-1} \leqslant \omega(z,\zeta) \leqslant \Omega R^{-1} \quad [\omega = \omega(a,\delta) > 0,\ \Omega = \Omega(a,\delta) > 0].$$

This result is as obvious from potential theory considerations as from the explicit formulae, and from it and the equality (70) we derive the two inequalities

$$(74)\quad \Omega R^{-1}\int_\Gamma \overset{+}{\log}|F(\zeta)|\lambda(\zeta)|d\zeta| - \omega R^{-1}\int_\Gamma \overset{+}{\log}|1/F(\zeta)|\lambda(\zeta)|d\zeta|$$

$$\geqslant \log|F(z)| + \sum_{|a_\mu|<R}\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right|$$

$$\geqslant \omega R^{-1}\int_\Gamma \overset{+}{\log}|F(\zeta)|\lambda(\zeta)|d\zeta| - \Omega R^{-1}\int_\Gamma \overset{+}{\log}|1/F(\zeta)|\lambda(\zeta)|d\zeta|.$$

We now make an assumption a little more general than (69), namely the existence of a point z_1 inside $D(a, \delta)$ such that

$$(75)\qquad \log|F(z_1)| > H_1 V(R)$$

while

$$(76)\qquad \log|F(\zeta)| < HV(R)$$

for all ζ on Γ. On these assumptions the first of the inequalities (74) with $z = z_1$ gives

$$(77)\qquad \omega\int_\Gamma \overset{+}{\log}|1/F(z)|\lambda(\zeta)|d\zeta| \leqslant (KH-H_1)RV(R) \quad [K = K(a,\delta)].$$

Using this result and the second of the inequalities (74), we should obtain a lower bound for $\log|F(z)|$ if we had sufficient information about the zeros of $F(z)$. To obtain this from our present assumptions we use methods similar to those of § 3 for a circle. There we had to introduce

a third concentric circle which is now replaced by the region $D(\frac{1}{2}a, \frac{1}{2}\delta)$, larger than $D(a, \delta)$. If z is in $D(a, \delta)$ and a_μ is outside $D(\frac{1}{2}a, \frac{1}{2}\delta)$, then

$$g(z, a_\mu) = \log \left| \frac{z-\bar{a}_\mu}{z-a_\mu} \frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z} \right| \tag{78}$$

has positive maxima and minima $G(a_\mu)$ and $g(a_\mu)$ depending on a_μ, a and δ. Also $G(a_\mu)/g(a_\mu)$ is less than a fixed positive bound, as a_μ varies outside $D(\frac{1}{2}a, \frac{1}{2}\delta)$, dependent only on a and δ. These remarks are obvious from potential-theory considerations, but a direct proof from the explicit formulae would be tedious.

It follows that, if

$$\Lambda(z) = \Sigma' \log \left| \frac{z-\bar{a}_\mu}{z-a_\mu} \frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z} \right| \tag{79}$$

is that part of the sum from the a_μ outside $D(\frac{1}{2}a, \frac{1}{2}\delta)$, then

$$\max \Lambda(z) \leqslant M \min \Lambda(z) \quad [M = M(a, \delta)], \tag{80}$$

for z restricted to lie in $D(a, \delta)$. From (74), (75) and (76) follows

$$\Lambda(z_1) \leqslant (KH - H_1) V(R) \tag{81}$$

and hence, from (80),

$$\Lambda(z) \leqslant K'(KH - H) V(R) \quad [K' = K'(a, \delta)], \tag{82}$$

for all z in $D(a, \delta)$.

The remaining sum relating to the zeros a_μ in $D(\frac{1}{2}a, \frac{1}{2}\delta)$ which we denote by Σ'' has evidently no upper bound. On the other hand we can limit the number N of such a_μ, for the lower bound of each term is positive, say

$$\log \left| \frac{z-\bar{a}_\mu}{z-a_\mu} \frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z} \right| \geqslant p \tag{83}$$

with z in $D(a, \delta)$ and a_μ in $D(\frac{1}{2}a, \frac{1}{2}\delta)$. Thus again from (74), (75) and (76)

$$N \leqslant K''(KH - H_1) V(R). \tag{84}$$

We have now assembled the necessary material for arguments like those of § 3. We prove first, however, a result about $\log |F(z)|$, due to Miss Cartwright* and generalizing (1).

---

* 6, a generalization of Theorem 1, 164-166.

From the second of the inequalities (74) combined with (76), and (82) we have, for all z in $D(a, \delta)$,

$$\log|F(z)| \geqslant -K'(KH-H_1)V(R)-\Sigma'' \log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right|. \tag{85}$$

Now when z is in $D(a, \delta)$ and a_μ is in $D(\frac{1}{2}a, \frac{1}{2}\delta)$ it is clear that

$$-\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right| \geqslant \log\left|\frac{z-a_\mu}{AR}\right| \quad [A = A(a, \delta)], \tag{86}$$

and there follows at once from Cartan's lemma (15) and (84) the inequality

$$\log|F(z)| \geqslant -K'(KH-H_1)V(R) \tag{87}$$

$$[K = K(a, \beta, h), \quad K' = K'(a, \beta, h)],$$

for all z in $D(a, \delta)$ except for a set of at most $KV(R)$ circles, the sum of whose radii is at most hR.

To obtain results about the logarithmic derivatives, we may argue as follows. From the various inequalities obtained we can see that, for z in $D(a, \delta)$,

$$-K'(KH-H_1)V(R) \leqslant \log|F(z)|-\Sigma'' \log\left|\frac{z-a_\mu}{AR}\right| \tag{88}$$

$$\leqslant K'(KH-H_1)V(R),$$

or, if we put

$$\psi(z) = \log F(z)-\Sigma'' \log\left(\frac{z-a_\mu}{AR}\right), \tag{89}$$

we have

$$|R\{\psi(z)\}| \leqslant K'(KH-H_1)V(R). \tag{90}$$

Our inequalities have been asserted for z in $D(a, \delta)$, but at a cost of increasing the value of K', the last may be obtained for z in $D(\frac{3}{4}a, \frac{3}{4}\delta)$. Hence, from well-known modifications of Schwarz's lemma, we have

$$|D^q\psi(z)| \leqslant K'(KH-H_1)\frac{V(R)}{R^q} \tag{91}$$

for z in $D(a, \delta)$, so that

$$|D^q \log F(z)| \leqslant K'(KH-H_1)\frac{V(R)}{R^q}+\left|\Sigma\frac{(q-1)!}{(z-a_\mu)^q}\right|, \tag{92}$$

and applying the lemmas of § 2 we have

THEOREM 9. *If $F(z)$ is regular in the half-plane $I(z) \geqslant 0$ and satisfies* (75) *and* (76) *for a sequence of R tending to infinity, then, for this sequence of R, $F(z)$ satisfies the inequalities, for R sufficiently large,*

$$(93) \qquad \begin{cases} \left|\dfrac{F'(z)}{F(z)}\right| < (KH-K'H_1)\dfrac{V(R)}{R}, \\ |D^2 \log F(z)| < (KH-K'H_1)\dfrac{V(R)\log V(R)}{R^2}, \\ |D^q \log F(z)| < (KH-K'H_1)\dfrac{V(R)^{\frac{1}{2}q}}{R^q} \quad (q>2), \end{cases}$$

for all z in $D(\alpha, \delta)$ except in at most $KV(R)$ circles of total area not exceeding hR^2,

$$(94) \qquad \begin{cases} \left|\dfrac{F'(z)}{F(z)}\right| < (KH-K'H_1)\dfrac{V(R)\log V(R)}{R}, \\ |D^q \log F(z)| < (KH-K'H_1)\dfrac{V(R)^q}{R^q} \quad (q>1), \end{cases}$$

for all z in $D(\alpha, \delta)$ except in at most $KV(R)$ circles the sum of whose radii does not exceed hR and

$$(95) \qquad |D^q \log F(iy)| < (KH-K'H_1)\frac{V(R)^q}{R^q} \quad (q \geqslant 1),$$

for all y in $0 < y < R$ except in at most $KV(R)$ intervals the sum of whose lengths does not exceed hR, where the constants K, K' are determined by q and prescribed positive h, α, δ.

10. *The use of Nevanlinna's characteristic. Density results.*

In the last article we used the inequalities (74) in two ways. First we had to obtain limitations on $\log|1/F(z)|$ on the boundary of our semi-circle from assumptions about $\log|F(z)|$ on the boundary and at a point inside. Afterwards we applied them and (60) to the discussion of $F(z)$ inside $D(\alpha, \delta)$. As explained in § 7, we can replace the first stage by the use of Nevanlinna's characteristic $S(R)$. We discussed there the relation between the integrals $A(R)$ and $\mathcal{A}(R)$. There is a similar relation between the two sums

$$(96) \qquad \Sigma \log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right|, \qquad \Sigma\left(1-\frac{|a_\mu|^2}{R^2}\right)\sin\alpha_\mu.$$

Following Nevanlinna, we have*

$$(97)\quad \left|\frac{R^2-a_\mu z}{R^2-\bar{a}_\mu z}\right|-1=\frac{|R^2-a_\mu z|-|R^2-\bar{a}_\mu z|}{|R^2-\bar{a}_\mu z|}\leqslant\frac{2r|a_\mu|\sin\alpha_\mu}{R(R-r)}\leqslant\frac{2r}{R-r}\sin\alpha_\mu,$$

and hence

$$(98)\qquad 0\leqslant\log\left|\frac{R^2-a_\mu z}{R^2-\bar{a}_\mu z}\right|\leqslant\log\left(1+\frac{2r}{R-r}\sin\alpha_\mu\right)\leqslant\frac{2r}{R-r}\sin\alpha_\mu.$$

Similarly

$$(99)\quad \left|\frac{z-\bar{a}_\mu}{z-a_\mu}\right|-1=\frac{|z-\bar{a}_\mu|^2-|z-a_\mu|^2}{|z-a_\mu|\{|z-a_\mu|+|z-\bar{a}_\mu|\}}=\frac{4r|a_\mu|\sin\theta\sin\alpha_\mu}{|z-a_\mu|\{|z-a_\mu|+|z-\bar{a}_\mu|\}},$$

and hence

$$(100)\qquad 0\leqslant\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\right|\leqslant K\sin\alpha_\mu$$

if z is inside $D(a,\delta)$ and a_μ outside $D(\frac{1}{2}a,\frac{1}{2}\delta)$. Hence with Σ' defined as before in (78), and z in $D(a,\delta)$, we have

$$(101)\qquad 0\leqslant\Sigma'\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right|\leqslant K\Sigma'\sin\alpha_\mu\leqslant Kc(R,0),$$

while for the remaining sum

$$(102)\quad -Kc(R,0)\leqslant\Sigma''\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{R^2-\bar{a}_\mu z}{R^2-a_\mu z}\right|-\Sigma''\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\right|\leqslant 0.$$

Similar inequalities hold for the corresponding sums formed from the poles.

Thus from (59) we have, with z in $D(a,\delta)$,

$$(103)\quad \log|F(z)|\geqslant\frac{K}{R}\mathcal{A}(R,F)-\frac{K}{R}\mathcal{A}(R,1/F)+KRB(R,F)$$
$$-KRB(R,1/F)-Kc(R,0)+\Sigma''\log\left|\frac{z-a_\mu}{z-\bar{a}_\mu}\right|$$
$$+Kc(R,\infty)-\Sigma''\log\left|\frac{z-b_\nu}{z-\bar{b}_\nu}\right|.$$

* 14, 30.

The number of terms N in Σ'' is evidently at most

$$(104)\quad \frac{c(R,\,0)+c(R,\,\infty)}{\sin\frac{1}{2}\alpha} \leqslant KR[C(kR,\,0)+C(kR,\,\infty)] \leqslant KRS(kR)\ \ (k>1).$$

From (103), (68), (63) and Cartan's lemma (15) with $H = hR$ it follows that

$$(105)\qquad \log|F(z)| \geqslant -KRS(kR,\,F)$$

for all z in $D(a,\,\delta)$ except for the a set of at most $KRS(kR)$ circles the sum of whose radii is at most hR.

This result has been obtained by Miss Cartwright* in the case when $F(z)$ is regular, and we may use it in combination with the last paragraph to show that the results there obtained are valid for all R if $V(R)$ is taken as $RS(kR)$. We can, however, deduce them directly. It is clear that

$$(106)\qquad \psi(z) = \log F(z) - \Sigma'' \log\left(\frac{z-a_\mu}{z-\bar{a}_\mu}\right) + \Sigma'' \log\left(\frac{z-b_\nu}{z-\bar{b}_\nu}\right)$$

is a regular function in $D(a,\,\delta)$ and indeed in $D(\frac{3}{4}a,\,\frac{3}{4}\delta)$, and that its real part is constrained to lie between $-KRS(R)$ and $KRS(R)$. It follows from Schwarz's lemma that in $D(a,\,\delta)$ its derivatives satisfy

$$(107)\qquad |D^q\psi(z)| \leqslant K\frac{RS(kR)}{R^q}.$$

We now apply the results of § 2 to the first two sums in

$$(108)\quad D^q \log F(z) = D^q\psi(z) + \Sigma'' \frac{(-)^{q-1}(q-1)!}{(z-a_\mu)^q} - \Sigma'' \frac{(-)^{q-1}(q-1)!}{(z-b_\nu)^q}$$
$$- \Sigma'' \frac{(-)^{q-1}(q-1)!}{(z-\bar{a}_\mu)^q} + \Sigma'' \frac{(-)^{q-1}(q-1)!}{(z-\bar{b}_\nu)^q},$$

while for the last two we remark that

$$(109)\qquad \begin{cases} |D^q\Sigma'' \log(z-\bar{a}_\mu)| \leqslant \dfrac{KN}{R^q} \leqslant \dfrac{KRS(kR)}{R^q}, \\ |D^q\Sigma'' \log(z-\bar{b}_\nu)| \leqslant \dfrac{KRS(kR)}{R^q}, \end{cases}$$

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* 8, 508, Theorem II.

and so obtain

THEOREM 10. *If (Fz) is meromorphic in $I(z) \geqslant 0$ then for all R sufficiently large the inequalities of Theorem 9 hold with $KRS(kR)$ in place of $(KH-K'H_1)V(R)$.*

11. *Functions regular and of finite order in an angle.*

In this paragraph we examine the forms taken by Theorems 9 and 10 when the original hypotheses about $F(z)$ are that it is regular and of a prescribed order in a half-plane. In order to simplify our results we take the order of magnitude assumption in the form

$$\log|F(re^{i\theta})| \leqslant Hr^{\rho} \quad (r > r_0,\ \rho > 0). \tag{110}$$

From these assumptions we evidently have, remembering the definitions (64),

$$\begin{aligned} A(R, F) &\leqslant \frac{4H}{\pi(\rho^2-1)} R^{\rho-1}+A \quad && (\rho > 1), \\ &\leqslant \frac{2H}{\pi} \log R+A && (\rho = 1,\ R > 1), \\ &\leqslant A && (\rho < 1), \end{aligned} \tag{111}$$

and

$$B(R, F) \leqslant \frac{4H}{\pi} R^{\rho-1} \quad (R > r_0), \tag{112}$$

while, since $F(z)$ is regular, $C(R, \infty) = 0$. Thus $S(R)$ is $O(R^{\rho-1})$, $O(\log R)$, or $O(1)$ as the case may be. We state only three of the resulting eighteen inequalities, those corresponding to Theorem 3, and then only for the direction of the imaginary axis.

THEOREM 11. *If $F(z)$ is regular in the half-plane $I(z) \geqslant 0$ and satisfies* (110), *then, for sufficiently large R,*

$$\begin{aligned} \left|\frac{F'(iy)}{F(iy)}\right| &\leqslant KH^{\rho-1} && (\rho > 1), \\ &\leqslant KH \log R && (\rho = 1), \\ &\leqslant A && (\rho < 1), \end{aligned} \tag{113}$$

for all y in $0<y<R$ except possibly for a set of at most KR^ρ ($\rho>1$), $KR\log R$ ($\rho=1$), or KR ($\rho<1$) intervals of total length not exceeding hR.

From Theorem 9, however, we obtain a sharper inequality if $\rho<1$, although it is not valid for all R. We need as well as (110) an additional assumption such as

$$\overline{\lim_{y\to\infty}}\frac{\log|F(iy)|}{y^\rho}=H'>-\infty \tag{114}$$

to obtain

THEOREM 12. *If $F(z)$ is regular in the half-plane $I(z)>0$, and satisfies* (110) *and* (114), *then there exist arbitrarily large values of R such that*

$$\left|\frac{F'(iy)}{F(iy)}\right|\leqslant K(H-H')R^{\rho-1} \tag{115}$$

for all y in $0<y<R$ except for a set of at most $K(H-H')R^\rho$ intervals of total length at most hR. Further, for these values of R, the number of zeros of $F(z)$ in $D(\alpha,\delta)$ is not more than $K(H-H')R^\rho$ and the remaining inequalities of Theorem 9 are satisfied with $V(R)=(H-H'+\epsilon)R^\rho$.

It is also possible to improve Theorem 11 if we make some supplementary hypothesis. The factor $\log R$ in (113) could be removed if we knew that $S(R)$ was bounded, and this will follow from the additional assumption*

$$\int_{-\infty}^{\infty}\log^+|F(t)|\frac{dt}{1+t^2}<\infty. \tag{116}$$

We can, however, do a little better if we do assume (116); for, if $S(R)$ is bounded, it is clear that $B(R,F)=S(R)-A(R,F)$ has a definite limit and that $B(R,1/F)=S(R)-A(R,1/F)-C(R,0)$ also has a definite limit as R tends to infinity. The formulae (59), (60), (61) and (62) then take limiting forms as R tends to infinity. This fact was exploited by Nevanlinna† in the case of (59) which becomes

$$\log|F(z)|=\frac{1}{\pi}\int_{-\infty}^{\infty}\log|F(t)|\frac{r\sin\theta\,dt}{r^2+t^2-2rt\cos\theta}+\eta r\sin\theta-\Sigma\log\left|\frac{z-\bar{a}_\mu}{z-a_\mu}\right|, \tag{117}$$

* See remarks following (62).

† **14**, 28–45.

in which η is the real constant

$$\eta = \lim_{R\to\infty} \{B(R, F) - B(R, 1/F)\}. \tag{118}$$

From (117) [or (60)] we deduce that

$$\log F(z) = iC + \frac{1}{\pi i}\int_{-\infty}^{\infty} \log|F(t)|\,\frac{dt}{t-z} + \frac{\eta z}{i} - \Sigma \log\left(\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{a_\mu}{\bar{a}_\mu}\right). \tag{119}$$

It is only necessary to discuss the convergence of the sum; that the sum is convergent follows because

$$\frac{z-\bar{a}_\mu}{z-a_\mu}\,\frac{a_\mu}{\bar{a}_\mu} - 1 = \frac{z(\bar{a}_\mu - a_\mu)}{(z-a_\mu)\bar{a}_\mu}$$

and $\Sigma|a_\mu|^{-1}\sin\alpha_\mu$ converges since $C(R, 0)$ is bounded.

We can differentiate (119) with respect to z and obtain

$$\frac{F'(z)}{F(z)} = \frac{1}{\pi i}\int_{-\infty}^{\infty} \log|F(t)|\,\frac{dt}{(t-z)^2} + \frac{\eta}{i} - \Sigma\,\frac{1}{z-\bar{a}_\mu} + \Sigma\,\frac{1}{z-a_\mu}. \tag{120}$$

It is clear that, if z is confined to $D(\alpha, \delta)$ formed with a given R, the integral in (120) is $o(1)$. Similarly the infinite sum

$$\Sigma'\left(\frac{1}{z-a_\mu} - \frac{1}{z-\bar{a}_\mu}\right) \tag{121}$$

for those a_μ outside $D(\frac{1}{2}\alpha, \frac{1}{2}\delta)$ is also $o(1)$. The number of a_μ inside $D(\frac{1}{2}\alpha, \frac{1}{2}\delta)$ is $o(R)$ and hence the associated $\Sigma''\,1/(z-\bar{a}_\mu)$ is $o(1)$.

For the remaining sum $\Sigma''\,1/(z-a_\mu)$ we use the results quoted in § 2, and hence evidently, since similar results follow for the higher derivatives, we have

THEOREM 13. *If $F(z)$ is regular in the half-plane $I(z) > 0$ and if its characteristic $S(R)$ is bounded, a condition ensured by* (110) *with $\rho = 1$ and* (116), *then there exists a real constant η defined by* (118) *such that*

$$\left|\frac{F'(iy)}{F(iy)} + i\eta\right| < \epsilon \tag{122}$$

for all y in $0 < y < R$ (R sufficiently large), except for a set of at most ϵR intervals the sum of whose lengths is at most hR. In addition, the other results of Theorem 10 *on higher derivatives hold when $KS(kR)$ is replaced by any positive number ϵ.*

Theorem 13 is of particular interest if $\eta = 0$ as follows from the assumptions

$$\log|F(z)| < \epsilon r \quad (r > r_0), \tag{123}$$

$$\overline{\lim_{y\to\infty}} \frac{\log|F(iy)|}{y} = 0, \tag{124}$$

and (116). From (123) it follows that the limit of $B(R, F)$ is zero and hence that $\eta \leqslant 0$. The converse inequality $\eta \geqslant 0$ easily follows from (124) and (117), as was observed by Miss Cartwright*. We thus have

THEOREM 14. *If $F(z)$ is regular and of minimum type, order* 1, *in the half-plane $I(z) > 0$ and if*

$$\int_{-\infty}^{\infty} \frac{+}{\log|F(t)|} \frac{dt}{1+t^2}$$

is convergent, then for all sufficiently large R determined by given $\epsilon > 0$, $h > 0$

$$\left|\frac{F'(iy)}{F(iy)}\right| < \epsilon \tag{125}$$

for all y in $0 < y < R$ except possibly for a set of at most ϵR intervals of total length not exceeding hR.

Since ϵ and h are arbitrary positive numbers we may say that (124) holds for almost all y. A similar result is evidently true for functions meromorphic in the angle concerned.

12. *Maximal density results.*

For functions satisfying (110) and (114) a special interest attaches to the "indicator"

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{1}{r^\rho} \log|F(re^{i\theta})|, \tag{126}$$

and the behaviour of $F(z)$ is characterised to a considerable degree by the nature of $h(\theta)$. In this paragraph we examine the effect on our earlier results of a certain hypothesis on the behaviour of $h(\theta)$. It will be more convenient now to suppose that $\rho = 1$ and allow variations in the angle of regularity, instead of keeping the angle fixed at a half-plane and considering different values of ρ. If we use the geometrical interpretation of $h(\theta)$, the case in which we are interested is that in which the indicator

* 6, 170.

diagram of $F(z)$ has a vertex in a given direction, say $\theta = 0$. Alternatively, remembering that $h(\theta)$ has both a right-handed and a left-handed derivative, we oberve that Theorem 16 is concerned with the values of θ for which these derivatives are equal. We shall take the vertex concerned to be the origin and the tangent there to be the imaginary axis. In the work itself, however, the corresponding hypotheses on $F(z)$ are stated directly, their precise form being determined by certain applications. We give first a result proved by a slight modification of the method used for Theorem 8, namely,

THEOREM 15. *If $F(z)$ is regular in the angle $|\arg z| \leqslant \alpha$ $(< \frac{1}{2}\pi)$ and satisfies*

$$\overline{\lim_{|z|\to\infty}} \frac{1}{|z|} \log|F(z)| \leqslant \delta \tan\alpha \tag{127}$$

and

$$\overline{\lim_{r\to\infty}} \frac{1}{r} \log|F(r)| = 0, \tag{128}$$

then, when h and θ are given positive numbers less than unity, there exist arbitrarily large values of r such that $F(z)$ satisfies

$$\left|\frac{F'(x)}{F(x)}\right| \leqslant K\delta \sec\alpha \tag{129}$$

for all x in

$$r-\theta r \sin\alpha \leqslant x \leqslant r+\theta r \sin\alpha \tag{130}$$

except possibly for at most $K_1 \delta r \tan\alpha$ intervals the sum of whose lengths does not exceed $hr \sin\alpha$. K_1 depends only on θ and K on θ and h.

The proof is quite simple. From (128) we have

$$\log|F(r)| > -\tfrac{1}{2}\delta \tan\alpha(1+\sin\alpha)r \tag{131}$$

for some arbitrarily large values of r. If r is sufficiently large, we have also, from (127),

$$\log|F(z)| < (\delta \tan\alpha + \tfrac{1}{2}\delta \tan\alpha)(r+r\sin\alpha) \tag{132}$$

throughout the circle $|z-r| < r \sin\alpha$. Hence for $|z| < r \sin\alpha$ the function

$$G(z) = \frac{F(z+r)}{F(r)} \tag{133}$$

satisfies the inequality

$$\log|G(z)| < 2\delta \tan\alpha(1+\sin\alpha)r < 4\delta r \tan\alpha. \tag{134}$$

On applying Theorem 3a to $G(z)$ with $r\sin\alpha$ in place of r, we have Theorem 15.

We are interested in the case when α can be found so that the hypotheses of Theorem 15 hold no matter how small the positive number δ is chosen. The situation could only be described in terms of great complexity were it not for Pólya's conception of "maximal density".

If we confine our attention to "satisfactory intervals" of the form $n \leqslant x \leqslant n+1$, where n is a positive integer, and throughout which

$$\left|\frac{F'(z)}{F(z)}\right| < \epsilon, \tag{135}$$

then, however small ϵ may be, δ (and α) can be chosen so that this inequality results from (129). Hence, from Theorem 15, there exist arbitrarily large values of r such that the number of "satisfactory intervals" in

$$r-\tfrac{1}{2}r\sin\alpha \leqslant x \leqslant r+\tfrac{1}{2}r\sin\alpha \tag{136}$$

is at least

$$r(\sin\alpha - K_1\delta\tan\alpha - h\sin\alpha). \tag{137}$$

By choice of h (first) and δ we can make the expression (137) greater than $(1-\epsilon)r\sin\alpha$. We describe this situation, following Pólya*, by saying that the maximal density of the "satisfactory intervals" is unity and have

THEOREM 16. *If $F(z)$ is regular in an angle containing the real axis and α exists such that the conditions of Theorem* 15 *are satisfied for any positive number δ, then, ϵ being any positive number, the inequality*

$$\left|\frac{F'(x)}{F(x)}\right| < \epsilon \tag{138}$$

is satisfied for a sequence of intervals $n \leqslant x \leqslant n+1$ whose maximal density is unity.

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* 17, 559–561.

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ON SETS OF FRACTIONAL DIMENSIONS (III)

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1. In this paper we discuss the set $E_{r,s}$ of all numbers in (0, 1), which, when they are expressed as decimals in the scale of r ($r > 2$), do not contain the digit s ($s < r$). We shall prove that the dimension of $E_{r,s}$ depends only on r and not on s. We shall determine also the measure of the set $E_{r,s}$ in some particular cases, namely $r = 3$, $s = 0, 1, 2$; $r = 4$, $s = 0, 1, 2, 3$; $s = 0$ or $r-1$, r any value. As will be apparent later, there is an essential difference between the cases $r = 3$ and $r > 3$; the case $r = 4$ contains all the peculiarities of the general case. We begin with a number of definitions.

2. We are concerned with the measure theory formulated by Hausdorff† and we give a definition of this.

Suppose that $h(t)$ is any function of t which in some interval $0 \leqslant t \leqslant t_0$ is continuous, concave and strictly increasing and such that $h(0) = 0$. Let E be any linear set. Given a positive number $\rho \leqslant t_0$, we denote by $I(E, \rho)$ any set of intervals $\{I_r\}$, of respective lengths $\{a_r\}$, such that

(i) $a_r < \rho$ for each r, and

(ii) each point of E is interior to at least one interval I_r.

Then we denote‡

$$\text{lower bound} \sum_{I(E,\rho)} h(a_r)$$

by $h-m_\rho E$, and $\lim_{\rho \to 0} (h-m_\rho E)$ (which obviously exists) is denoted by $h-m^* E$ and is called the *exterior h-measure* of E. It is easily verifiable that this satisfies all the standard requirements of a measure function

† F. Hausdorff, "Dimension und äusseres Mass", *Math. Annalen*, 79 (1919), 157–179.

‡ A symbol like $\sum_{I(E,\rho)}$ denotes a summation over the members of the set $I(E, \rho)$.

and such notions as *measurability*, etc., now enter quite naturally. If E is *measurable*, in particular if it is closed, we speak of its *h-measure* and denote the latter by $h-mE$.

We say that the *dimension* of a given set E is $[\lambda(t)]$ if and only if

$$0 < \lambda-mA < \infty$$

(strict inequality must be adhered to).

If $\lambda(t)$ is of the form

$$t^a(\log 1/t)^{-p_1}(\log\log 1/t)^{-p_2}\ldots(\log\log\ldots\log 1/t)^{-p_r}$$

with $0 \leqslant a \leqslant 1$, then we say that $\lambda(t)$ is of dimensions $[a, p_1, p_2, \ldots, p_r]$ in the logarithmic scale of dimensions.

Further, if $\lambda(t) = t^a$, we write $a-mA$ in place of $\lambda-mA$, or even mA if no confusion can arise.

We quote one theorem of which we make use later.

THEOREM†. *Given a closed linear set E of finite h-measure in $0 \leqslant x \leqslant 1$, a necessary and sufficient condition that $h-mE > 0$ is that there exists a closed subset E_1 of E and a function $\phi(x)$ such that*

(i) *$\phi(x)$ is continuous and non-decreasing in* (0, 1),

(ii) *$\phi(x)$ is constant in each contiguous interval of E_1,*

(iii) *as x describes E, $\phi(x)$ describes a set of positive Lebesgue measure K, and*

(iv) *if x, x' belong to E_1, then $|\phi(x)-\phi(x')| \leqslant h(|x-x'|)$.*

If we replace (iv) by

(iv)′ *if x, x' belong to E_1, then $|\phi(x)-\phi(x')| \leqslant \mu h(|x-x'|)$ where μ is a positive constant,*

then we can assert that $h-mE \geqslant K/\mu$.

3. In any scale some numbers can be expressed in two different ways; in the scale of 10 the number 0·600... can also be expressed as 0·5999..., *i.e.* as a repeating decimal. Now if we are considering $E_{10,6}$ or $E_{10,5}$ it seems important that we should know in which way we are to express such a number; in actual fact, however, it is unimportant, for these

† J. Gillis, "Note on a theorem of Myrberg", *Proc. Cambridge Phil. Soc.*, 33 (1937), 419–424.

numbers form a subset of the set of rational numbers and are consequently denumerable; thus by the usual properties of measure functions it does not matter whether we assume $E_{r,s}$ to contain one, both, or neither representations. To make $E_{r,s}$ closed we assume that it contains both representations.

4. We obtain the set in the following manner. Consider the interval (0, 1); divide it into r equal portions; then all the decimals in scale r which have s in the first place are contained in the $(s+1)$-th interval; therefore we remove that interval. We have now $r-1$ equal intervals left. We divide each of these into r equal portions which will each be of length $1/r^2$; then all decimals which have the digit s in the second but not in the first place lie in the $(s+1)$-th interval of length $1/r^2$ from the left-hand end of each interval of length $1/r$; we remove these intervals. We have now $(r-1)^2$ intervals left, each of length $1/r^2$, and we repeat the process of division; and so on. Since by § 3 we remove always open intervals, the set of points left at every stage is closed.

We see that $E_{r,s}$ may, at the n-th stage, be covered with $(r-1)^n$ intervals, each of length $1/r^n$. Hence

$$h-mE_{r,s} \leqslant (r-1)^n h(1/r^n).$$

Take $$h(t) = t^a$$

and then $$h-mE_{r,s} \leqslant (r-1)^n/r^{na}.$$

We wish this to be finite and not zero. This means that we must have

$$(r-1)^n \sim r^{na}$$

i.e. $$a = [\log (r-1)]/\log r,$$

and then† $$a-mE_{r,s} \leqslant 1.$$

We take the function $h(t) = t^a$ and prove that it is actually the dimension function of the set. We can obtain a better value for an upper bound for the measure.

5. We consider first the case $s = 0$, $r \geqslant 3$. In this case we remove at the first stage the extreme left-hand end portion of length $1/r$; at the next stage we remove a length $1/r^2$ abutting this, a length $1/r^3$ abutting

† Writing $a-mE_{r,s}$ instead of $t^a-mE_{r,s}$.

this at the third stage, and so on. In this way we remove an interval of length

$$\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\ldots=\frac{1}{r-1}.$$

Similarly, we remove at the points $2/r$, $3/r$, ... $(r-1)/r$ intervals of length $1/r(r-1)$, and so on. By joining these removed intervals into a complete interval we affect only an enumerable sub-set of $E_{r,s}$ and therefore do not affect the measure. Our construction is ultimately equivalent to the removal of $r-2$ portions, each of length $1/\{r^{n-1}(r-1)\}$, from an interval of length $(r-2)/\{r^{n-2}(r-1)\}$, and these intervals of length $1/\{r^{n-1}(r-1)\}$ are symmetrically placed and equidistant within the larger interval of length $(r-2)/\{r^{n-2}(r-1)\}$. This set $E_{r,0}$ is a generalisation of a set discussed by Hausdorff in the above-mentioned paper. In a similar manner to his proof we show that its dimension is

$$\alpha=\left[\frac{\log(r-1)}{\log r}\right]$$

and measure

$$\left(\frac{r-2}{r-1}\right)^{\alpha}.$$

6. We now return to the general case. We suppose that

$$0<s\leqslant\left[\frac{r-1}{2}\right],$$

which we may obviously do from symmetry.

We had, at the n-th stage, $(r-1)^n$ intervals, each of length $1/r^n$; but it is easy to see that many of these intervals adjoin; however, only three types of intervals occur, those of lengths $(r-1)/r^n$, $(r-s-1)/r^n$, s/r^n. On proceeding to the $(n+1)$-th stage each of these intervals gives rise to 1 interval each of lengths $(r-s-1)/r^{n+1}$ and s/r^{n+1}, and to $(r-2)$, $(r-s-2)$ and $(s-1)$, respectively, intervals of length $(r-1)/r^{n+1}$. We assert that at the n-th stage there are

$$\left.\begin{array}{lll}\dfrac{(r-1)^{n-1}-1}{r-2}(r-3) & \text{intervals of length} & \dfrac{r-1}{r^n},\\[2ex] \dfrac{(r-1)^{n-1}-1}{r-2}+1 & \text{intervals of length} & \dfrac{r-s-1}{r^n},\\[2ex] \dfrac{(r-1)^{n-1}-1}{r-2}+1 & \text{intervals of length} & \dfrac{s}{r^n}.\end{array}\right\} \qquad (1)$$

These results can be easily proved by induction.

We can now see the essential difference between the cases $r = 3$ and $r \geqslant 4$. In the case $r = 3$ all the intervals are of the same length. In all the other cases there are three types of intervals.

The case $r = 3$, $s = 1$ is a particular case of the general example given by Hausdorff in the paper mentioned above.; we have $mE_{3,1} = 1$ and the dimension of E is

$$\left[\frac{\log 2}{\log 3}\right] = 0{\cdot}63093.$$

7. We confine ourselves henceforth to the case $r \geqslant 4$, $s \neq 0$.

We first divide in the usual way up to the $(m-1)$-th stage; when we proceed to the m-th stage, let us divide intervals of lengths $(r-s-1)/r^{m-1}$, s/r^{m-1} in the usual way; but we divide an interval of length $(r-1)/r^{m-1}$ into an interval of length s/r^m and one of length $[(r-2)r+r-s-1]/r^m$; we can do this by grouping together the $r-2$ intervals of length $(r-1)/r^m$ and the interval $(r-s-1)/r^m$ into which it would have been divided. (We have, of course, to cover also some of the parts which would have been omitted according to the usual covering.) As we continue to divide we do not divide the interval of length $[(r-2)r+r-s-1]/r^m$, but let it remain as it is; we divide the other intervals in the usual way. We treat each interval of length $(r-1)/r^{m+p}$ $(p \geqslant 0)$ in the same way from now on. Let, now, X_n, Y_n, Z_n, W_n be the number of intervals at the $(n+m)$-th stage of respective lengths s/r^{n+m}, $(r-s-1)/r^{n+m}$, $(r-1)/r^{n+m}$, d/r^{n+m} where $d = (r-2)r+(r-s-1)$. Then $W_n = Z_{n-1}$. We have also, as we divide,

$$X_n = X_{n-1}+Y_{n-1}+Z_{n-1},$$

$$Y_n = X_{n-1}+Y_{n-1},$$

$$Z_n = (s-1)\,X_{n-1}+(r-s-2)\,Y_{n-1}.$$

Hence we get without difficulty

$$X_n = 2X_{n-1}+(s-1)\,X_{n-2}+(r-2s-1)X_{n-3}.$$

This gives $$X_n = Au^n+Bv^n+Cw^n,$$

where u, v, w are the roots of

$$x^3 = 2x^2+(s-1)x+(r-2s-1)$$

and A, B, C are constants depending on X_0, Y_0, Z_0.

We have similarly

$$Y_n = 3Y_{n-1} + (s-3)Y_{n-2} + (r-3s)Y_{n-3} - (r-2s-1)Y_{n-4},$$

$$Y_n = A'\alpha^n + B'\beta^n + C'\gamma^n + D'\delta^n,$$

where A', B', C', D' are constants and α, β, γ, δ are roots of

$$x^4 - 3x^3 + (s-3)x^2 - (r-3s)x + (r-2s-1) = 0$$

i.e. of $$(x-1)[x^3 - 2x^2 - (s-1)x - (r-2s-1)] = 0$$

and therefore

$$Y_n = A'u^n + B'v^n + C'w^n + D'.$$

Using again the relation

$$Y_n - Y_{n-1} = X_{n-1},$$

we have

$$A'u^{n-1}(u-1) + B'v^{n-1}(v-1) + C'w^{n-1}(w-1) = Au^{n-1} + Bv^{n-1} + Cw^{n-1}.$$

Writing down the three expressions of this type with $n = 1, 2, 3$ and solving simultaneously, we find that

$$A' = \frac{A}{u-1}, \quad B' = \frac{B}{v-1}, \quad C' = \frac{C}{w-1},$$

so that, writing $D = D'$,

$$Y_n = \frac{A}{u-1}u^n + \frac{B}{v-1}v^n + \frac{C}{w-1}w^n + D.$$

Now we also have

$$Z_n = X_{n-1} - Y_{n+1};$$

therefore

$$Z_n = A\frac{u-2}{u-1}u^n + B\frac{v-2}{v-1}v^n + C\frac{w-2}{w-1}w^n - D.$$

By (1) we see that, if we make the new covering, we have

$$\frac{(r-1)^{m-1}-1}{r-2} + 1 \quad \text{intervals of length} \quad \frac{s}{r^m},$$

$$\frac{2(r-1)^{m-2}-2}{r-2} + 2 \quad \text{intervals of length} \quad \frac{r-s-1}{r^m},$$

$$(r-3)\left[\frac{(r-1)^{m-2}-1}{r-2} + 1\right] \quad \text{intervals of length} \quad \frac{r-1}{r^m},$$

$$\frac{(r-1)^{m-2}-1}{r-2}(r-3) \quad \text{intervals of length} \quad \frac{d}{r^m}.$$

Hence

$$X_0 = A+B+C = \frac{(r-1)^{m-1}-1}{r-2}+1,$$

$$X_1 = Au+Bv+Cw = \frac{(r-1)^{m-1}-1}{r-2}+1+(r-3)\left[\frac{(r-1)^{m-2}-1}{r-2}+1\right]$$
$$+2\frac{(r-1)^{m-2}-2}{r-2}+2,$$

$$X_2 = Au^2+Bv^2+Cw^2 = (r-1)\left[\frac{(r-1)^{m-2}-1}{r-1}+1\right]$$
$$+\frac{(r-1)^{m-1}-1}{r-2}+1+\frac{(r-1)^{m-1}-1}{r-2}+1+\frac{2(r-1)^{m-2}-2}{r-2}$$
$$+2+(s-1)\left[\frac{(r-1)^{m-1}-1}{r-2}+1\right]+2\left[\frac{(r-1)^{m-2}-1}{r-2}+1\right](r-s-2),$$

whence $$A+B+C = \frac{(r-1)^{m-1}}{r-2}+\frac{r-3}{r-2},$$

$$Au+Bv+Cw = \frac{2(r-1)^{m-1}}{r-2}+r\frac{r-3}{r-2},$$

$$Au^2+Bv^2+Cw^2 = \frac{(r-1)^{m-2}}{r-2}(rs+4r-3s-4)+(3r-s-2)\frac{r-3}{r-2}.$$

Eliminating A and B we obtain

$$C(w-u)(w-v) = \frac{(r-1)^{m-2}}{r-2}[rs+4r-3s-4+(r-1)(vu-2u-2v)]$$
$$+\frac{r-3}{r-2}(3r-s-2-ru-rv+uv),$$

and similarly for A and B. Further,

$$Y_0 = \frac{A}{u-1}+\frac{B}{v-1}+\frac{C}{w-1}+D = \frac{2(r-1)^{m-2}-2}{r-2}+2;$$

hence $$D = \frac{2(r-1)^{m-2}-1}{r-2}+2-\Sigma\frac{A}{u-1}.$$

We get, therefore,

$$mE_{r,s} \leqslant \lim_{m\to\infty}\left\{X_n\frac{s^\alpha}{r^{(n+m)\alpha}}+Y_n\frac{(r-s-1)^\alpha}{r^{(n+m)\alpha}}+Z_n\frac{(r-1)^\alpha}{r^{(n+m)\alpha}}\right.$$
$$\left.+\frac{d^\alpha}{r^{(n+m)\alpha}}[W_n+W_{n-1}(r-1)+\ldots+W_0(r-1)^n]\right\}$$

giving, with $r^\alpha = r-1$,

$$mE_{r,s} \leqslant \lim_{m\to\infty} \Big\{ \Sigma \frac{w^n}{(w-u)(w-v)} \frac{s^\alpha}{(r-1)^{n+m}} \times \Big[\frac{(r-1)^{m-2}}{r-2} \{rs+4r-3s+4(r-1)(vu-2u-2v)\} + \frac{r-3}{r-2}(3r-s-2-ru-rv+uv) \Big]$$

$$+\Sigma \frac{w^n}{(w-u)(w-v)} \frac{(r-s-1)^\alpha}{(r-1)^{m+n}} (w-1) \times \Big[\frac{(r-1)^{m-2}}{r-2} \{rs+4r-3s+4(r-1)(vu-2u-2v)\} + \frac{r-3}{r-2}(3r-s-2-ru-rv+uv) \Big] + D \frac{(r-s-1)^\alpha}{(r-1)^{n+m}}$$

$$+\Sigma \frac{w^{n+1}}{(w-u)(w-v)} \frac{(r-1)^\alpha}{(r-1)^{m+n}} \frac{w-2}{w-1} \times \Big[\frac{(r-1)^{n-2}}{r-2} \{rs+4r-3s+4(r-1)(vu-2u-2v)\} + \frac{r-3}{r-2}(3r-s-2-ru-rv+uv) \Big] - D \frac{(r-1)^\alpha}{(r-1)^{n+m}}$$

$$+ \frac{d^\alpha}{(r-1)^{n+m}} [W_n + W_{n-1}(r-1) + \ldots + W_0(r-1)^n] \Big\},$$

where Σ refers to summation over u, v, w.

We assert now that

$$\frac{|t|^n}{(r-1)^n},$$

where t is any one of u, v, w, tends to zero as n tends to infinity.

The equation

$$x^3-2x^2-(s-1)x-(r-2s-1)=0$$

has roots u, v, w, and so

$$uvw = r-2s-1.$$

Suppose that u is a real root. Then v and w are either conjugate complex numbers or both real. Let

$$y = x^3-2x^2-(s-1)x-(r-2s-1);$$

then $$\frac{dy}{dx} = 3x^2-4x-(s-1),$$

and we have extreme values where

$$x = \frac{2\pm\sqrt{(1+3s)}}{3};$$

and, since $$\frac{d^2y}{dx^2} = 6x-4,$$

there is a maximum at $x = \frac{1}{3}[2-\sqrt{(1+3s)}] > -(r-1)$ and a minimum at $x = \frac{1}{3}[2+\sqrt{(1+3s)}] < r-1$.

When $x = r-1$,

$$\begin{aligned} y &= r^3-5r^2+7r-sr+3s-3 \\ &\geqslant r^3-5r^2+7r-\tfrac{1}{2}r^2, \quad \text{since} \quad 1 \leqslant s \leqslant \tfrac{1}{2}[r-1] \\ &= \tfrac{1}{2}r(2r-7)(r-2) \\ &\geqslant 0 \quad \text{if} \quad r \geqslant 4. \end{aligned}$$

When $x = -(r-1)$,

$$y = -(r^3-r^2+r-sr-s-1) < 0,$$

and, since maximum and minimum lie between $x = \pm(r-1)$, all real roots must lie between $\pm(r-1)$.

There may, however, be one real root u and two conjugate complex roots v, w. Then

$$|vw| = |v^2|$$

and $$|uvw| = r-2s-1 = u|v|^2.$$

We wish to prove that

$$|v^2| < (r-1)^2$$

which is equivalent to $$u > \frac{r-2s-1}{(r-1)^2};$$

and this is true, since $y \leqslant 0$ when $x = (r-2s-1)/(r-1)^2$. Hence we have $|t| < r-1$, and our assertion is proved.

Returning now to our formula for $mE_{r,s}$ we have

$$mE_{r,s} \leqslant \lim_{m\to\infty} \left\{ \frac{d^a}{(r-1)^{n+m}} [W_n+W_{n-1}(r-1)+\ldots+W_0(r-1)^n] + \lambda\Sigma \frac{w^n}{(r-1)^n} + \mu\Sigma \frac{w^n}{(r-1)^{n+m}} \right\}$$

where λ, μ are functions of r, s, u, v, w, but not of n, m. Hence, if we now choose $n = n(m)$ so that n tends to infinity with m,

$$mE_{r,s} \leqslant \lim_{m\to\infty} \left\{ \frac{d^\alpha}{(r-1)^{n+m}} [W_n + W_{n-1}(r-1) + \ldots + W_0(r-1)^n] \right\}.$$

Now
$$W_n = Z_{n-1} = A\frac{u-2}{u-1}u^n + B\frac{v-2}{v-1}v^n + C\frac{w-2}{w-1}w^n - D$$

$$= \Sigma A\frac{u-2}{u-1}u^n - \frac{2(r-1)^{m-2}-2}{r-2} - 2 + \Sigma\frac{A}{u-1}.$$

Therefore

$$mE_{r,s} \leqslant \lim_{m\to\infty} \frac{d^\alpha}{(r-1)^m} \left\{ \left[\frac{1}{(r-1)^n} \Sigma A\frac{u-2}{u-1}u^n + \frac{1}{(r-1)^{n-1}} \Sigma A\frac{u-2}{u-1}u^{n-1} + \ldots \right] - \left[\frac{2(r-1)^{m-2}-2}{r-2} + 2 - \Sigma\frac{A}{u-1} \right] \left[\frac{1}{(r-1)^n} + \frac{1}{(r-1)^{n-1}} + \ldots + 1 \right] \right\}$$

$$= \lim_{m\to\infty} \frac{d^\alpha}{(r-1)^m} \left\{ \Sigma A\frac{u-2}{u-1}\frac{1-\{u/(r-1)\}^{n+1}}{1-\{u/(r-1)\}} - \left[\frac{2(r-1)^{n-2}-2}{r-2} + 2 - \Sigma\frac{A}{u-1} \right] \frac{1-\{1/(r-1)^{n+1}\}}{1-\{1/(r-1)\}} \right\}.$$

Hence, remembering the values of A, B, C, we have

$$mE_{r,s} \leqslant \frac{d^\alpha}{(r-1)(r-2)} \Sigma \frac{(u-2)[rs+4r-3s-4+(r-1)(vw-2w-2v)]}{(u-1)(r-1-u)(u-w)(u-v)}$$

$$- \frac{2d^\alpha}{(r-2)^2(r-1)} + \frac{d^\alpha}{(r-1)(r-2)^2} \Sigma \frac{rs+4r-3s-4+(r-1)(vw-2w-2v)}{(u-w)(u-v)(u-1)}.$$

From the definitions of u, v, w we have

$$u+v+w = 2,$$

$$uv+vw+wu = -(s-1),$$

$$uvw = r-2s-1,$$

These give us

$$\Sigma(w^2-v^2)=0, \quad \Sigma(u-v)uv=\Pi(u-w),$$

$$\Sigma(v-u)=0, \quad \Sigma(v-u)v^2u^2=(s-1)\,\Pi(u-w),$$

$$\Sigma(w^2-v^2)\,u=\Pi(u-w), \quad \Sigma(w^2-v^2)\,u^2=0,$$

$$\Sigma(w^2-v^2)\,u^3=(s-1)\,\Pi(u-w), \quad \Sigma(w^2-v^2)\,vw=-2\Pi(u-w),$$

$$\Sigma(w^2-v^2)\,w^2v^2=(r-3)\,\Pi(u-w).$$

We use all these results constantly in what follows. We calculate $mE_{r,s}$ term by term.

$$\Sigma\frac{rs+4r-3s-4+(r-1)(vw-2w-2v)}{(u-w)(u-v)(u-1)}$$

$$=\frac{1}{\Pi(u-w)\,\Pi(w-1)}\,\Sigma\,(v-1)(u-1)(v-u)$$
$$\times[rs+4r-3s-4+(r-1)(vu-2u-2v)]$$

$$=\frac{1}{(uvw-\Sigma uv+\Sigma u-1)\,\Pi(u-w)}\{(rs+4r-3s-4)\,\Sigma(uv-u-v+1)(v-u)$$
$$+(r-1)\,\Sigma vu(v-1)(u-1)(v-u)$$
$$-2(r-1)\,\Sigma(v-u)(v+u)(v-1)(u-1)\}$$

$$=\frac{1}{(r-s-1)\,\Pi(u-w)}\,\{-(rs+4r-3s-4)\,\Pi(u-w)$$
$$+(r-1)\,\Sigma vu(vu-v-u+1)(v-u)$$
$$-2(r-1)\,\Sigma(v^2-u^2)(vu+w-1)\}$$

$$=\frac{1}{r-s-1}\{-(rs+4r-3s-4)+(r-1)(s-1)$$
$$+2(r-1)-(r-1)+4(r-1)-2(r-1)\}$$

$$=-2.$$

Hence this term and the preceding term in the last expression for the upper bound of $mE_{r,s}$ cancel out and we are left with

$$mE_{r,s}\leqslant-\frac{d^a\,\Sigma(u-2)(w-v)(v-1)(w-1)(r-1-v)(r-1-w)}{(r-1)(r-2)\,\Pi(u-1)\,\Pi(r-1-u)\,\Pi(u-w)}$$
$$\times[rs+4r-3s-4+(r-1)(vw-2w-2v)]$$

We have

$$vw=u^2-2u-s+1,$$

$$v^2w^2=-(r-2s-1)(v+w)-vw(s-1).$$

Using these, writing

$$H = \Sigma(u-2)(w-v)(v-1)(w-1)(r-1-v)(r-1-w)$$
$$\times [rs+4r-3s-4+(r-1)(vw-2w-2v)]$$

and reducing, we get

$$H = -\Sigma(v+w)(w-v)(v^2-rv+r-1)(w^2-rw+r-1)$$
$$\times [rs+4r-3s-4+(r-1)(vw-2w-2v)]$$
$$= -\Sigma(w^2-v^2)\,[v^2w^2-rvw(v+w)+r^2vw+(r-1)(v^2+w^2)$$
$$-(r-1)r(v+w)+(r-1)^2][rs+4r-3s-4+(r-1)(vw-2w-2v)]$$
$$= -\Sigma(w^2-v^2)[-(r-2s-1)(v+w)-vw(s-1)-rvw(2-u)+r^2vw$$
$$+(r-1)(v+w)^2-2(r-1)vw-r(r-1)(v+w)+(r-1)^2]$$
$$\times [rs+4r-3s-4+(r-1)(vw-2w-2v)]$$
$$= -\Sigma(w^2-v^2)[-vw(s-3+4r-r^2)-(v+w)(r^2-2s-1)+(r-1)(v+w)^2$$
$$+r(r-2s-1)+(r-1)^2][rs+4r-3s-4+(r-1)(vw-2w-2v)]$$
$$= -\Sigma(w^2-v^2)[-vw(s-3+4r-r^2)+(u-2)(r^2-2s-1)$$
$$+(r-1)(u^2-4u+4)+2r^2-2rs-3r+1]$$
$$\times [rs+4r-3s-4+(r-1)(vw-2w-2v)]$$
$$= -\Sigma(w^2-v^2)[u(r^2-4r-2s+3)+(r-1)u^2+(r^2-4r-s+3)vw$$
$$-2rs+r+4s-1][(r-1)vw+2(r-1)u+rs-3s]$$
$$= -\Sigma(w^2-v^2)\{(r-1)[(r-2s-1)(r^2-4r-2s+3)+u(r-1)(r-2s-1)$$
$$+(r^2-4r-s+3)v^2w^2-(2rs-r-4s+1)vw]$$
$$+2(r-1)[u^2(r^2-4r-2s+3)+(r-1)u^3+(r^2-4r-s+3)(r-2s-1)$$
$$-(2rs-r-4s+1)u]+(rs-3s)[u(r^2-4r-2s+3)$$
$$+(r-1)u^2+(r^2-4r-s+3)vw-(2rs-r-4s+1)]\}$$
$$= -\{(r-1)[(r-1)(r-2s-1)+(r-3)(r^2-4r-s+3)$$
$$+2(2rs-r-4s+1)]+2(r-1)[(s-1)(r-1)-(2rs-r-4s+1)$$
$$+(rs-3s)[(r^2-4r-2s+3)-2(r^2-4r-s+3)]\}\,\Pi(u-w),$$

which reduces to

$$H = -(r-1)(r-3)(r-2)(r-1-s)\,\Pi(u-w).$$

Also $$\Pi(u-1) = r-s-1,$$

and $$\Pi(r-1-u) = (r-3)(r^2-2r-s+1).$$

Hence $$mE_{r,s} \leqslant \frac{d^\alpha}{r^2-2r-s+1} = \frac{(r^2-r-s-1)^\alpha}{r^2-2r-s+1}.$$

In the particular case $r = 4$, $s = 1$, we have

$$mE_{4,1} \leqslant \frac{10^\alpha}{8}.$$

8. We now find a positive lower bound for the measure. To do this we use the theorem quoted in § 2. We define a function $f_n(x)$ with $f_n(0) = 0$, $f_n(1) = 1$ and such that it is linear and continuous everywhere; in addition it is to be constant in the removed intervals and will have a gradient of $1/(r-1)^n$ elsewhere in the intervals obtained at the n-th stage of division which is done according to the original method of § 6 and not that of § 7. $f_n(x)$ is monotonic increasing. Since at the n-th stage there are $(r-1)^n$ intervals of length $1/r^n$, $f_n(1) = 1$ as defined.

It is easy to see that $f_n(x)$ converges uniformly and we define

$$f(x) = \lim_{n\to\infty} f_n(x).$$

The function $f(x)$ obviously satisfies the first three conditions of the theorem of § 2. We proceed now to the consideration of the fourth.

We take any two points x, x', which we suppose to belong to what remains of (0, 1) after the n-th division. We let the distance between x and x' be

$$\frac{a_0 r^\lambda + a_1 r^{\lambda-1} + \ldots + a_q r^{\lambda-q}}{r^n},$$

where the a_i are integers and $a_i \leqslant r-1$. In considering the fourth condition of the theorem the least favourable case for the inequality obviously occurs when x and x' are at the ends of their intervals which are farthest apart. The total increment of $f_n(x)$ in an interval of length $1/r^{n-1}$ $(i \neq 0)$ is $1/(r-1)^{n-i}$; and the total increment in an interval of length a_n/r^n is at most $a_n/(r-1)^n$. Hence, dividing the distance between x and x' into sections of lengths $1/r^{n-\lambda}$, $1/r^{n-\lambda+1}$, ..., $1/r^{n-\lambda+q}$, we see that the total

increment between x and x' is, at most,

$$\frac{a_0}{(r-1)^{n-\lambda}}+\frac{a_1}{(r-1)^{n-\lambda+1}}+\ldots+\frac{a_q}{(r-1)^{n-\lambda+q}}$$

$$=[a_0t^{\lambda}+a_1t^{\lambda-1}+\ldots+a_qt^{\lambda-q}]/(r-1)^n,$$

where $t=r-1=r^{\alpha}$.

We now try to show that, for some constant μ,

$$|f_n(x)-f_n(x')|\leqslant\mu|x-x'|^{\alpha}$$

or that

$$\frac{a_0t^{\lambda}+a_1t^{\lambda-1}+\ldots+a_qt^{\lambda-q}}{(r-1)^n}\leqslant\mu\left[\frac{a_0r^{\lambda}+a_1r^{\lambda-1}+\ldots+a_qr^{\lambda-q}}{r^n}\right]^{\alpha}$$

i.e. that

$$\frac{(a_0r^{\lambda}+a_1r^{\lambda-1}+\ldots+a_qr^{\lambda-q})^{\alpha}}{a_0t^{\lambda}+a_1t^{\lambda-1}+\ldots+a_qt^{\lambda-q}}\geqslant\frac{1}{\mu}.$$

The result when x_0, x' belong to the same interval of length $(r-1)/r^n$ or $(r-s-1)/r^n$ or s/r^n is easily seen to be true with $\mu\geqslant(r-1)^{1-\alpha}$.

We now consider the expression

$$y=\frac{(a_0r^{\lambda}+\ldots+a_qr^{\lambda-q})^{\alpha}}{a_0t^{\lambda}+\ldots+a_qt^{\lambda-q}}.$$

We assert that

$$\frac{(a_0r^{\lambda}+\ldots+a_qr^{\lambda-q})^{\alpha}}{a_0t^{\lambda}+\ldots+a_qt^{\lambda-q}}\geqslant\frac{(a_0r^{\lambda+1}+\ldots+a_qr^{\lambda-q+1}+a_{q+1}r^{\lambda-q})^{\alpha}}{a_0t^{\lambda+1}+\ldots+a_qt^{\lambda-q+1}+a_{q+1}t^{\lambda-q}}.$$

If this is so, then, remembering that $r^{\alpha}=t$,

$$\left(1+\frac{a_{q+1}r^{\lambda-q}}{r\sum_0^q a_ir^{\lambda-i}}\right)^{\alpha}\leqslant1+\frac{a_{q+1}t^{\lambda-q}}{t\sum_0^q a_it^{\lambda-i}}.$$

Now $\alpha<1$, and therefore

$$\left(1+\frac{a_{q+1}r^{\lambda-q}}{r\sum_0^q a_ir^{\lambda-i}}\right)^{\alpha}\leqslant1+\frac{a_{q+1}r^{\lambda-q}}{r\sum_0^q a_ir^{\lambda-i}};$$

and hence our assertion will be true if

$$\frac{a_{q+1}r^{\lambda-q}}{r\sum_0^q a_ir^{\lambda-i}}\leqslant\frac{a_{q+1}t^{\lambda-q}}{t\sum_0^q a_it^{\lambda-i}}.$$

This is so, since $r > t$. Hence the assertion is true. Further, we can prove that the larger a_{q+1} is, the smaller y will be; in essence this inequality amounts to this: if we have two expressions for y in which a certain number of corresponding terms are equal and if in our expression the remaining terms are all greater than the corresponding terms in the other expression, then the first expression will be the smaller.

In order to calculate μ, we desire to find the minimum value of y. It would appear from what we have just proved that this should occur when all the a_i are equal to $r-1$. This case cannot, however, arise in practice; for if a_0, say, is $r-1$ and some other a_i is not zero, then, while we actually traverse a distance greater than $(r-1)/r^{n-\lambda}$ and less than $1/r^{n-\lambda-1}$, we must remember by our construction of the intervals that this distance must contain an interval of length $1/r^{n-\lambda}$ of the type which we remove and therefore the increment corresponding to $(r-1)a_0$ will not be $(r-1)/(r-1)^{n-\lambda}$ but $(r-2)/(r-1)^{n-\lambda}$. Hence y is not a minimum with all the a_i equal to $r-1$; y is certainly less than this with all the a_i equal to $r-2$ since the denominators are then the same while the numerator in the latter case is less. Hence the minimum of y is either greater than this or greater than in the case $a_0 = r-1$, $a_i = 0$ $(i \neq 0)$. Intermediate cases with some a_i non-zero obviously need not be considered. Hence the minimum for y is at least equal to the smaller of

$$\lim_{q\to\infty} \frac{[(r-2)(r^\lambda+\ldots+r^{\lambda-q})]^\alpha}{(r-2)(t^\lambda+\ldots+t^{\lambda-q})} \quad \text{with} \quad \lambda-q \geqslant 0,$$

and

$$\frac{[(r-1)\,r^\lambda]^\alpha}{(r-1)\,t^\lambda},$$

that is of $(r-2)^\alpha/(r-1)^\alpha$ and $(r-1)^\alpha/(r-1)$. Simple algebra shows that the first of these two is the smaller. Hence, if y_0 is the minimum value of y,

$$y_0 \geqslant \left(\frac{r-2}{r-1}\right)^\alpha,$$

and equality holds† for

$$\mu_0 = \frac{1}{y_0} = \left(\frac{r-1}{r-2}\right)^\alpha.$$

This gives

$$|f_n(x)-f_n(x')| \leqslant \left(\frac{r-1}{r-2}\right)^\alpha |x-x'|^\alpha,$$

† This value obviously also satisfies the inequality for single intervals.

and hence, proceeding to the limit, we have

$$|f(x)-f(x')| \leqslant \left(\frac{r-1}{r-2}\right)^{\alpha} |x-x'|^{\alpha}.$$

Thus all four conditions of the theorem of § 2 are satisfied and we can assert that

$$\alpha - mE_{r,s} \geqslant \left(\frac{r-2}{r-1}\right)^{\alpha}.$$

This means that

$$\frac{(r^2-r-s-1)^{\alpha}}{r^2-2r-s+1} \geqslant \alpha - mE_{r,s} \geqslant \left(\frac{r-2}{r-1}\right)^{\alpha},$$

and therefore that $E_{r,s}$ is of dimension

$$\left[\frac{\log(r-1)}{\log r}\right]$$

in the logarithmic scale and that the dimension depends only on r and not on s.

9. In the special case $r = 4$ we can find the actual value of $\alpha - mE_{4,1}$.

(i) $a_0 = r-1 = 3$. Only two cases need be considered.

(a) $a_i = 0, \quad i \neq 0.$

(b) $a_i = 2$ $(i \geqslant 1)$ in both numerator and denominator. It might seem that there would be many intermediate values of the a_i to consider, *e.g.* $a_i = 3$ $(i < k)$ in the numerator and either $a_i = 2$ or $a_i = 0$ $(i \geqslant k)$; but then $a_i = 2$ $(i < k)$ in the first case and $a_i = 2$ $(i < k-1)$, $a_{k-1} = 3$ in the second in the denominator and this is obviously greater than (b).

(ii) $a_0 = r-2 = 2$. The whole distance x to x' must lie within an interval of length $3/4^{n-\lambda-1}$ (containing no intervals of the type of length $1/4^{n-\lambda-1}$ which are removed). On dividing this interval we obtain, going from left to right, first an interval of length $1/4^{n-\lambda}$, then a removed portion of length $1/4^{n-\lambda}$, then two intervals of length $3/4^{n-\lambda}$, each followed by a removed portion of length $1/4^{n-\lambda}$, and lastly an interval of length $2/4^{n-\lambda}$; each end of the interval of length $3/4^{n-\lambda-1}$ has at its outer side a removed portion of length $3/4^{n-\lambda-1}$.

If, now, $a_1 \geqslant 2$ and not all the succeeding a_i are zero, we must include in (x, x') an interval of the type which we remove of length $1/4^{n-\lambda}$ and

therefore we replace a_1 in the denominator by a_1-1. This gives us the possibility of

(*a*) $a_1=3$, $a_i=2$ $(i\geqslant 2)$ in the numerator and $a_i=2$, $a_i=2$ $(i\geqslant 1)$ in the denominator.

(*b*) $a_1=3$, $a_2=3$, $a_i=0$ $(i\geqslant 3)$ in the numerator and $a_1=2$, $a_2=3$, $a_i=0$ $(i\geqslant 3)$ in the denominator. It is obvious that we need not consider intermediate cases such as $a_1=3$, $a_i=2$ $(2\leqslant i<k)$, $a_k=3$, for these form a monotonic sequence as k increases and therefore lie in value between (*a*) and (*b*); so they are irrelevant for the determination of the minimum of y. We use this principle throughout.

(*c*) $a_1=2$, $a_i=0$ $(i\geqslant 2)$ in both denominator and numerator.

Remembering now the constitution of the interval of length $3/4^{n-\lambda-1}$ and taking x (or x') at the extreme right end of the interval we have three possible cases left:

(*d*) $a_1=1$, $a_2=3$, $a_i=2$ $(i\geqslant 3)$ in the numerator and $a_1=1$, $a_i=2$ $(i\geqslant 2)$ in the denominator.

(*e*) $a_1=1$, $a_2=2$, $a_i=0$ $(i\geqslant 3)$ in both numerator and denominator.

(*f*) $a_i=1$, $i\geqslant 1$ in both numerator and denominator. As before, intermediate cases form a monotonic sequence.

(iii) $a_0=r-3=1$. One extreme value is again

(*a*) $a_1=r-1=3$ in both numerator and denominator. Here, where $a_0=r-3$, we take x (or x') as the left hand end point of the left-hand interval of length $3/4^{n-\lambda}$. The other cases are given by

(*b*) $a_1=2$, $a_2=3$, $a_3=2$ in the numerator and $a_1=a_2=a_3=2$ in the denominator.

(*c*) $a_1=2$, $a_i=3$ $(i\geqslant 2)$ in the numerator and $a_i=2$ $(i\geqslant 1)$ in the denominator. The sequence $a_i=2$ $(i\geqslant 2)$ would require $a_i=1$ $(i\geqslant 2)$ in the denominator and need not therefore be considered.

(*d*) $a_1=a_2=2$ in both numerator and denominator.

(*e*) $a_1=2$, $a_i=1$ $(i\geqslant 3)$ in both numerator and denominator.

(*f*) $a_1=2$, $a_2=1$, $a_3=2$ in both numerator and denominator and once again intermediate cases give a monotonic sequence.

Working out these cases we obtain

(i) (*a*) $(r-1)^\alpha/(r-1) = 3^\alpha/3 = 0{\cdot}796.$

(*b*) $\lim_{q\to\infty} \dfrac{(3.4^\lambda+2.4^{\lambda-1}+\ldots+2.4^{\lambda-q})^\alpha}{2.3^\lambda+2.3^{\lambda-1}+\ldots+2.3^{\lambda-q}} = \dfrac{11^\alpha}{3^{1+\alpha}} = 0{\cdot}933.$

(ii) (*a*) $\lim_{q\to\infty} \dfrac{(2.4^\lambda+3.4^{\lambda-1}+2.4^{\lambda-2}+\ldots+2.4^{\lambda-q})^\alpha}{2.3^\lambda+2.3^{\lambda-1}+2.3^{\lambda-2}+\ldots+2.3^{\lambda-q}} = \dfrac{35^\alpha}{9.3^\alpha} = 0{\cdot}7788.$

(*b*) $\dfrac{(2.4^\lambda+3.4^{\lambda-1}+3.4^{\lambda-2})^\alpha}{2.3^\lambda+2.3^{\lambda-1}+3.3^{\lambda-2}} = \dfrac{47^\alpha}{27} = 0{\cdot}781.$

(*c*) $\dfrac{(2.4^\lambda+2.4^{\lambda-1})^\alpha}{2.3^\lambda+2.3^{\lambda-1}} = \dfrac{10^\alpha}{8} = 0{\cdot}7752.$

(*d*) $\lim_{q\to\infty} \dfrac{(2.4^\lambda+1.4^{\lambda-1}+3.4^{\lambda-2}+2.4^{\lambda-3}+\ldots+2.4^{\lambda-q})^\alpha}{2.3^\lambda+1.3^{\lambda-1}+2.3^{\lambda-2}+2.3^{\lambda-3}+\ldots+2.3^{\lambda-q}}$

$= \dfrac{125^\alpha}{24.3^\alpha} = 0{\cdot}798.$

(*e*) $\dfrac{(2.4^\lambda+1.4^{\lambda-1}+2.4^{\lambda-2})^\alpha}{2.3^\lambda+1.3^{\lambda-1}+2.3^{\lambda-2}} = \dfrac{38^\alpha}{23} = 0{\cdot}7763.$

(*f*) $\lim_{q\to\infty} \dfrac{(2.4^\lambda+1.4^{\lambda-1}+\ldots+1.4^{\lambda-q})^\alpha}{2.3^\lambda+1.3^{\lambda-1}+\ldots+1.3^{\lambda-q}} = \dfrac{2.7^\alpha}{5.3^\alpha} = 0{\cdot}783.$

(iii) (*a*) $\dfrac{(1.4^\lambda+3.4^{\lambda-1})^\alpha}{1.3^\lambda+3.3^{\lambda-1}} = \dfrac{7^\alpha}{6} = 0{\cdot}7791.$

(*b*) $\dfrac{(1.4^\lambda+2.4^{\lambda-1}+3.4^{\lambda-2}+2.4^{\lambda-3})^\alpha}{1.3^\lambda+2.3^{\lambda-1}+2.3^{\lambda-2}+2.3^{\lambda-3}} = \dfrac{110^\alpha}{53} = 0{\cdot}781.$

(*c*) $\lim_{q\to\infty} \dfrac{(1.4^\lambda+2.4^{\lambda-1}+3.4^{\lambda-2}+\ldots+3.4^{\lambda-q})^\alpha}{1.3^\lambda+2.3^{\lambda-1}+2.3^{\lambda-2}+\ldots+2.3^{\lambda-q}} = \dfrac{7^\alpha}{6} = 0{\cdot}7791.$

(*d*) $\dfrac{(1.4^\lambda+2.4^{\lambda-1}+2.4^{\lambda-2})^\alpha}{1.3^\lambda+2.3^{\lambda-1}+2.3^{\lambda-2}} = \dfrac{26^\alpha}{17} = 0{\cdot}7764.$

(*e*) $\lim_{q\to\infty} \dfrac{(1.4^\lambda+2.4^{\lambda-1}+1.4^{\lambda-2}+\ldots+1.4^{\lambda-q})^\alpha}{1.3^\lambda+2.3^{\lambda-1}+1.3^{\lambda-2}+\ldots+1.3^{\lambda-q}} = \dfrac{2.19^\alpha}{11.3^\alpha} = 0{\cdot}786.$

(*f*) $\dfrac{(1.4^\lambda+2.4^{\lambda-1}+1.4^{\lambda-2}+2.4^{\lambda-3})^\alpha}{1.3^\lambda+2.3^{\lambda-1}+1.3^{\lambda-2}+2.3^{\lambda-3}} = \dfrac{102^\alpha}{50} = 0{\cdot}782.$

The minimum therefore is that of (ii) (*c*). Hence we have

$$mE_{4,1} \geqslant \frac{10^\alpha}{8}.$$

But we have already proved that $mE_{4,1} \leqslant 10^{\alpha}/8$; hence

$$mE_{4,1} = \frac{10^{\alpha}}{8}.$$

10. Summing up our results we have in the general case that the dimension depends only on r and is

$$\frac{\log(r-1)}{\log r}.$$

In particular cases we have

$$mE_{r,0} = \left(\frac{r-2}{r-1}\right)^{\alpha},$$

which gives

$$mE_{3,0} = 0{\cdot}647,$$

$$mE_{4,0} = 0{\cdot}725,$$

and also we have

$$mE_{3,1} = 1,$$

$$mE_{4,1} = \frac{10^{\alpha}}{8} = 0{\cdot}775.$$

Hence

$$mE_{3,1} \neq mE_{3,0},$$

$$mE_{4,1} \neq mE_{3,0}.$$

Thus the measure depends on s as well as on r; it would seem that the measure increases as s increases from zero to $[(r-1)/2]$.

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NOTES ON A NET OF QUADRIC SURFACES: (V) THE PENTAHEDRAL NET

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Introduction.

The four preceding Notes of this series* have, save for a few paragraphs at the end of Note IV, been concerned with a general net of quadric surfaces; in them enough properties of the general net have been found to justify an examination of special nets, and it may be hoped that, from this standpoint, these special nets will be seen in their proper perspective, more of their properties discovered, and those of their properties which are already known better appreciated. It would be a long undertaking to consider all the different specialisations of a net of quadrics, and the importance of some of these would be small and out of all proportion to the labour involved in investigating them thoroughly. Some few of the special nets, however, would well repay study, and of these the pentahedral net is outstanding.

Contrary to what was first believed, a net of quadric surfaces does not, in general, possess any self-conjugate pentahedron. Reye showed that the existence of such a pentahedron demanded a particularisation of the net†, and that the condition thus imposed was poristic, the existence of one such pentahedron involving that of a singly-infinite set; the resulting net of quadrics may therefore fittingly be described as *pentahedral*. The faces of the pentahedra all belong to a developable ω of the third class, and so osculate a twisted cubic γ; each plane of ω belongs to one and

* (I) "The Cremona transformation", *Proc. London Math. Soc.* (2), 43 (1937), 302–315. (II) "Anharmonic covariants", *Journal London Math. Soc.*, 12 (1937), 276–280. (III) "The scroll of trisecants of the Jacobian curve", *Proc. London Math. Soc.* (2), 44 (1938), 466–480. (IV) "Combinantal covariants of low order", *Proc. London Math. Soc.* (2), 47 (1941), 123–141.

† *Journal für Math.*, 82 (1877), 75. Reye proved that a net of quadric envelopes does not, in general, possess a self-conjugate pentad of points.

only one of the pentahedra, which therefore constitute the sets of a linear series g_5^1 among the planes of ω or, what is the same thing, the sets of osculating planes of γ at the pentads of a g_5^1 on the curve.

It was suggested a few years ago* that it is preferable to define the pentahedral net of quadrics in this way rather than in the way in which it originally presented itself—as the net of polar quadrics of the points of a plane with respect to a cubic surface. It is now proposed to pursue this suggestion further, and use it, in conjunction with the results obtained in the preceding Notes of the series, to give an exposition of the properties of the pentahedral net. This net will be denoted throughout by the symbol $\mathcal{P}$, and the linear complex to which the tangents of γ belong by the symbol $\mathcal{L}$.

Among the advantages of this approach to the study of the pentahedral net is one which is exploited in the third section (§§ 13–22) of this Note. A g_5^1 on a rational curve is given algebraically by a pencil of binary quintics. Any condition which is imposed on this pencil must be reflected in some corresponding particularisation of the net of quadrics; in some further particularisation, that is, in addition to the one to which it has already been subjected in order to be pentahedral. For our purpose any particularisation of the g_5^1 is admissible so long as there is no member common to all its sets; *i.e.* so long as the quintics of the pencil do not have a common factor. One particular pencil of binary quintics, in some ways the simplest of all, is that constituted by those quintics which are linearly dependent on two fifth powers; and the special pentahedral net which arises from this pencil is studied in detail. It will be found, *inter alia*, that, while this net does not contain any plane-pairs, it has only two distinct base points, each of which must be counted four times to make up the eight which a net of quadrics in general possesses; that the Jacobian curve breaks up into two twisted cubics and its trisecant scroll into two quartic scrolls. The simplification of the algebra consequent upon the specialisation of the g_5^1 allows forms for some concomitants to be obtained which do not yield themselves so readily for a general pentahedral net; in particular an explicit expression is found for Gundelfinger's contravariant.

The first section of the Note (§§ 1–8) is concerned to set up the configuration and obtain equations for some of the loci which play principal parts in it. In § 1 and § 2 equations are given for the quadrics of $\mathcal{P}$ and for their Jacobian curve ϑ, these being the particular instances in three

* *Proc. Edinburgh Math. Soc.* (2), 4 (1936), 185. Some references to papers concerned with the pentahedral net are given there.

dimensions of equations that have previously been given for n dimensions. In § 3 the plane equation of the scroll generated by the trisecants of ϑ is obtained. In § 4 certain apolarity relations between sets of points on γ are described; these are used, in § 5, to obtain a quartic surface combinantally covariant for $\mathcal{P}$ (the equation of another such surface is given in passing), and, in § 6, to establish a (1, 1) correspondence between the quadrics of $\mathcal{P}$ and the axes of ω. At the end of § 6 a further quartic covariant appears. § 7 is concerned with the base points of the net, while the geometry in § 8 is preparatory to the second section as well as an ending for the first.

In the second section (§§ 9–12) the net $\mathcal{P}$ is considered in the light of some of the results of Notes III and IV. It will be remembered that, in Note III, a five-dimensional configuration consisting of a quadric and two Veronese surfaces was encountered; this configuration is markedly particularised for the net $\mathcal{P}$, and is described in outline in § 9 and § 10. In § 11 the net of quadric envelopes is identified which is the special instance, for $\mathcal{P}$, of the contravariant net discovered by W. P. Milne, and the geometrical identification which we give is corroborated by using Milne's canonical form. In § 12 the special forms assumed, for the pentahedral net, by one or two of its covariants are given.

The classical results in the geometry of the twisted cubic, and of the linear complex associated with it, are assumed throughout.

I.

1. The plane whose equation is

$$P \equiv x_0+\theta x_1+\theta^2 x_2+\theta^3 x_3 = 0,$$

varying with the parameter θ, belongs to a developable ω of class 3, and osculates a twisted cubic γ.

Take any two sets of five of these planes; the parameters of the five planes of the first set are the roots of a quintic

$$f(\theta) \equiv a_0+a_1\theta+a_2\theta^2+a_3\theta^3+a_4\theta^4+a_5\theta^5 = 0,$$

while the second set of planes is similarly determined by a quintic equation

$$g(\theta) \equiv b_0+b_1\theta+b_2\theta^2+b_3\theta^3+b_4\theta^4+b_5\theta^5 = 0.$$

Then, as the ratio $\lambda:\mu$ varies, the equation

$$\lambda f(\theta)+\mu g(\theta) = 0 \qquad (1)$$

determines an involution, or g_5^1, of pentahedra whose faces are planes of ω.

These pentahedra are known to be self-conjugate for the quadrics of a net $\mathfrak{P}$; if the roots of $f(\theta)=0$ are $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, then $\mathfrak{P}$ is the net determined by the three quadrics

$$Q_0 \equiv \sum_{i=1}^{5} \frac{P_i^2}{f'(\theta_i)\,g(\theta_i)} = 0, \quad Q_1 \equiv \sum_{i=1}^{5} \frac{\theta_i P_i^2}{f'(\theta_i)\,g(\theta_i)} = 0, \quad Q_2 \equiv \sum_{i=1}^{5} \frac{\theta_i^2 P_i^2}{f'(\theta_i)\,g(\theta_i)} = 0, \quad (2)$$

where $$P_i \equiv x_0 + \theta_i x_1 + \theta_i^2 x_2 + \theta_i^3 x_3.$$

The left-hand sides of the equations of these quadrics are functions of the roots of $f(\theta)=0$, these roots being the parameters of the faces of a definite pentahedron. But if the corresponding functions of the roots of any other of the quintic equations (1) are taken, the three quadrics so obtained are always* the quadrics (2). Every quadric of $\mathfrak{P}$ is linearly dependent on the quadrics (2), and is outpolar to every quadric inscribed in ω.

2. Each pentahedron is self-conjugate for each quadric of $\mathfrak{P}$, so that the polar plane of the vertex of a pentahedron with respect to any quadric of the net passes through the opposite edge of this pentahedron. Whence the Jacobian curve ϑ of $\mathfrak{P}$ is the locus of vertices of the pentahedra, while the scroll R^8 of trisecants of ϑ is generated by their edges.

Suppose then that x is a point of ϑ; the three planes of ω which pass through x all belong to the same pentahedron; hence there exist constants $\lambda, \mu, \alpha, \beta, \gamma$ such that

$$\lambda f(\theta) + \mu g(\theta) \equiv (x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3)(\alpha + \beta\theta + \gamma\theta^2).$$

Equating the different powers of θ on the two sides of this identity, and eliminating the five constants from the six linear equations so arising, we have, for the equations of ϑ,

$$\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ x_0 & x_1 & x_2 & x_3 & . & . \\ . & x_0 & x_1 & x_2 & x_3 & . \\ . & . & x_0 & x_1 & x_2 & x_3 \end{Vmatrix} = 0.$$

* These results are included in the more general results for space of any number of dimensions which are obtained in the paper, already referred to, in the *Proc. Edinburgh Math. Soc.*

An alternative form for the equations of ϑ, obtained directly as the Jacobian of Q_0, Q_1, Q_2, is

$$\begin{Vmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \end{Vmatrix} = 0,$$

where

$$\sigma_p = \sum_{i=1}^{5} \frac{\theta_i^p P_i}{f'(\theta_i) g(\theta_i)}.$$

3. It is also easy to obtain a form for the plane equation of R^8. For if the plane

$$l_0 x_0 + l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$$

touches R^8, it contains an edge of one of the pentahedra, and so has an equation of the form

$$\kappa_i(x_0 + \phi_i x_1 + \phi_i^2 x_2 + \phi_i^3 x_3) + \kappa_j(x_0 + \phi_j x_1 + \phi_j^2 x_2 + \phi_j^3 x_3) = 0,$$

where both ϕ_i and ϕ_j are roots of an equation $\lambda f(\theta) + \mu g(\theta) = 0$. Thus, since

$$l_0 = \kappa_i + \kappa_j, \quad l_1 = \kappa_i \phi_i + \kappa_j \phi_j, \quad l_2 = \kappa_i \phi_i^2 + \kappa_j \phi_j^2, \quad l_3 = \kappa_i \phi_i^3 + \kappa_j \phi_j^3,$$

we have

$$1 : \phi_i + \phi_j : \phi_i \phi_j = L_0 : -L_1 : L_2,$$

where

$$L_0 \equiv l_0 l_2 - l_1^2, \quad L_1 \equiv l_1 l_2 - l_0 l_3, \quad L_2 \equiv l_1 l_3 - l_2^2.$$

Hence there must be constants λ, μ, α, β, γ, δ such that

$$\lambda f(\theta) + \mu g(\theta) \equiv (L_2 + L_1 \theta + L_0 \theta^2)(\alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3).$$

Equating the coefficients of the different powers of θ on the two sides of this identity, and eliminating the six constants from the six equations which arise, we find

$$\begin{vmatrix} a_0 & b_0 & L_2 & . & . & . \\ a_1 & b_1 & L_1 & L_2 & . & . \\ a_2 & b_2 & L_0 & L_1 & L_2 & . \\ a_3 & b_3 & . & L_0 & L_1 & L_2 \\ a_4 & b_4 & . & . & L_0 & L_1 \\ a_5 & b_5 & . & . & . & L_0 \end{vmatrix} = 0.$$

This form of the plane equation of R^8 shows most clearly a particular feature of the pentahedral net, namely, that every plane of ω is a quadritangent plane of R^8. For the planes of ω satisfy the equations

$$L_0 = L_1 = L_2 = 0,$$

and, when these are simultaneously satisfied, the determinant has rank two only.

4. The quadrics of $\mathfrak{P}$ cut out a g_6^2 on γ, the six points of the set cut out by the quadric $\xi Q_0 + \eta Q_1 + \zeta Q_2 = 0$ being given by

$$\sum_{i=1}^{5} \frac{\xi + \eta\theta_i + \zeta\theta_i^2}{f'(\theta_i)\, g(\theta_i)} (\theta - \theta_i)^6 = 0.$$

This sextic polynomial, being a linear combination of the five sixth powers $(\theta - \theta_i)^6$, is apolar to the product of the five factors $\theta - \theta_i$; it follows similarly, from the other forms for the equations of the quadrics, that it is apolar to every quintic giving a set of the g_5^1. Thus the g_6^2 is particularised since, in general, there is no quintic apolar to all the sets of a g_6^2. Now those binary quintics which are apolar to the binary quintics of a given pencil constitute a triply infinite set, so that the g_5^1 has an apolar g_5^3. Moreover the sets of any g_5^3 consist of the third polars of a definite octavic, the only exception to this being when the g_5^3 includes a set which consists of a single element counted five times over*. But this exception cannot arise here, since such an element would belong to every set of the original g_5^1, and it is supposed always that these sets do not have a common element; thus this unique octavic always exists for a pentahedral net. Its second polars are then apolar to g_5^1, and constitute the g_6^2 cut out by the quadrics of $\mathfrak{P}$ on γ. The particularisation of the g_6^2 therefore consists in its property, not possessed by a general g_6^2, of being the set of second polars of a binary octavic. This has recently been pointed out by B. Ramamurti†, and we will call this octavic by his name. If Ramamurti's octavic, written with the usual binomial coefficients, is

$$(A, B, C, D, E, F, G, H, K\chi\theta, 1)^8,$$

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* Grace, *Proc. London Math. Soc.* (2), 28 (1928), 423–425.

† *Proc. Indian Academy of Sciences*, 9 (1939), 316.

then the conditions of apolarity are expressed by

$$\begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 & a_4 & -a_5 \\ b_0 & -b_1 & b_2 & -b_3 & b_4 & -b_5 \end{pmatrix} \begin{pmatrix} A & B & C & D \\ B & C & D & E \\ C & D & E & F \\ D & E & F & G \\ E & F & G & H \\ F & G & H & K \end{pmatrix} = 0,$$

the mutual ratios of the nine coefficients of the octavic being uniquely determined by this set of eight homogeneous linear equations.

5. The g_6^2 on γ, with the corresponding g_6^2 of planes of ω, is covariant for $\mathfrak{P}$, and loci which are covariantly related to it must themselves be covariants of $\mathfrak{P}$; for instance, the g_6^2 of planes of ω yields a set of hexahedra whose vertices lie on a surface, while, reciprocating with respect to $\mathfrak{L}$, the hexads formed by the sets of the g_6^2 on γ give, as the envelope of the planes joining groups of three points in the same set, a contravariant of $\mathfrak{P}$.

The equations of this surface and envelope are obtainable forthwith. If a point lies on the surface, then, since it is a vertex of a hexahedron, those three planes of ω which pass through it must all belong to the same set of g_6^2; hence there must exist constants $\lambda, \mu, \nu, \alpha, \beta, \gamma, \delta$ such that

$$\lambda\phi(\theta)+\mu\psi(\theta)+\nu\chi(\theta) \equiv (x_0+\theta x_1+\theta^2 x_2+\theta^3 x_3)(\alpha+\beta\theta+\gamma\theta^2+\delta\theta^3),$$

where $\phi(\theta)$, $\psi(\theta)$, $\chi(\theta)$ are any three linearly independent sextics belonging to the g_6^2. Equating the coefficients of the different powers of θ on the two sides of this identity, and eliminating the seven constants from the resulting seven equations, we obtain

$$\begin{vmatrix} A & B & C & . & . & . & x_3 \\ B & C & D & . & . & x_3 & x_2 \\ C & D & E & . & x_3 & x_2 & x_1 \\ D & E & F & x_3 & x_2 & x_1 & x_0 \\ E & F & G & x_2 & x_1 & x_0 & . \\ F & G & H & x_1 & x_0 & . & . \\ G & H & K & x_0 & . & . & . \end{vmatrix} = 0.$$

This is, then, a quartic covariant of $\mathcal{P}$. Reciprocating with respect to $\mathcal{L}$ we obtain a contravariant, of class four, whose equation arises from the above determinant on replacing x_0, x_1, x_2, x_3 respectively by $-l_3, 3l_2, -3l_1, l_0$.

Another quartic surface which, as its definition shows, is a combinantal covariant of $\mathcal{P}$ is the locus of a point O such that the intersection, O', of its polar planes with respect to all the quadrics of $\mathcal{P}$ lies in its polar plane with respect to $\mathcal{L}$. This surface must contain the Jacobian curve and the eight base points of $\mathcal{P}$; its equation is

$$\begin{vmatrix} 3x_3 & -x_2 & x_1 & -3x_0 \\ \frac{\partial Q_0}{\partial x_0} & \frac{\partial Q_0}{\partial x_1} & \frac{\partial Q_0}{\partial x_2} & \frac{\partial Q_0}{\partial x_3} \\ \frac{\partial Q_1}{\partial x_0} & \frac{\partial Q_1}{\partial x_1} & \frac{\partial Q_1}{\partial x_2} & \frac{\partial Q_1}{\partial x_3} \\ \frac{\partial Q_2}{\partial x_0} & \frac{\partial Q_2}{\partial x_1} & \frac{\partial Q_2}{\partial x_2} & \frac{\partial Q_2}{\partial x_3} \end{vmatrix} = 0.$$

6. Each quadric of $\mathcal{P}$ cuts out on γ a set of six points which is the polar set of some pair of points on γ with respect to Ramamurti's octavic; the two planes which osculate γ at this pair of points intersect in an axis of ω. Conversely: given an axis of ω, the two planes of ω which pass through it determine a pair of points on γ; if Ramamurti's octavic is polarised by this pair of points a set of six points is obtained in which γ is met by a quadric of $\mathcal{P}$. Thus there is* a (1, 1) correspondence between the quadrics of $\mathcal{P}$ and the axes of ω.

Suppose that, to speak the language more appropriate to binary forms, we take for the moment the ratio $\theta : \theta'$ in place of the parameter θ. Then the sets of points in which γ is met by the three quadrics $Q_0 = 0$, $Q_1 = 0$, $Q_2 = 0$ respectively are, on referring to (2), seen to be given by

$$\sum_{i=1}^{5} a_i \theta_i'^2 (\theta\theta_i' - \theta'\theta_i)^6 = 0, \quad \sum_{i=1}^{5} a_i \theta_i \theta_i' (\theta\theta_i' - \theta'\theta_i)^6 = 0, \quad \sum_{i=1}^{5} a_i \theta_i^2 (\theta\theta_i' - \theta'\theta_i)^6 = 0,$$

* This correspondence has, under another aspect, been noticed before. When $\mathcal{P}$ is regarded as a net of polar quadrics of a cubic surface, the point of which a given quadric of $\mathcal{P}$ is the polar must lie on a definite line, namely that axis of ω which corresponds to the quadric. See Turnbull (who refers also to Töplitz), *Proc. Cambridge Phil. Soc.*, 31 (1935), 177–179. The argument used by Ramamurti on 318 of his paper is also relevant.

where the a_i's are certain constants, the same in each of the three equations. The sextics on the left of these equations are second polars of an octavic $r_6{}^8$, and their form shows that they must respectively be

$$\frac{\partial^2 r}{\partial\theta^2}, \quad -\frac{\partial^2 r}{\partial\theta\,\partial\theta'}, \quad \frac{\partial^2 r}{\partial\theta'^2}.$$

The sextic which gives the six points cut out on γ by the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ is therefore

$$\xi\frac{\partial^2 r}{\partial\theta^2}-\eta\frac{\partial^2 r}{\partial\theta\,\partial\theta'}+\zeta\frac{\partial^2 r}{\partial\theta'^2},$$

and this is obtained by polarising r with the pair of points whose parameters are the roots of the quadratic $\xi\theta'^2+\eta\theta\theta'+\zeta\theta^2=0$.

So, returning to the non-homogeneous parameter, the (1, 1) correspondence between the quadrics of $\mathfrak{P}$ and the axes of ω is given by the fact that to the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ there corresponds that axis of ω which is the line common to those two planes whose parameters satisfy $\xi+\eta\theta+\zeta\theta^2=0$.

To this result two corollaries may be appended.

Take an edge e of one of the pentahedra; it is an axis of ω, and as such corresponds to a quadric of $\mathfrak{P}$. The intersections of this quadric with γ are obtained on polarising Ramamurti's octavic by means of two points belonging to the same set of $g_5{}^1$. It may be supposed that the quintic $f(\theta)$, whose five roots θ_i appear in (2), is the one which gives this set, so that the group of intersections is obtained by polarisation of the octavic with two of these roots, say θ_1 and θ_2. These are the roots of the quadratic $\theta_1\theta_2-\theta(\theta_1+\theta_2)+\theta^2=0$, so that the quadric of $\mathfrak{P}$ which corresponds to e is $\theta_1\theta_2 Q_0-(\theta_1+\theta_2)Q_1+Q_2=0$; this is a linear combination of $P_3{}^2$, $P_4{}^2$, $P_5{}^2$, being without the terms in $P_1{}^2$ and $P_2{}^2$; hence it is that cone of $\mathfrak{P}$ whose vertex is that vertex of the pentahedron opposite to e. The correspondence between quadrics of $\mathfrak{P}$ and axes of ω is therefore such that to the cones of $\mathfrak{P}$ there correspond those trisecants of ϑ which are respectively conjugate to the vertices of the cones.

Secondly: *those quadrics of $\mathfrak{P}$ which correspond to tangents of γ are such that the quadratic $\xi+\eta\theta+\zeta\theta^2=0$ has equal roots; hence they are those quadrics $\xi Q_0+\eta Q_1+\zeta Q_2=0$ for which $\eta^2=4\zeta\xi$. The envelope of these*

quadrics is the octadic surface* $Q_1^2 = Q_0 Q_2$; *it is a covariant of the net of quadrics, and is the third covariant quartic surface that we have obtained.*

7. No mention seems ever to have been made of the base points of the pentahedral net; they may be found as follows.

The identity $\sum_{i=1}^{5} \frac{\theta_i^p}{f'(\theta_i)} \equiv 0$ is true for $p = 0, 1, 2, 3$. Hence, if it is possible to find constants, κ and ρ say, such that

$$P_i^2 = (\kappa + \rho\theta_i)\, g(\theta_i)$$

for each of the five values of i, the equations (2) will all be satisfied. But this demands that the equation

$$(x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3)^2 = (\kappa + \rho\theta)\, g(\theta)$$

should be satisfied by each of the five roots of $f(\theta) = 0$, and hence there will be, corresponding to each base point, an identity

$$(x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3)^2 \equiv (\kappa + \rho\theta)\, g(\theta) + (\sigma + \tau\theta) f(\theta), \qquad (3)$$

where σ, τ are two more constants. The form of this identity shows that its existence or non-existence depends on the g_5^1 as a whole, and not on the choice of some particular set belonging to it.

The existence of such identities is best established by consideration of linear series. The planes of the pentahedra osculate γ at points forming sets of a g_5^1; those sets of six points which consists of a set of this g_5^1 and a point of γ constitute a (non-linear) doubly-infinite aggregate which belongs to that g_6^3 whose different sets are found by equating the right-hand side of (3) to zero and varying the constants κ, ρ, σ, τ. If it is possible to choose these constants so that the right-hand side of (3) becomes a perfect square, then the g_6^3 contains a corresponding set of points which consists of three points of γ, taken twice over. The corresponding base point (x_0, x_1, x_2, x_3) of $\mathcal{P}$ is then the intersection of the osculating planes of γ at these three points. Thus it is to be expected that the g_6^3 contains eight such double sets of three points, and this expectation is soon confirmed. If γ is transformed, by means of the g_6^3,

* See the paper "Octadic surfaces and plane quartic curves", *Proc. London Math. Soc.* (2), 34 (1932) 492–525.

into a rational twisted sextic s, the sets of the g_6^3 become the sections of s by the planes of the space in which it lies, and the number of double sets of these points is therefore the number of planes which are tritangent to s. Known formulae show, however, that s has precisely eight tritangent planes. It is true that s is not a general twisted sextic, since the g_6^3 is not itself general; through every point of s there pass the planes of a pencil cutting out a g_5^1, each set of which may be obtained in the same way from every point of s; thus s lies on a quadric, meeting the lines of one regulus each in one point and the lines of the complementary regulus each in five points. But this specialisation of s does not affect the validity of the formula for the number of its tritangent planes.

The g_6^3 is that constituted by those sextics which are apolar to all the sextics of the g_6^2 cut out on γ by the quadrics of $\mathfrak{P}$; for each member of the g_6^2 is apolar to any quintic of the g_5^1, and so to every sextic of which this quintic is a factor.

8. Consider now the relation between the surface R^8, generated by the edges of the pentahedra, and the developable surface F^4 generated by the tangents of γ.

A pentahedron consists of those five planes which osculate γ at the points of a set of the g_5^1. Were two of these planes to meet in a tangent of γ, the corresponding set of g_5^1 would have a double point at the point of contact of this tangent. Conversely: if a set of g_5^1 has a double point, the osculating plane of γ at this point counts for two among the faces of the pentahedron to which it belongs, and the tangent of γ there is an edge of this pentahedron and so a generator of R^8. Thus R^8 and F^4 have eight common generators, namely the tangents of γ at the points of the Jacobian set J of g_5^1.

Having shown that R^8 and F^4 have these eight common generators, let us consider the residual curve in which they meet. Any generator of R^8 is common to two planes of ω, and an intersection of this generator with F^4 lies on a tangent of γ. But through a point on a tangent of γ there passes only one plane of ω other than that in which the tangent itself lies; hence the only tangents of γ which the generator of R^8 can meet are those which lie in the two planes of ω intersecting in the generator. Thus the generators of R^8 are bitangents of F^4. On the other hand, let t be a tangent of γ at a point not belonging to J. Through a point of t there passes only one plane of ω other than that in which t itself lies; hence, since a generator of R^8 is the intersection of two planes of ω which belong to the same set of g_5^1, R^8 cannot meet t except in the four points

where t is met by those planes of ω which are the four remaining planes of that set of g_5^1 to which the plane of ω through t belongs. Conversely, each of these four points does lie on R^8. Hence t is a quadritangent line of R^8.

Thus, apart from their eight common generators, R^8 and F^4 touch wherever they meet.

Those four generators of R^8 on which lie its points of contact with a generator of F^4 are, as has just been shown, the four lines in which one face of a pentahedron is met by its remaining four faces; they are conjugate to the vertices of the tetrahedron formed by these remaining four faces. Now the equations (2) show clearly that the tetrahedron which is obtained by omitting any face of one of the pentahedra is self-conjugate for a pencil of quadrics of $\mathfrak{P}$, and so its vertices form a canonical set on ϑ. Thus those four generators of R^8 on which lie its points of contact with any generator of F^4 form a canonical set of generators of R^8. Now it was shown in Note III that, of the ∞^2 canonical sets of generators of R^8, ∞^1 are such that one of their singly-infinite set of transversal lines is a quadritangent line of R^8; and it was further shown that these quadritangent lines of R^8 generate a scroll ρ^8. Hence, for the net $\mathfrak{P}$, the covariant ρ^8 must include the scroll F^4; it will indeed be seen that ρ^8 is simply F^4 itself, counted twice over.

II.

9. It was shown in Note III that, when the lines of [3] are represented by the points of a quadric Ω in [5], the generators of R^8 are represented by the curve in which Ω is met by a Veronese surface.

For the net $\mathfrak{P}$ it is seen immediately that the curve on Ω which represents R^8 lies on a Veronese surface v; for the generators of R^8 are all axes of ω, and it is known* that the axes of a cubic developable are represented on Ω by the points of a Veronese surface. But now, since *all* points of this surface v represent lines of [3], *v lies entirely on Ω*. Here, then, is a property which sharply distinguishes the pentahedral net $\mathfrak{P}$ from a general net; the expression $\Xi' M' A M \Xi$ of Note III must, in consequence of the form of the matrix M which corresponds to a pentahedral net, vanish identically. The fact that v lies entirely on Ω, however, in no way prevents it from being reciprocated with respect to Ω,

* Baker, *Principles of geometry*, 4 (Cambridge, 1925), 52.

and so a second Veronese surface w, the locus of points whose polar primes with respect to Ω touch v along conics, is obtained as in Note III.

10. An axis of ω is the intersection of two planes

$$x_0+\phi_1 x_1+\phi_1^2 x_2+\phi_1^3 x_3=0,$$

$$x_0+\phi_2 x_1+\phi_2^2 x_2+\phi_2^3 x_3=0,$$

so that its line coordinates are

$$\{\phi_1\phi_2,\ -(\phi_1+\phi_2),\ 1,\ -(\phi_1^2+\phi_1\phi_2+\phi_2^2),\ -\phi_1\phi_2(\phi_1+\phi_2),\ -\phi_1^2\phi_2^2\}. \quad (4)$$

If the quadratic of which ϕ_1 and ϕ_2 are roots is $\xi+\eta\theta+\zeta\theta^2=0$, these are proportional to

$$\{\zeta\xi,\ \eta\zeta,\ \zeta^2,\ \zeta\xi-\eta^2,\ \xi\eta,\ -\xi^2\}, \quad (5)$$

which are therefore the coordinates of that axis of ω which, in the manner described in § 6, corresponds to the quadric whose equation is

$$\xi Q_0+\eta Q_1+\zeta Q_2=0.$$

Since they are homogeneous quadratic polynomials in ξ, η, ζ the points in [5] which represent the axes of ω lie on a Veronese surface v. When ξ, η, ζ are so restricted that the quadric is a cone the axis of ω, as was proved in § 6, is a generator of R^8, and so the representative point in [5] describes, on v, the curve Γ which represents this scroll.

It was proved in Note III that the coordinates of a point in [5] which represents a generator of R^8 are given by

$$y=M[\xi^2,\ \eta^2,\ \zeta^2,\ \tau\eta\zeta,\ \tau\zeta\xi,\ \tau\xi\eta]'=M\Xi,$$

where M is a non-singular matrix and $\tau^2=2$. The set of coordinates (5), when written as a matrix of one column, is in agreement with this, and has

$$M=\begin{pmatrix} . & . & . & . & \tau^{-1} & . \\ . & . & . & \tau^{-1} & . & . \\ . & . & 1 & . & . & . \\ . & -1 & . & . & \tau^{-1} & . \\ . & . & . & . & . & \tau^{-1} \\ -1 & . & . & . & . & . \end{pmatrix}.$$

The equation of Ω, the quadric in [5] whose points represent the lines of [3], is, here, $y_0y_3+y_1y_4+y_2y_5=0$, which may be written $y'Ay=0$ with

$$A=\begin{pmatrix} . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \\ 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \end{pmatrix}.$$

These forms of M and A give

$$M'AM=\begin{pmatrix} . & . & -1 & . & . & . \\ . & . & . & . & -\tau^{-1} & . \\ -1 & . & . & . & . & . \\ . & . & . & . & . & \frac{1}{2} \\ . & -\tau^{-1} & . & . & 1 & . \\ . & . & . & \frac{1}{2} & . & . \end{pmatrix},$$

and it is now easily verified that $\Xi'M'AM\Xi$ vanishes identically, in conformity with the fact that v lies entirely on Ω.

It was also shown in Note III that the points of the Veronese surface w, being the poles with respect to Ω of those primes which touch v along conics, are given parametrically by

$$y=(M'A)^{-1}[l^2,\ m^2,\ n^2,\ \tau mn,\ \tau nl,\ \tau lm]'$$

$$=\begin{pmatrix} . & -1 & . & . & . & . \\ . & . & . & . & . & \tau \\ -1 & . & . & . & . & . \\ . & 1 & . & . & \tau & . \\ . & . & . & \tau & . & . \\ . & . & 1 & . & . & . \end{pmatrix}\begin{pmatrix} l^2 \\ m^2 \\ n^2 \\ \tau mn \\ \tau nl \\ \tau lm \end{pmatrix}$$

$$=[-m^2,\ 2lm,\ -l^2,\ m^2+2nl,\ 2mn,\ n^2]'.$$

This surface does not lie entirely on Ω; those of its points which do lie on Ω satisfy

$$-m^2(m^2+2nl)+(2lm)(2mn)-l^2n^2=0,$$

$$(m^2-nl)^2=0.$$

Thus Ω meets w twice in that curve Δ on w for which $m^2 = nl$; this is the section of w by the prime L whose equation is $3y_0 + y_3 = 0$.

The intersection of L with v is seen, on referring to (5), to be given by taking $\eta^2 = 4\zeta\xi$. Indeed L meets both v and w in the same curve Δ, as is seen by taking

$$\xi : \eta : \zeta = n : -2m : l = \psi^2 : 2\psi : 1;$$

for this gives the parametric form

$$(\psi^2,\ 2\psi,\ 1,\ -3\psi^2,\ 2\psi^3,\ -\psi^4)$$

for the coordinates of a point of Δ. Any such point represents a tangent of γ, for it is obtained by putting $\phi_1 = \phi_2 = -\psi$ in (4).

It was seen in Note III that the curve of intersection of w and Ω was, for a general net of quadrics, of the eighth order, and that it represented the scroll ρ^8 generated by quadritangent lines of R^8. Thus, as was anticipated at the end of § 8, ρ^8 becomes, for the pentahedral net, F^4 taken twice.

11. Let us now consider the net $\mathfrak{P}$ from the standpoint adopted in Note IV. It was there pointed out that, for a general net of quadrics, R^8 has eight tritangent planes; these are associated, and so determine a net ν of quadric envelopes. For $\mathfrak{P}$, however, R^8 has not merely eight, but an infinite number of tritangent planes, and, indeed, of quadritangent planes, namely the planes of ω. Thus, in order to identify for $\mathfrak{P}$ those eight planes which correspond to the tritangent planes of R^8 for a general net of quadrics, some further property would have to be appealed to. This has been done previously*; but it is not necessary to know the result in order to identify the net which, for $\mathfrak{P}$, plays the part of the net ν. For every tritangent plane of R^8 is now (a quadritangent plane and) a plane of ω, so that every quadric envelope to which any eight tritangent planes of R^8 belong is inscribed in ω. The net ν is therefore now the net ϖ of quadrics which are inscribed in ω.

The existence of the net ν, as acknowledged in Note IV, was established by W. P. Milne†; moreover Milne gives a set of equations for three linearly independent envelopes $Q = 0$, $R = 0$, $S = 0$ of ν. The condition for $\kappa Q + \rho R + \sigma S = 0$ to consist of planes which pass through the tangents

* The eight planes are those which constitute the Jacobian set of the g_5^1 given by the pentahedra; see *Proc. London Math. Soc.* (2), 34 (1932), 521–522.

† *Journal London Math. Soc.*, 8 (1933), 211–216.

of a conic is that the discriminant D of $\kappa Q+\rho R+\sigma S$ should vanish. The sets of values of $\kappa:\rho:\sigma$ for which this happens are in (1, 1) correspondence with the planes of the conics; these planes form, in general, a developable of the sixth class. When the coefficients, in the equations of the quadrics of ν, are so specialised that ν becomes a net ϖ of quadrics inscribed in a cubic developable, the envelope of the planes of the conics, instead of being of the sixth class, is the cubic developable taken twice over; D must become a perfect square. This can be verified for Milne's canonical form. For this canonical form is derived from a canonical form for a net of quadric loci, and, from what has been said above, when the coefficients in the equations of these loci are so chosen that they yield a pentahedral net, ν must thereby become a net ϖ. Now the condition for Milne's equations (1) to represent a pentahedral net is easily shown to be $a_2b_3c_1=a_3b_1c_2$; and, *when this is satisfied*, the discriminant of the quadratic form $\kappa Q+\rho R+\sigma S$ is found, in Milne's notation, to be the square of the expression

$$k[\rho\sigma\{ka_2a_3(c_2r-b_3s)+b_1c_1(a_2^2c_2-a_3^2b_3)\}$$

$$+\sigma\kappa\{kb_3b_1(a_3s-c_1q)+c_2a_2(b_3^2a_3-b_1^2c_1)\}$$

$$+\kappa\rho\{kc_1c_2(b_1q-a_2r)+a_3b_3(c_1^2b_1-c_2^2a_2)\}].$$

12. After what has been written in Note IV, some of the concomitants of $\mathfrak{P}$ may be readily identified.

The surface S^8, generated by the conics belonging to ν, is here the surface generated by the conics belonging to ϖ; it is known, from the geometry of the twisted cubic, that these conics all lie on F^4. Hence, for the pentahedral net, S^8 is the surface F^4 taken twice.

The quartic surface G^4, dual to Gundelfinger's contravariant, contains the eight curves along which S^8 is touched by the base planes of ν. Thus G^4 now contains eight quartic curves in which F^4 is met by planes of ω, and must therefore be the same surface as F^4. The property, possessed by G^4, of having all the twenty-eight lines of intersection of pairs of base planes of ν for bitangents, is quickly verified. For the curve in which F^4 is met by a plane of ω consists of a tangent of γ and a conic, the tangent of γ counting twice because the plane touches F^4 all along it. The remaining planes of ω meet this one in the tangents of the conic, which are all bitangents of the composite quartic curve in which the plane meets F^4.

The identification of Gundelfinger's contravariant itself is not so simple; but it is obtained below for the specialised form of $\mathcal{P}$ to the consideration of which we now proceed.

III.

13. A pentahedral net is determined when a twisted cubic γ and a g_5^1 thereon have been assigned; and numerous specialisations of $\mathcal{P}$ arise corresponding to the different special types of g_5^1. Let us then take what is, in one way, the extreme specialisation; namely, a g_5^1 two of whose sets consist of single points counted five times; each other set of g_5^1 is linearly dependent on these two. Using the notation and coordinate system of the earlier part of this Note, and choosing the parameter θ to have the values 0 and ∞ at the two points, we take

$$f(\theta) \equiv \theta^5, \qquad g(\theta) \equiv 1.$$

The expressions appearing in the equations (2) of § 1 then become, with ϵ denoting a primitive fifth root of unity,

$$\left.\begin{aligned} Q_0 &\equiv \sum_{i=1}^{5} \frac{P_i^2}{5\theta_i^4} \equiv \sum_{i=1}^{5} \frac{(x_0+\epsilon^i x_1+\epsilon^{2i}x_2+\epsilon^{3i}x_3)^2}{5\epsilon^{4i}} \equiv x_2^2+2x_1x_3; \\ Q_1 &\equiv \sum_{i=1}^{5} \frac{\theta_i P_i^2}{5\theta_i^4} \equiv \sum_{i=1}^{5} \frac{(x_0+\epsilon^i x_1+\epsilon^{2i}x_2+\epsilon^{3i}x_3)^2}{5\epsilon^{3i}} \equiv 2x_0x_3+2x_1x_2; \\ Q_2 &\equiv \sum_{i=1}^{5} \frac{\theta_i^2 P_i^2}{5\theta_i^4} \equiv \sum_{i=1}^{5} \frac{(x_0+\epsilon^i x_1+\epsilon^{2i}x_2+\epsilon^{3i}x_3)^2}{5\epsilon^{2i}} \equiv x_1^2+2x_0x_2. \end{aligned}\right\} \quad (6)$$

Here the five roots θ_i may be the parameters of any set of the g_5^1 which consists of five distinct points, and so the roots of any equation $\lambda\theta^5+\mu=0$ which has five distinct roots; we have therefore taken them to be the roots of $\theta^5=1$. The net determined by (6) will be denoted by the symbol $\wp$.

If the coordinates of a point on γ had been given the homogeneous form $(\theta^3, -3\theta^2\theta', 3\theta\theta'^2, -\theta'^3)$, then the g_5^1 would be determined by the two binary quintics

$$f(\theta, \theta') \equiv \theta^5, \quad g(\theta, \theta') \equiv \theta'^5.$$

Ramamurti's octavic, apolar both to $f(\theta, \theta')$ and $g(\theta, \theta')$, is then*

$$r_\theta^8 \equiv \theta^4\theta'^4.$$

* For this special pencil of binary quintics, Ramamurti's octavic is the same as the Jacobian; this is due to the specialisation of the pencil, and is not true in general.

The g_6^2 cut out by the quadrics on γ must be the set of second polars of r, and, from the forms obtained above for Q_0, Q_1, Q_2 it is seen that a point of γ lies on the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ provided that its parameter satisfies

$$\theta^2\theta'^2(15\xi\theta'^2-20\eta\theta\theta'+15\zeta\theta^2)=0.$$

And this sextic is obtainable by polarisation from r, being in fact

$$\tfrac{5}{4}\left(\xi\frac{\partial^2 r}{\partial\theta'^2}-\eta\frac{\partial^2 r}{\partial\theta\,\partial\theta'}+\zeta\frac{\partial^2 r}{\partial\theta^2}\right),$$

in agreement with § 6. Thus, for the net $\wp$, the g_6^2 cut out by the quadrics on γ has four fixed points, namely the two vertices X_0 and X_3 of the tetrahedron of reference each counted twice; and the sets of g_6^2 are got by adding the pairs of points of γ to this fixed group of four.

14. That X_0 and X_3 are both base points of $\wp$ is obvious from (6). It may be confirmed, and more completely explained, by the considerations of § 7. For the g_6^3 which is apolar to the g_6^2 cut out on γ by the quadrics is that one of which four sets are determined by the sextics θ^6, $\theta^5\theta'$, $\theta\theta'^5$, θ'^6. The only linear combinations of these which are perfect squares are θ^6 and θ'^6; there are no others, and these two must count each for four of the eight which would arise, as explained in § 7, for a more general g_6^3. Hence *the base points of $\wp$ coincide four at each of two points*, namely X_0 and X_3.

Conversely: the argument of § 7 shows that if $\mathcal{P}$ has a base point on γ this point, when counted six times, makes up a set of the g_6^3; it then counts for four among the eight sets of repeated triads which a general g_6^3 possesses, and for four of the eight base points of $\mathcal{P}$.

15. Let us now address ourselves to the study of the Jacobian curve ϑ and of the scroll R^8 which is generated by its trisecants. These break up, each into two component parts. The plane equation of R^8 is, in the notation of § 3,

$$\begin{vmatrix} L_1 & L_2 & . & . \\ L_0 & L_1 & L_2 & . \\ . & L_0 & L_1 & L_2 \\ . & . & L_0 & L_1 \end{vmatrix}=0,$$

that is

$$L_1^4-3L_0L_2L_1^2+L_0^2L_2^2=0,$$

and so it factorises. Thus, for the net $\wp$, R^8 consists of two quartic scrolls.

Since the quadric envelopes

$$L_0 = 0, \quad L_1 = 0, \quad L_2 = 0,$$

are all inscribed in ω, each quartic scroll has ω for its bitangent developable; this is in accordance with the fact that every plane of ω is a quadritangent plane of R^8. Moreover, since the simultaneous interchange of l_0 with l_3 and l_1 with l_2 interchanges L_0 with L_2 and leaves L_1 unaltered, it is seen that each quartic scroll is symmetrically related to the two base points.

It is now to be expected that the sextic curve ϑ, whose trisecants generate R^8, will also break up; the presumption is that ϑ will consist of two twisted cubics, those chords of either which meet the other being trisecants of the composite curve and generating a quartic scroll. In order that these two scrolls should each be of the fourth order, it will be necessary for the two cubics to have four intersections.

These expectations are easily confirmed. The equations of ϑ are

$$\begin{Vmatrix} 0 & x_3 & x_2 & x_1 \\ x_3 & x_2 & x_1 & x_0 \\ x_2 & x_1 & x_0 & 0 \end{Vmatrix} = 0;$$

it is thus the intersection, apart from the twisted cubic $x_0/x_1 = x_1/x_2 = x_2/x_3$, of the two surfaces

$$2x_0x_1x_2 = x_1^3 + x_0^2x_3 \quad \text{and} \quad 2x_1x_2x_3 = x_2^3 + x_3^2x_0;$$

these are Cayley scrolls, their nodal lines being, respectively, $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$, the tangents of γ at X_3 and X_0. They intersect, apart from the twisted cubic which has already been excluded, in the two twisted cubics

$$\frac{qx_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{qx_3} \quad \text{and} \quad \frac{x_0}{qx_1} = -\frac{x_1}{x_2} = \frac{qx_2}{x_3},$$

where q is written for $\frac{1}{2}(1+\sqrt{5})$, so that $q^2 = q+1$; the two curves are obtained, each from the other, on replacing q by* $-q^{-1} = \frac{1}{2}(1-\sqrt{5})$. They will be called γ_1 and γ_2 respectively; the coordinates of their points are given parametrically by

$$(1, q\theta, q\theta^2, \theta^3) \quad \text{and} \quad (1, -q^{-1}\theta, -q^{-1}\theta^2, \theta^3),$$

* The primitive fifth root ϵ may be chosen so that $\epsilon + \epsilon^4 = q^{-1}$ and $\epsilon^2 + \epsilon^3 = -q$.

so that they both pass through X_0 and X_3, and have the same tangents and osculating planes there as does γ. Incidentally it is confirmed that γ_1 and γ_2 have four intersections, these being X_0 and X_3, each taken twice.

16. Those chords of γ_2 which meet γ_1 generate a quartic scroll Γ_2; two of them pass through each point of γ_2, which is therefore the nodal curve of Γ_2. Hence the equation of Γ_2 is obtained by equating to zero some homogeneous quadratic polynomial in the three expressions

$$x_0x_2+qx_1^2, \quad x_0x_3-q^2x_1x_2, \quad x_1x_3+qx_2^2;$$

for these, when equated to zero, give the linearly independent quadrics through γ_2. The quadratic polynomial is identified by the fact that it must vanish at every point of γ_1, and thus the equation of Γ_2 is found to be

$$(x_0x_3-q^2x_1x_2)^2 = (x_0x_2+qx_1^2)(x_1x_3+qx_2^2).$$

The equation of the scroll Γ_1, generated by those chords of γ_1 which meet γ_2, is obtained from this on replacing q by $-q^{-1}$, and so is

$$(x_0x_3-q^{-2}x_1x_2)^2 = (x_0x_2-q^{-1}x_1^2)(x_1x_3-q^{-1}x_2^2).$$

The equation of R^8 is found by taking the product of the equations of Γ_1 and Γ_2. The factorised form of the *plane* equation, found above, for R^8 is

$$(L_1^2-q^2L_0L_2)(L_1^2-q^{-2}L_0L_2) = 0.$$

17. We now have a Jacobian curve consisting of two twisted cubics γ_1 and γ_2, and its trisecant scroll consisting of two quartic scrolls Γ_1 and Γ_2; the generators of Γ_1 are chords of γ_1 and secants of γ_2, while the generators of Γ_2 are chords of γ_2 and secants of γ_1.

For a general net of quadrics, any point of the Jacobian curve ϑ is conjugate to a trisecant of ϑ; thus the points of γ_1 are conjugate to the generators of one of the two scrolls Γ_1, Γ_2, while the points of γ_2 are conjugate to the generators of the other scroll. On examination it is seen that the scroll whose generators are conjugate to the points of γ_1 is Γ_1, while the points of γ_2 are conjugate to the generators of Γ_2. For consider any pentahedron which is self-polar for $\wp$; each face has three intersections with γ_1 and three with γ_2, the ten vertices lying five on γ_1 and five on γ_2. An edge of the pentahedron is a chord of one of the two curves and a secant of the other. Let an edge e be a chord of γ_1, and so meet γ_2 in one

point. Any plane through e has one further intersection with γ_1 and two further intersections with γ_2; this applies, in particular, to either of the two faces of the pentahedron which intersect in e, so that these two faces together account for four points of γ_1 and five points of γ_2. The remaining vertex of the pentahedron must therefore be a point of γ_1. Now this vertex is the one opposite to e; thus an edge of a pentahedron is a chord of that twisted cubic, γ_1 or γ_2, as the case may be, which passes through its opposite vertex. This proves that the lines conjugate to the points of γ_1 are the generators of Γ_1, while the lines conjugate to the points of γ_2 are the generators of Γ_2.

It is easily verified that the points of γ_1 whose parameters are θ_1 and θ_2 are conjugate with respect to every quadric of $\wp$ provided that

$$\theta_1^2+q\theta_1\theta_2+\theta_2^2=0,$$

the corresponding relation between conjugate points of γ_2 being

$$\theta_1^2-q^{-1}\theta_1\theta_2+\theta_2^2=0.$$

18. The breaking up of the cones belonging to $\wp$ into two families can be seen otherwise; for the discriminant of the quadratic form

$$\xi Q_0+\eta Q_1+\zeta Q_2\equiv\xi(x_2^2+2x_1x_3)+2\eta(x_0x_3+x_1x_2)+\zeta(x_1^2+2x_0x_2)$$

is

$$\begin{vmatrix} 0 & 0 & \zeta & \eta \\ 0 & \zeta & \eta & \xi \\ \zeta & \eta & \xi & 0 \\ \eta & \xi & 0 & 0 \end{vmatrix} = \zeta^2\xi^2-3\zeta\xi\eta^2+\eta^4\equiv(\zeta\xi-q^{-2}\eta^2)(\zeta\xi-q^2\eta^2).$$

Thus the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ is a cone either when $\zeta\xi=q^{-2}\eta^2$ or when $\zeta\xi=q^2\eta^2$, the existence of two families of cones being thus demonstrated. When the quadrics of $\wp$ are represented by the points (ξ, η, ζ) of a plane the quartic curve whose points represent the cones of the net breaks up into two conics having double contact, the points of contact of the conics representing the cones, $Q_0=0$ and $Q_2=0$, which are common to both families.

The locus of vertices of the cones of one family will be the twisted cubic γ_1, while the cones of the other family will have their vertices on γ_2. It is seen that those cones which are represented by the points of the

conic $\zeta\xi = q^{-2}\eta^2$ have their vertices on γ_1. For the coordinates of a point on this conic are expressible parametrically as $\xi : \eta : \zeta = 1 : -q\theta : \theta^2$, and the vertex of the cone

$$x_2{}^2 + 2x_1x_3 - 2q\theta(x_0x_3 + x_1x_2) + \theta^2(x_1{}^2 + 2x_0x_2) = 0 \qquad (7)$$

is found to be the point $(1, q\theta, q\theta^2, \theta^3)$, which is on γ_1. Similarly, those cones which are represented by the points of the conic $\zeta\xi = q^2\eta^2$ have their vertices on γ_2.

19. We now proceed to obtain, for the net $\wp$, the form of the identity

$$\pi^8 \equiv (\phi^4)^2 + \lambda\sigma^8$$

which occurred in Note IV, and was there denoted by (G). This identity was not obtained for $\mathfrak{P}$; but the specialisation that has now been imposed on the pentahedral net renders its derivation more practicable.

First let us obtain the contravariant σ^8 which, when equated to zero, gives the envelope, of class eight, of the cones belonging to $\wp$. Here σ^8 will be the product of two factors Σ_1 and Σ_2, where $\Sigma_1 = 0$ is the envelope of those cones whose vertices are on γ_1 while $\Sigma_2 = 0$ arises similarly from those cones whose vertices are on γ_2. Both Σ_1 and Σ_2 will be homogeneous quartic polynomials in the plane coordinates l_0, l_1, l_2, l_3, either being obtained from the other by changing q into $-q^{-1}$ wherever it occurs in the coefficients.

The condition for a plane to touch a quadric whose point equation is given is obtained, in all the text-books on three-dimensional analytical geometry, as the vanishing of the determinant that arises when the matrix of the coefficients in the point equation of the quadric is bordered by a row and column of plane coordinates. When a plane touches a quadric, it passes through its own pole; there thus arises a system of five homogeneous linear equations having a non-zero solution, and so the matrix of these equations, which is precisely the matrix of the above bordered determinant, must have rank 4 instead of its full rank 5.

Now, though the books generally omit to say so*, this argument can be carried further. For if a plane touches a quadric at every point of a line, then there is a singly-infinite set of points, linearly dependent on any two of them, which are poles of the plane with respect to the quadric

* Sommerville, in particular, in his *Analytical geometry of three dimensions* (Cambridge, 1934), seems strangely to miss the opportunity on 217–218.

and which also lie on the plane itself. The above set of five homogeneous equations must now have two independent solutions, and so the rank of the matrix of the bordered determinant must reduce to 3. This gives a necessary and sufficient set of conditions for a plane to touch a quadric (then necessarily a cone) along a generator*.

This being so, the plane

$$l_0x_0+l_1x_1+l_2x_2+l_3x_3=0$$

is a tangent plane of the cone (7) when and only when the matrix

$$\begin{pmatrix} 0 & 0 & \theta^2 & -q\theta & l_0 \\ 0 & \theta^2 & -q\theta & 1 & l_1 \\ \theta^2 & -q\theta & 1 & 0 & l_2 \\ -q\theta & 1 & 0 & 0 & l_3 \\ l_0 & l_1 & l_2 & l_3 & 0 \end{pmatrix}$$

has rank 3. The determinant of the matrix obtained by omitting the last row and column vanishes identically, since it is the discriminant of a cone. The determinant of the whole matrix is $(l_0+l_1q\theta+l_2q\theta^2+l_3\theta^3)^2$, and its vanishing is the condition for the plane to pass through the vertex of the cone. If the last column and any row other than the last, or the last row and any column other than the last, are omitted, the determinant of the resulting matrix is a multiple of $l_0+l_1q\theta+l_2q\theta^2+l_3\theta^3$. But, if the last row and column are both retained, and some other row and column omitted, the resulting determinant is always quadratic in l_0, l_1, l_2, l_3 and, after taking account of the vanishing of $l_0+l_1q\theta+l_2q\theta^2+l_3\theta^3$ and perhaps dividing through by some power of θ, in θ. The vanishing of such a determinant causes the plane, already constrained to pass through the vertex of the cone, to touch the cone along a generator. For example: omitting the first row and column a determinant is obtained whose value is $l_2^2+2l_3l_1+2l_2l_3q\theta+l_3^2q\theta^2$; omitting the central row and fourth column a determinant is obtained whose value is

$$l_0\theta(l_0+l_1q\theta+l_2q\theta^2+l_3\theta^3)-q\theta\{ql_0^2+2ql_0l_1\theta+(l_1^2+2l_0l_2)\theta^2\};$$

and so on. The envelope of all planes which touch cones whose vertices

* An alternative form of these conditions, but not so suitable for our present purpose, is given by Bertini: *Geometria proiettiva degli iperspazi* (Messina, 1923), 148.

are on γ_1 is found by the elimination of θ between the cubic

$$l_0+l_1q\theta+l_2q\theta^2+l_3\theta^3=0$$

and any one of the quadratics. When this elimination is carried out, for example by the dialytic method (the routine algebra may be passed over), the result is found always to be

$$\Sigma_1\equiv 2l_0l_1l_2l_3+l_1^2l_2^2+2(l_0l_2^3+l_1^3l_3)+q^5l_0^2l_3^2=0.$$

Let us write $\Sigma_1\equiv\Lambda+q^5l_0^2l_3^2.$

Then $\Sigma_2\equiv\Lambda-q^{-5}l_0^2l_3^2,$

and so $\sigma^8\equiv\Sigma_1\Sigma_2\equiv\Lambda^2+11\Lambda l_0^2l_3^2-l_0^4l_3^4.$

20. Having calculated σ^8, we proceed to calculate ϕ^4. This contravariant, as explained in Note IV, is the envelope of those planes which cut φ in nets of conics for which Sylvester's invariant T vanishes. Now the plane

$$l_0x_0+l_1x_1+l_2x_2+l_3x_3=0$$

meets the quadrics

$$x_2^2+2x_3x_1=0,\quad 2x_0x_3+2x_1x_2=0,\quad x_1^2+2x_0x_2=0,$$

in conics which are projected from X_0 by the cones

$$\begin{aligned} x_2^2 \qquad\qquad &+2x_3x_1 \qquad\qquad &=0,\\ 2l_3x_3^2+2l_2x_2x_3&+2l_1x_3x_1-2l_0x_1x_2=0,\\ -l_0x_1^2+2l_2x_2^2 \quad &+2l_3x_2x_3 \quad &+2l_1x_1x_2=0. \end{aligned}$$

The value of the invariant T is now obtained forthwith from the expression given in § 389 of Salmon's *Conic sections*; omitting again a few lines of routine algebra, we find that

$$T=-16(2\Lambda+11l_0^2l_3^2),$$

where Λ is as before. So we may take

$$\phi^4\equiv 2\Lambda+11\,l_0^2l_3^2,$$

Gundelfinger's contravariant for φ.

21. The identity (G) satisfied by the contravariants of a net of quadrics tells us that there must be some linear combination of σ^8 and of the square of ϕ^4 which splits up into eight linear factors, these factors, when equated to zero, giving the base points of the net. But the eight base points of $\wp$ consist of the two points X_0 and X_3, each counted four times; it is therefore to be expected that the expression which, for a general net, is the product of eight linear factors is here a numerical multiple of $l_0^4 l_3^4$. The forms found for σ^8 and ϕ^4 clearly admit this, since

$$(2\Lambda+11 l_0^2 l_3^2)^2-4(\Lambda^2+11\Lambda l_0^2 l_3^2-l_0^4 l_3^4)\equiv 125 l_0^4 l_3^4.$$

22. An alternative derivation of the forms Σ_1 and Σ_2 is of sufficient interest to be described before the Note ends.

The cones of $\wp$ whose vertices lie on γ_1 are those quadrics

$$\xi Q_0+\eta Q_1+\zeta Q_2=0$$

for which $\zeta\xi=q^{-2}\eta^2$; since the conic $\zeta\xi=q^{-2}\eta^2$ has the line equation $4\nu\lambda=q^2\mu^2$, the envelope of these cones is the quartic surface*

$$4Q_0 Q_2=q^2 Q_1^2.$$

This is then the surface whose plane equation is $\Sigma_1=0$, and is thus of the fourth order as well as of the fourth class. It may therefore be suspected that it is a scroll, and this suspicion is confirmed by noticing the identity

$$q^3\Sigma_1\equiv(q^4 l_0 l_3-l_1 l_2)^2+2q(l_1^2+q^2 l_0 l_2)(l_2^2+q^2 l_1 l_3),$$

which shows that $\Sigma_1=0$ has the bitangent developable, of the third class, $q^2 l_0/l_1=-l_1/l_2=l_2/q^2 l_3$.

The nodal curve of the surface is put in evidence by observing that

$$\begin{aligned}\tfrac{1}{4}(q^2 Q_1^2-4Q_0 Q_2)&\equiv q^2(x_0x_3+x_1x_2)^2-(x_2^2+2x_3x_1)(x_1^2+2x_0x_2)\\&\equiv q^2\{(x_0x_3-q^{-2}x_1x_2)^2+2q^{-1}(x_0x_2-q^{-1}x_1^2)(x_1x_3-q^{-1}x_2^2)\},\end{aligned}$$

which shows that the nodal curve is, as will be expected, γ_1. The cones of $\wp$ whose vertices lie on γ_1 are the tangent cones to this scroll from the points of its nodal curve.

The plane equation, $\Sigma_1=0$, of the scroll can now be deduced from its point equation. For it is found that the chord which joins the two

* See the paper, "Octadic surfaces and plane quartic curves", quoted in § 6.

points of γ_1 whose parameters are θ_1 and θ_2 lies on the scroll provided that $(\theta_1+\theta_2)^2+2q\theta_1\theta_2=0$. Hence we require the envelope of a plane, of coordinates l_0, l_1, l_2, l_3, such that the cubic equation

$$l_0+l_1 q\theta+l_2 q\theta^2+l_3\theta^3=0$$

has two of its three roots connected by the above relation. Thus, if the three roots of the cubic are called θ_1, θ_2, θ_3, the symmetric function

$$\{(\theta_2+\theta_3)^2+2q\theta_2\theta_3\}\{(\theta_3+\theta_1)^2+2q\theta_3\theta_1\}\{(\theta_1+\theta_2)^2+2q\theta_1\theta_2\}$$

must vanish. This symmetric function is found, save for a factor which is a numerical multiple of a power of l_3, to be identical with the form found for Σ_1 in § 19.

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